Presenter: Dr Norman Do - ***2016***

Title: The hitchhikers guide to geometry. ***(18:28)***

Time	*Dialogue*
00:010 00:37	So the story of mathematics begins a long time ago and it really begins with humans collecting together bits of information they found useful and odd assortments of curios if you will. Now as the curiosity and the natural curiosity of humans took over this wealth of knowledge began to grow and take shape. A pivotal time in mathematics and indeed the sciences was when Euclid arrived. This was about 300 BC and Euclid today is famous for writing a series of thirteen books which are known as The Elements. In these thirteen books he collects together a large proportion of the amount of knowledge in mathematics at the time. You could think of the Elements as the Wikipedia for ancient Greek mathematics. Now these have arguably been called the most successful and influential textbooks of all history. In fact they are estimated to be the most published book apart from the Bible.
01:09 01:29	Now it's not actually what is true in mathematics that we learn from Euclid but why something is true in mathematics. You see let me explain it in this way. Let's say I give you a maths problem and I know you didn't come here for that, but say you wanted to show that a triangle of three equal side lengths also has three equal angles. How would you do it? Well we mathematicians usually take a complicated statement like that and try to break it down into smaller pieces. Those smaller pieces are then reconstructed using logic to explain the more complicated fact. But this begs the question, how do you know that these smaller facts are actually true? Well you repeat the process. You say can I break those smaller pieces into simpler pieces still? But where does it all end?
02:01 02:23	How do we know that we are not caught in some infinite regress of ever increasingly simpler facts? And if that is all we mathematicians did look at simpler and simpler facts then it would be a pretty boring game. Let me tell you that mathematics is interesting and I hope you will agree. So what Euclid did was to turn the whole game of mathematics on its head. He said instead of taking a complicated fact and trying to break it down, let's start at the bottom and let's use the reductionist approach. He enunciated some basic building blocks, some very very simple facts in mathematics from which he would try to deduce everything that was known at the time. In the world of geometry Euclid started off with 5 of these basic facts which he called axioms.
02:46 03:15	I call them Euclid's pencil case. Here is a picture of four of them. And these are so basic they are irrefutable essentially. Everyone has to agree that they are true and if you agree with Euclid's axioms then you must necessarily agree with everything that Euclid deduces. So the first axiom is very simple. It says …Euclid says I have a ruler, if you give me two points I can join them with a line segment. OK. There's not much happening there. The next axiom is not only do I have a ruler I have a longer ruler. If you give me two points not only can I joint them, I can extend that line in both directions. Euclid's third axiom, there is such a thing as a set square. I have one of these in my pencil case. What it means is that right angles are always the same.
03:32 03:55	If you have a right angle in your pocket and I have a right angle in my pocket we can always line them up so they look exactly the same. And the forth axiom just says we can draw circles. Give me some centre and some radius and I can swing that compass around and I can draw a circle. When things start to get more complicated is when we get to Euclid's fifth axiom. This is called the parallel postulate and it's complicated because it doesn't correspond to something that you carry around in your pencil case. What Euclid said is if you give Euclid a line and some point then he can always find a line through the point parallel to the original but doesn't touch the original. So mathematicians observed then and I think you will observe now that this fifth axiom is actually much more complicated than the previous four.

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04:24 04:50	In fact mathematicians wondered can we do away with it completely. Could we not include it in our list of atomic facts in our list of axioms and can we prove it from the other four? So the challenge was set for mathematicians and many mathematicians tried and failed to prove this complicated parallel postulate from the proceeding four axioms. And it wasn't until the 19th century, over two millennia after the time of Euclid that the puzzle was resolved. It turns out that we can't actually prove this complicated postulate from the other 4 axioms. And the general reason is as follows that Euclid's geometry is not the only game in town. There are many flavours or versions of geometry and in some of them this parallel postulate isn't true. And in fact one flavour of this geometry should be very familiar to you. It's the geometry of the earth … of the sphere.
05:24 06:02	The parallel postulate doesn't hold here because parallel lines don't even exist. One way to think about it is if you have two planes flying in straight lines over the earth is those flight paths are always going to cross sooner or later. So we even know now from Einstein's Theory of General Relativity that the space that we inhabit, the universe doesn't obey Euclid's rules, even though say drawings that you might make of a piece of paper or architectural plans or other geometric figures on our worldly scale appear Euclidian to us physics dictates that on large scales or near massive objects that's not the case. So in some sense we could say the Euclid was a little bit wrong as amazing as he was. So I would like to leave Euclid behind and jump to another period in mathematical history. Now mathematical history is intricately tied to the history of the sciences. You can think of science as the process of observing phenomena in nature, then measuring those phenomena and trying to explain what you see.
06:37 07:08	The fundamental question to ask is - how do you measure stuff? I would like to break it down and think of a simpler question. How would we measure the area of Australia? So it turns out that mathematicians have actually not come a long way in terms of measurement. We do it the same way you might have learnt it in Primary School. What you would do is draw the map of Australia on a grid of squares and you would start counting. Now we all know Australia is not some pixelated country made up of squares. There are "inny" bits and "outy" bits and "wiggly" bits. So you have to count parts of squares, but how much of that part do you count? So a way forward is to actually not try and get the exact area but to think about estimating the area. How does this grid of squares help us? Well we can get an underestimate. That's pretty easy. You just count those squares that lie completely within the map.
07:34 08:06	On the other hand you can always get an over estimate as well. Don't just count those squares that lie inside but count those squares around the edges that just touch the map. This is not particularly satisfying because we've got this underestimate over here. This overestimate over here and your answer could be any of these numbers in between. But what we do now is we just play the game again using a smaller grid of squares. What you'll find is you will get a better underestimate and a better overestimate but you still haven't honed in on your answer so what do you do? You play the game again, and again, and again and again. Infinitely often and sooner or later those over and under estimates will sandwich in the correct answer. To a mathematician this is thoroughly unsatisfying. Why when you need the area of a shape you need to do something infinitely often? Who's got that sort of time?
08:36 09:06	So a way around it was proposed. A way to try and compare the area of two different shapes without using some infinite procedure came about and the method is this. It's clear that for a shape in terms of it area, the shape is always the sum of its parts. If you can cut it up into pieces then the area of the original matches the sum of the area of those little pieces. So if I give you two shapes and you can cut one using scissors and rearrange those shapes to make the other then those two shapes that you started with have to have the same area. Let's see an example. So this is just a square and an equilateral triangle of the same area and you can see from this picture that the square can be cut into 4 pieces which rearrange to make that triangle.

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09:30 10:00	And actually it was proven by mathematicians that this is true for any two shapes you begin with with the same area as long as they are made out of straight edges. You can always cut one using straight lines into a bunch of puzzle pieces, rearrange them to get the other thereby proving that they have the same area without resorting to some infinite procedure. It turns out that this mathematical theory of measurements has a lot of counter intuitive and rather amazing facts. So let me tell you about a couple more. You see we could have instead of asking for the area of Australia asked for the length of its coastline. And I'm sure that if I asked you that question, what you would do is pull out your phone and look it up on Wikipedia and what you'd find is that the answer according to Wikipedia is 25, 760km. That's all well and good but actually if you read Wikipedia a little bit more it also says that the coastline of Australia has the length 66, 530km.
10:34 11:05	What's going on Wikipedia? Like these two answers are completely disparate. So it turns out that measuring the coastline of a country is a much more mathematically subtle issue than you might at first suspect because it all comes down to the fact that these coastlines are incredibly "wiggly". In mathematics we would say they exhibit fractal behaviour. So trying to measure the length of a coastline actually comes down to how you do it. If you use a shorter ruler you will pick up more of the wiggles in the coastline and hence you will get a longer answer. So I would like to jump to another point in mathematical history. I would like to tell you about another field of mathematics in particular. The field of mathematics is something that is very near and dear to my heart. It's what I do research in. It's called Topology. Now you can think of Topology and Geometry's 'wacky' cousin.
11:32 12:03	Like Geometry, Topology studies shapes but in Topology we play by a very different set of rules. You see in Topology you are allowed to take a shape and you're allowed bend it, stretch it, squish it, flex it, squash it and still consider it to be the same. So for instance I could take a square … and take a circle and in the eyes of the topologist these are exactly the same. The reason being that you take that square and start padding in the corners. You start rounding it out and sooner or later you end up with a circle. In fact I could have started with a triangle or a hexagon or something like that and still ended up with a circle. So in the eyes of a topologist, squares, circles, these are all boring. We don't care about them so much. Where things start to get interesting is when you think about surfaces.
12:26 12:52	To understand what a surface is pretend that you are this little ant and you are crawling around and everywhere you crawl around you notice that you are surrounded by this little patch of land just like I am now. Well what surface could you be on? Humans were confronted with this very same problem thousands of years ago. Humans realised that wherever you walked or sailed you were always surrounded by this little patch of earth around you. And humans therefore decided, rather naively that they were probably living on some infinite expanse of land that stretched infinitely far in every direction. Now we know that is not the case. As previously discussed w live on a sphere. So if you are this little ant walking around you might also be living on a sphere but what other options are there? It turns out that you could actually be living on a donut.
13:19 13:47	Or even a two holed donut or something much more mathematically crazy still. To give you more of the flavour of Topology I want to take you back many years to a computer game. A computer game called Pacman (*Microsoft Pac-man was released in 1982*) and some of you are too young to see what this game is about. It's OK. All you need to know is that Pacman's universe, Pacman's world is your computer screen. Your rectangular computer screen … but with one interesting feature. When Pacman goes off one side of the screen he comes back on the other side. Let's suppose that when Pacman goes off the top of the screen he comes up through the bottom. What we observe now is that Pacman lives on a surface and that everywhere he stands he is always surrounded by a two dimensional patch of land. He never runs into any boundaries.

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14:11 13:03	He doesn't see the side of the screen he just goes straight though it and comes in the other side. So the question we would like to ask and answer is what surface does Pacman live on? And this is how a Topologist would solve this problem. You see lets concentrate on the edges of the screen. In particular look at that top edge and that bottom edge. Remember that when Pacman goes up the top he comes in through the bottom. For Pacman he doesn't know the difference between the top edge and the bottom edge. For Pacman they are the same thing and if they are the same thing then let's put them in the same place. We are going to roll up our computer screen and glue the top edge to the bottom edge. Don't try this at home. Of course in Topology though we're allowed to bend and flex and stretch and squish our shapes. And this is what you will see.
15:02 15:34	You will see that the shape you will obtain is what a mathematician might call a cylinder and you might call a toilet paper roll. Similarly though to Pacman he doesn't see the left and right edge of the toilet paper roll. He just goes straight through them and comes back on the other side. So we should do as we did before and take those two ends and glue them together and the picture you will obtain is the following. You recognise this of course as the surface of a donut. And so hopefully you have learned something new today. You have learned that Pacman lives on a donut. (*Laughter*). We are going to shift gears now. Surfaces in some sense are two dimensional. An ant can walk around on them. Let's consider now an ant flying around in a space ship and let's suppose that our ant is in this universe where everywhere it is flying around it notices this three dimensional blob of space around it.
16:03 16:32	In some sense this is a three dimensional analogue of a surface and we mathematicians call these objects three manifolds. Now the same questions remain. What sort of universe could our ant be living in? Naively you might think that the universe then expands infinitely far in every direction, but that would be falling into the same trap as our human ancestors who thought the world was flat. Because it could actually be that the universe that our ant lives in, indeed the universe that we live in is much more complicated than that. It could be just as you can sail around the earth and come back to where you started; it could be that if you travelled far enough in a particular direction you actually came back to earth.
16:52 17:18	It could be that the universe that we live in is actually a three dimensional analogue of a sphere. Or indeed we could be living in a three dimensional analogue of a donut or something much crazier still. Whereas physicist are yet to understand exactly the shape of the universe that we live in though mathematicians aren't so constrained by reality. In mathematics we explore all of these universes and this theory of three manifolds has continued with breakthroughs happening all the time to this day. So what we have seen today is a few different sections in the history of geometry. We learnt about Euclid and what he told us about mathematical truth.
17:38 18:14	We have seen how the theory of measurement from the mathematical viewpoint can come up with quite thorny and quite counterintuitive issues. And we have seen how in mathematics we are allowed to change the rules. Bend the rules and bend our shapes. So these are just some key chapters in the story of geometry. The story of geometry plays a very important role in the story of mathematics more generally. And at least in my opinion the story of mathematics plays a crucial role in the greater story of humankinds' quest for knowledge. Thank-you.