Abstract

We study super-replication of contingent claims in markets with delay. This can be viewed as a stochastic target problem with delayed filtration. First, we establish a duality result for this setup. Our second result says that the scaling limit of super-replication prices for binomial models with a fixed number moments of delay $H$ is equal to the $G$–expectation with volatility uncertainty interval $[0, \sigma \sqrt{H + 1}]$. 
MARKET DELAY AND \textit{G–}EXPECTATIONS

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ABSTRACT. We study super-replication of contingent claims in markets with delay. This can be viewed as a stochastic target problem with delayed filtration. First, we establish a duality result for this setup. Our second result says that the scaling limit of super-replication prices for binomial models with a fixed number moments of delay $H$ is equal to the \textit{G}–expectation with volatility uncertainty interval $[0, \sigma \sqrt{H} + 1]$.

1. Introduction

This paper deals with super-replication of European options in financial markets with delay. Namely, we consider the following setup. At time $t$ the investor gives an order to buy $\gamma$ shares, however the order is executed only at time $t + h$. Thus, $h > 0$ is the market delay. Clearly, in a continuous–time model, if the risky asset has unbounded fluctuation on any finite (deterministic) time interval, then super–replication in a delayed market in general is impossible. This is the case, for instance in the Black–Scholes model. Up to date, most papers dealing with hedging of derivative securities in markets with restricted information considered the risk minimization approach, see [5, 8, 13, 14, 18].

Our approach is to study the limiting behavior of super–replication prices in the Cox-Ross–Rubinstein (CRR) binomial models of [3]. We fix a natural number $H$ and assume that the delay equals to $H$ trading times. With this type of scaling we prove that the super–replication prices in the CRR models with constant volatility $\sigma > 0$ converge to the \textit{G}–expectation of Peng [15] with uncertainty interval $[0, \sigma \sqrt{H} + 1]$. We prove this result under quite general assumptions on the payoff which in particular allows to consider path dependent payoffs. In particular, when the payoff is convex (can be path dependent) we converge to the Black–Scholes price with increased volatility $\sigma \sqrt{H} + 1$.

The proof of the above convergence theorem relies heavily on a duality result which we derive for super–replication in discrete–time financial markets with delay. As we will see, hedging in delayed markets is mathematically equivalent to hedging with delayed filtration. There are few papers [11, 12] which studied no–arbitrage criteria for hedging with restricted information and, to best of our knowledge, duality theory for super–replication of contingent claims was not studied before.

We prove a general duality result in a discrete–time financial market which lives on a finite probability space. The fact that the probability space is finite allows us to apply linear programming theory. It seems that our duality result can be
extended to a general probability space, however since our motivation comes from scaling limits of binomial models, we leave this extension for future research.

This paper is inspired by a recent work of Ichiba and Mousavi [9] where the authors considered super-replication in binomial models with delay. The authors setup was a bit different and was restricted to contingent claims with convex and path-independent payoffs. The novelty of our approach is to establish a duality theory and apply it for studying the asymptotic behaviour of the super-replication prices. This approach allows to treat a much more general setup.

The paper is organized as follows. In Section 2 we introduce the setup. Section 3 provides the duality result. In Section 4 we give the scaling limit of super-replication prices.

2. Preliminaries and the model

Let \( \bar{\Omega} = \{-1,1\}^N \) be the space of infinite sequences \( \omega = (\omega_1, \omega_2, \ldots) ; \omega_i \in \{-1,1\} \) with the product probability \( \mathbb{P} = \{\frac{1}{2}, \frac{1}{2}\}^N \). Define the canonical sequence of independent and identically distributed (i.i.d.) random variables \( \xi_1, \xi_2, \ldots \) by \( \xi_i(\omega) = \omega_i, \ i \in \mathbb{N} \), and consider the natural filtration \( \mathcal{F}_k = \sigma(\xi_1, \ldots, \xi_k), \ k \geq 1 \) and let \( \mathcal{F}_0 \) be trivial.

Next, we introduce a sequence of binomial models with a constant interest rate \( r > 0 \) and constant volatility \( \sigma > 0 \). For any \( n \) consider the \( n \)-step binomial model of a financial market which is active at times \( 0, 1/n, 2/n, \ldots, 1 \). We assume that the market consists of a savings account and of a stock. The price of a saving account at time \( k/n \) given by \( F_k^{(n)} = e^{rk/n} \), and the stock price at time \( k/n \) is given by

\[
S_k^{(n)} = s \exp \left( rk/n + \sigma \sqrt{\frac{T}{n}} \sum_{i=1}^{k} \xi_i \right), \quad k = 0, 1, \ldots, n,
\]

where \( s > 0 \) is the initial stock price.

2.1. Wealth dynamics and super-replication with delay. As in [9], we consider a situation where there is a delay of \( H \) periods in the execution of the trading orders. The investor is aware of the delay. We assume that \( H \) is constant which does not depend on the time step of the binomial models.

Next, we define the notion of a self-financing portfolio. Fix \( n > H \) (we assume that \( n \) is sufficiently large) and consider the \( n \)-step binomial model. A self-financing portfolio \( \pi \) with an initial capital \( x \) is a pair \( \pi = (x, (\beta_k)_{k=0}^{n-H}) \) where for any \( k, \beta_k \) is a \( \mathcal{F}_k \)-measurable random variable. Here \( \beta_k \) represents the number of stocks that the investor buy (negative \( \beta_k \) means selling) at the moment \( k/n \). The portfolio dynamics is modeled by a pair of adapted processes \( \{(Y_k^{\pi}, Z_k^{\pi})\}_{k=0}^{n} \), where \( Y_k^{\pi} \) is the amount on the saving account at time \( k \) and \( Z_k^{\pi} \) is the number of stocks at time \( k \). The dynamics is given by the following recursive relations

\[
Y_0^{\pi} = 0, \quad Y_k^{\pi} = e^{r/n}Y_{k-1}^{\pi} - \beta_{k-H}S_k^{(n)} \text{ for } k = 1, \ldots, n
\]

\[
Z_k^{\pi} = \sum_{i=0}^{k} \beta_{i-H}, \quad k = 0, 1, \ldots, n,
\]

where we set \( \beta_{-H}, \beta_{1-H}, \ldots, \beta_{-1} \equiv 0 \). We deal with a cash settled claim, and so at time \( 1 - H/n \) the investor gives an order to liquidate the portfolio (the order will be executed at the maturity date \( T = 1 \)). Formally, it means that \( Z_n^{\pi} = 0 \). The portfolio value at the maturity date equals to \( Y_n^{\pi} \). We denote by \( A_n(x) \) the set of all trading strategies with an initial capital \( x \).
If we discount all terms, i.e. we set \( \bar{Y}_k^n := e^{-r_k/n} Y_k^n \), \( \bar{S}_k^{(n)} := e^{-r_k/n} S_k^{(n)} \), then the discounted portfolio (at the maturity date) value is given by

\[
(2.1) \quad \bar{Y}_n^n = x - \sum_{k=H}^{n} \beta_{k-H} \bar{S}_k^{(n)} = x - \sum_{k=H}^{n} (Z_k^n - Z_{k-1}^n) \bar{S}_k^{(n)} = x + \sum_{k=H}^{n-1} Z_k^n (\bar{S}_k^{(n)} - \bar{S}_{k-1}^{(n)})
\]

where the last equality follows from the integration by parts formula and the fact that \( Z_n^n = 0 \). Thus, we see that the discounted portfolio value can be represented as a partial sum with respect to the discount stock price, where the integrand \( \{Z_k^n\}_{k=H}^{n-1} \) can be any stochastic process such that \( Z_k^n \) is \( \mathcal{F}_{k-H} \)-measurable. In other words, hedging with market delay is equivalent to hedging with delayed filtration.

We are interested in super-replication of European contingent claims. Formally, given a random variable \( F_n \) (which represents the payoff of the claim) which is \( \mathcal{F}_n \)-measurable, the super-replication price is given by

\[
V_n := V_n(F_n) = \inf \{ x \mid \exists \pi \in \mathcal{A}_n(x) \text{ such that } Y_n^n \geq F_n, \ P \text{-a.s.} \}.
\]

3. Duality

Our first result is the characterization of the dual problem. We believe that this simple result is of independent interest as well. Also it will be the essential tool to study the asymptotic behavior of the super-replication prices. During this section we fix \( n > H \).

**Theorem 3.1.** Let \( \mathcal{Q}_n^H \) be the set of all probability measures \( \mathcal{Q} \) on \( (\bar{\Omega}, \mathcal{F}_n) \) for which

\[
(3.1) \quad \mathbb{E}_\mathcal{Q}(\bar{S}_{k+H+1}^{(n)} - \bar{S}_{k+H}^{(n)} | \mathcal{F}_k) = 0, \quad k = 0, 1, \ldots, n-1-H
\]

where \( \mathbb{E}_\mathcal{Q} \) denotes the expectation with respect to the probability measure \( \mathcal{Q} \). Then

\[
(3.2) \quad V_n = e^{-r} \sup_{\mathcal{Q} \in \mathcal{Q}_n^H} \mathbb{E}_\mathcal{Q}[F_n].
\]

**Proof.** We model the \( n \)-step binomial model with delayed information as following. Consider a tree whose paths are sequences of the form \((a_1, \ldots, a_k) \in \{-1, 1\}^k \), \( 0 \leq k \leq n \). The set of all paths will be denoted by \( \mathbb{U} \). Let \( T = \{-1, 1\}^n \) be the set of all terminal nodes. The empty path (corresponds to the case \( k = 0 \)) is the root of the tree and will be denoted by \( \emptyset \). For \( u = (u_1, \ldots, u_k) \in \mathbb{U} \setminus \{\emptyset\} \) we define \( u^- := (u_1, \ldots, u_{k-1}) \) to be the unique direct predecessor of \( u \). Furthermore, for \( k > H \) we introduce the predecessor \( u^{(H)} = (u_1, \ldots, u_{k-1-H}) \). For \( u \in \mathbb{U} \), \( l(u) \) is the number of elements in the sequence \( u \), where we set \( l(\emptyset) = 0 \).

Next, define the functions \( \bar{S} \colon \mathbb{U} \to \mathbb{R} \) and \( \bar{F} \colon \mathbb{U} \to \mathbb{R}_+ \) by

\[
\bar{S}(u_1, \ldots, u_k) = s \exp \left( \frac{\sigma}{\sqrt{n}} \sum_{i=1}^{k} u_i \right), \quad \bar{F}(u_1, \ldots, u_n) = e^{-r} f_n \left( e^{r/n} \bar{S}(u_1), e^{2r/n} \bar{S}(u_1, u_2), \ldots, e^{r} \bar{S}(u_1, \ldots, u_n) \right)
\]

where \( f_n : \mathbb{R} \to \mathbb{R} \) is a measurable function such that \( F_n = f_n(S_1^{(n)}, \ldots, S_n^{(n)}) \). Recall (2.1) and set \( Z_k^n = Z_{k+H}^n \) for \( k = 0, \ldots, n-1-H \). The process \( \{Z_k^n\}_{k=0}^{n-1-H} \) can be any adapted stochastic process. Thus, the super-replication price \( V_n \) is the solution of the following linear programming problem

\[
(3.3) \quad \text{minimize } V(\emptyset)
\]
over all $V, \tilde{Z} : U \to \mathbb{R}$ subject to the constraints

$$V(u) - V(u^-) - \tilde{Z}(u^H)(\tilde{S}(u) - \tilde{S}(u^-)) \leq 0, \ \forall u \in U \setminus \{\emptyset\}$$

and

$$\tilde{F}(u) - V(u) \leq 0, \ \forall u \in \mathbb{T},$$

where $\tilde{Z}(u^H) \equiv 0$ for all $u$ with $l(u) \leq H$. Notice that the constraint (3.4) should be in fact an equality. However this modification of the constraint does not alter the value of the optimization problem.

From the standard theory of linear programming (see Corollary 7.1g in [17]) we obtain that the dual problem of the linear programming (3.3)–(3.5) is given by

$$\text{maximize} \sum_{u \in \mathbb{T}} q_u \tilde{F}(u)$$

over all $q : \mathbb{T} \to \mathbb{R}_+$ and $q : U \to \mathbb{R}_+$ subject to the constraints

$$q(\emptyset) = 1, \ q_u = \sum_{u,v = u} q_u \ \forall v \in U \setminus \mathbb{T}$$

$$q_u = \hat{q}_u \ \forall u \in \mathbb{T}$$

$$\sum_{u,v = u} q_u \tilde{S}(u) = \sum_{u,v = u} q_u \tilde{S}(u^-), \ \forall v \in U \text{ with } l(v) \leq n - H - 1.$$  

It remains to understand the dual problem. The first two relations in (3.7) imply (recall that $q \geq 0$) that $q : U \to \mathbb{R}_+$ corresponds to a probability measure $\mathbb{Q}$. The last equality in (3.7) implies

$$\mathbb{E}_Q(\tilde{Z}^{(n)}(\mathbb{F}_k) = \mathbb{E}_Q(\tilde{Z}^{(n)}(\mathbb{F}_k), \ k = 0, 1, \ldots, n - 1 - H.$$  

This together with (3.6) and the third equality in (3.7) gives that the value of the dual problem (3.6)–(3.7) equals to the right hand side of (3.2).

Thus in order to complete the proof, it remains to argue that in our case strong duality holds. In view of Corollary 7.1g in [17] we need to argue that in both of the above linear programming problems, there is an element which satisfies the constraints. Indeed, in (3.4)–(3.5) we take $\tilde{Z} \equiv 0$ and $V \equiv \max_{u \in \mathbb{T}} \tilde{F}(u)$. In (3.7) we take $q : U \to \mathbb{R}_+$ which corresponds to the unique martingale measure for the $n$–step binomial model (without delay). This completes the proof.

\[ \square \]

**Remark 3.2.** Clearly, the proof of Theorem 3.1 can be extended to a general discrete financial market which is defined on a finite probability space. A sufficient condition for duality to hold is that there exists a martingale measure for the un delayed financial market. Let us observe that (3.1) is equivalent to the fact that $M_k^{(n)} := \mathbb{E}_Q(S_k^{(n)}|\mathbb{F}_k), \ k = H, H + 1, \ldots, n$ is a $\mathbb{Q}$–martingale with respect to the delayed filtration $(\mathbb{F}_{k-H})_{k=H}^n$. Namely, the projection of the risky asset on the delayed filtration is a martingale. In [12], the authors proved that in a general discrete–time financial market with delayed filtration, this condition is equivalent to the absence of arbitrage.

4. **Asymptotic behaviour and $G$–expectations**

The main result of this paper is the identification of the limit (as time step goes to zero) of the super–replication prices as a $G$–expectation in the sense of Peng [15]. Formally, let $\Omega = C([0,1],\mathbb{R})$ be the space of continuous paths $f : [0,1] \to \mathbb{R}$ equipped with the topology of uniform convergence and the Borel $\sigma$–field $\mathcal{F} = \mathcal{B}(\Omega)$.
We denote by $\mathbb{B} = \mathbb{B}_t$, $t \geq 0$ the canonical process $\mathbb{B}_t(\omega) = \omega_t$. Introduce the following set of probability measures on $(\Omega, F)$

$$Q^H := \{Q : \text{is a } \mathbb{Q}\text{-martingale with } \mathbb{B}_0 = 0, \text{ } d\mathbb{B}/dt \leq \sigma^2(H + 1) \mathbb{Q} \otimes dt \text{-a.s.}\}$$

We assume the following.

**Assumption 4.1.** Let $F : \Omega \to \mathbb{R}_+$ be a continuous map such that there exists a constants $C, p > 0$ for which

$$F(\omega) \leq C(1 + \|\omega\|_p^p), \quad \forall \omega \in \Omega.$$  

For any $n \in \mathbb{N}$, let $W_n : \mathbb{R}^{n+1} \to \Omega$ be the linear interpolation operator given by

$$W_n(y)(t) := ([nt] + 1 - nt) y_{[nt]} + (nt - [nt]) y_{[nt]+1}, \quad \forall t \in [0, 1]$$

where $y = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}$ and $[z]$ denotes the integer part of $z$. In the $n$–step binomial model the payoff of the European contingent claim is given by

$$F_n := F\left(W_n(S^{(n)})\right)$$

where, by definition we consider, $W_n(S^{(n)})$ as a random element with values in $\Omega$.

Next, we formulate our main result.

**Theorem 4.2.** Assume **Assumption 4.1.** Then the limit of the super–replication prices is given by

$$\lim_{n \to \infty} V_n = e^{-r} \sup_{Q \in Q^n} \mathbb{E}_Q[F(S)]$$

where

$$S_t = \exp(rt + \mathbb{B}_t - \langle \mathbb{B}_t \rangle/2), \quad t \in [0, 1].$$

4.1. **Proof of the Upper Bound.** In this section we prove that

$$\lim_{n \to \infty} V_n \leq e^{-r} \sup_{Q \in Q^n} \mathbb{E}_Q[F(S)].$$

Without loss of generality (by passing to a subsequence) we assume that $\lim_{n \to \infty} V_n$ exists (it may be $\infty$). From the duality result it follows that for any $n \in \mathbb{N}$ there exists a probability measure $Q_n \in Q^n$ such that

$$V_n < \frac{1}{n} + e^{-r} \mathbb{E}_{Q_n}[F_n].$$

For any $n \in \mathbb{N}$ introduce the martingale $\{M_k^{(n)}\}_{k=0}^n$

$$M_k^{(n)} := \mathbb{E}_{Q_n}(S_k^{(n)}|F_{k-1}), \quad k \geq H$$

and we set $M_k^{(n)} = M_H^{(n)}$ for $k < H$. Clearly, there exists a constant $C$ such that

$$|M_k^{(n)} - S_k^{(n)}| \leq \frac{C}{\sqrt{n}} S_k^{(n)} \forall k \leq n.$$  

By applying Lemma 3.3 in [2] we obtain that $(W_n(M^{(n)})|Q_n)$ is tight on the space $\Omega = C([0, 1], \mathbb{R})$ and

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{Q_n}[\max_{0 \leq k \leq n} M_k^{(n)} + |\ln M_k^{(n)}|^{2m} < \infty, \quad \forall m > 0].$$

So there exists a subsequence (which, for ease of notation, we still index by $n$) which converges in distribution to a continuous stochastic process $M = \{M_t\}_{t=0}^1$. The
Next, from Assumption 4.1 and (4.2)–(4.4) we obtain
\[
\lim_{n \to \infty} V_n \leq e^{-\tau}E[F(\hat{M})]
\]
where \(\hat{M}_t = e^{\tau M_t}, \ t \in [0, 1]\). Introduce the continuous local martingale \(N_1 := \int_0^t \frac{dM_s}{M_s^2}, \ t \in [0, 1]\). Clearly,
\[
\hat{M}_t = \exp(rt + N_t - \langle N_t \rangle/2), \ t \in [0, 1].
\]

We conclude that in order to establish (4.1), it remains to prove the following lemma.

**Lemma 4.3.** The quadratic variation of the local martingale \(N\) satisfies
\[
\frac{d(N)}{dt} \leq \sigma^2(H + 1) \ \forall t \in [0, 1], \ a.s.,
\]
which in particular implies that \(N\) is a (true) martingale.

*Proof.* The implication that \(N\) is a martingale is clear from the Burkholder–Davis–Gundy inequality. Thus, let us prove (4.5).

Fix \(n \in \mathbb{N}\). From the Taylor expansion for \(e^x\) we get that for any \(k > H\),
\[
\tilde{S}^{(n)}_k - \tilde{S}^{(n)}_{k-1} = \tilde{S}^{(n)}_{k-H-1} \left(1 + \frac{\sigma_k}{\sqrt{n}} + O(1/n)\right)
\]
where the term \(O(1/n)\) is uniformly bounded by \(c/n\) for some constant \(c\). This together with (3.1) gives (recall that \(Q_n \in Q_n^H\))
\[
\mathbb{E}_{Q_n}(\xi_k|\mathcal{F}_{k-H-1}) = O(1/\sqrt{n}), \ k = H + 1, \ldots, n.
\]
For any \(j = 0, \ldots, H\) introduce the stochastic processes \(A^{n,j} = \{A^{n,j}_k\}_{k=0}^n\) and \(M^{n,j} = \{M^{n,j}_k\}_{k=0}^n\)
\[
A^{n,j}_k = \frac{\sigma}{\sqrt{n}} \sum_{i=1}^{[k/(H+1)]} \mathbb{E}_{Q_n} \left(\xi_{(i+1)(H+1)}|\mathcal{F}_{(i-1)(H+1)+j}\right),
\]
\[
M^{n,j}_k = \frac{\sigma}{\sqrt{n}} \left(\sum_{i=1}^{[k/(H+1)]} \xi_{(i+1)(H+1)+j} - \mathbb{E}_{Q_n} \left(\xi_{(i+1)(H+1)+j}|\mathcal{F}_{(i-1)(H+1)+j}\right)\right)
\]
where as before \([z]\) is the integer part of \(z\).

Fix \(j\). Clearly, \(M^{n,j}\) is a martingale with respect to the filtration generated by \(A^{n,j}\) and \(M^{n,j}\). Notice that for any \(i\), \(M^{n,j}_{i(H+1)} = M^{n,j}_{i(H+1)+1} = \ldots = M^{n,j}_{i(H+1)+H}\). Moreover, (4.6) implies that
\[
\mathbb{E}_{Q_n} \left((M^{n,j}_{(i+1)(H+1)} - M^{n,j}_{i(H+1)+j})^2|\mathcal{F}_{i(H+1)+j}\right) = \frac{\sigma^2}{n} + O(n^{-3/2}).
\]
Thus, from the Martingale Central Limit Theorem (Theorem 7.4.1 in [7]) it follows that
\[
\left\{M^{n,j}_{[nt]}\right\}_{t=0}^1 \Rightarrow \frac{\sigma}{\sqrt{H+1}}W,
\]
where \(W = \{W_t\}_{t=0}^1\) is a standard Brownian motion. From (4.6) we obtain that the sequence \(\left\{A^{n,j}_{[nt]}\right\}_{t=0}^1\), \(n \in \mathbb{N}\) is tight (on the Skorokhod space of right continuous with left limit functions), and any cluster point is a Lipschitz continuous
process. We conclude that there exists a subsequence (which, for ease of notation, we still index by $n$) such that we have the convergence of the joint distributions
\begin{equation}
\left(\left\{ M_{[nt]}^{n,j}_{1} \right\}_{t=0}^{1}, \left\{ A_{[nt]}^{n,j}_{1} \right\}_{t=0}^{1}\right), Q_{n} \Rightarrow \left( W^{(j)}, A^{(j)} \right)
\end{equation}
where $A^{(j)} = \left\{ A^{(j)}_{i} \right\}_{t=0}^{1}$ is a Lipschitz continuous process and $W^{(j)} = \left\{ W^{(j)}_{i} \right\}_{t=0}^{1}$ has the same distribution as $\frac{\sigma^{2}}{H+1} W$. Next, from the fact that for all $n$, $M^{n,j}$ is a martingale with respect to the filtration generated by $A^{n,j}$ and $M^{n,j}$, and its predictable variation is uniformly bounded (follows from (4.7)), we obtain that the limit process $W^{(j)}$ is a martingale with respect to the filtration generated by $W^{(j)}$ and $A^{(j)}$ (for details see Chapter 9 in [10]). Thus, from (4.8) it follows that
\begin{equation}
(W^{(j)} + A^{(j)})_{t} = \langle W^{(j)} \rangle_{t} = \frac{\sigma^{2}t}{H+1} \quad \forall t \in [0,1], \text{ a.s.}
\end{equation}
Now, we arrive to the final step of the proof. Without loss of generality (by passing to a subsequence), we assume that the sequence of the joint distributions
\begin{equation}
\left(\left\{ M_{[nt]}^{n,0}_{1} \right\}_{t=0}^{1}, \ldots, \left\{ M_{[nt]}^{n,H}_{1} \right\}_{t=0}^{1}, \left\{ A_{[nt]}^{n,0}_{1} \right\}_{t=0}^{1}, \ldots, \left\{ A_{[nt]}^{n,H}_{1} \right\}_{t=0}^{1}\right), Q_{n}
\end{equation}
converges. Observe that
\begin{equation}
\frac{\sigma}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_{i} = O(n^{-1/2}) + \sum_{j=0,1,\ldots,H} \left( A_{[nt]}^{n,j} + M_{[nt]}^{n,j} \right).
\end{equation}
This together with (4.9) and the weak convergence
\begin{equation}
\left(\left\{ \frac{\sigma}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_{i} \right\}_{t=0}^{1}, Q_{n} \right) \Rightarrow \left\{ N_{t} - \langle N \rangle_{t}/2 \right\}_{t=0}^{1}
\end{equation}
gives that the distribution of $\sum_{j=0,1,\ldots,H} \left( A^{(j)} + W^{(j)} \right)$ equals to the distribution of $N - \langle N \rangle/2$. Moreover, from the equality $\langle N \rangle \equiv \langle N - \langle N \rangle \rangle/2$ we deduce that, in (4.5) we can replace $N$ with $\sum_{j=0,1,\ldots,H} \left( A^{(j)} + W^{(j)} \right)$.

Finally, from (4.10) and the simple inequality
\begin{equation}
(H+1) \left( \sum_{i=1}^{H} a_{i} \right)^{2} \leq (H+1) \sum_{i=1}^{H+1} a_{i}^{2}, \quad a_{1}, \ldots, a_{H+1} \in \mathbb{R}
\end{equation}
we get that for any $s < t$
\begin{equation}
\langle \sum_{j=0,1,\ldots,H} \left( A^{(j)} + W^{(j)} \right) \rangle_{t} - \langle \sum_{j=0,1,\ldots,H} \left( A^{(j)} + W^{(j)} \right) \rangle_{s} \leq (H+1) \sum_{j=0,1,\ldots,H} \left( \left( A^{(j)} + W^{(j)} \right)_{t} - \left( A^{(j)} + W^{(j)} \right)_{s} \right) \leq \sigma^{2}(H+1)(t-s)
\end{equation}
and (4.5) follows.

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4.2. Proof of the Lower Bound. For an arbitrary complete probability space $(\Omega^{W}, F^{W}, \mathbb{P}^{W})$ carrying a standard Brownian motion $\{W_{t}\}_{t=0}^{1}$ we denote by $\Gamma$ the set of processes $\alpha = \{a_{t}\}_{t=0}^{1}$ which are progressively measurable with respect to the augmented filtration generated by $W$ with values in the set $[0, \sigma \sqrt{H+1}]$. Applying a randomization technique similar to Lemma 7.2 in [6] we obtain
\begin{equation}
\sup_{Q \in \mathcal{Q}^{W}} \mathbb{E}_{Q}[F(S)] = \sup_{\alpha \in \Gamma} \mathbb{E}_{\mathbb{P}^{W}} [F(S^{(\alpha)})]
\end{equation}
where
\[ S_t^{(\alpha)} = s \exp \left( rt + \int_0^t \alpha_u dW_u - \frac{1}{2} \int_0^t \alpha_u^2 du \right), \quad t \in [0, 1]. \]

Denote \( \tilde{S}_t^{(\alpha)} = e^{-rt} S_t^{(\alpha)}, \ t \in [0, 1]. \)

Next, standard density arguments (see Lemma 3.4 in [2] for the case \( d = 1 \)) imply that
\[ \sup_{\alpha \in \Gamma_c} \mathbb{E}_{\tilde{\rho}} \left[ F(S_t^{(\alpha)}) \right] = \sup_{\alpha \in \Gamma_c} \mathbb{E}_{\tilde{\rho}} \left[ F(S_t^{(\alpha)}) \right] \]
where \( \Gamma_c \subset \Gamma \) is the set of all processes \( \{\alpha_t\}_{t=0}^1 \) of the form
\[ \alpha_t = \sum_{j=0}^{J-1} \rho_j (\tilde{S}_{t_j}^{(\alpha)}, \ldots, \tilde{S}_{t_{j+1}}^{(\alpha)}) 1_{(t_j, t_{j+1}]}(t) \]
for some times \( 0 = t_0 < t_1 < \cdots < t_J = 1, \epsilon > 0 \), and continuous bounded functions \( \rho_j : \mathbb{R}^J \to [\epsilon, \sigma \sqrt{\mathbb{H} + 1}] \).

Observe that \( \rho_0 \) is a constant.

By applying Levy’s theorem in a similar way to Lemma 4.6 in [1] we arrive that \( M = S^{(\alpha)} \) is the unique (in law) martingale with initial value \( M_0 = s \) which satisfies
\[ \left\{ \begin{array}{l}
\sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} \frac{dM_u}{\rho_j(M_{t_j}, \ldots, M_{t_{j+1}})} \\
\end{array} \right\}_{t=0}^1 \]
\[ \text{is a standard Brownian motion.} \]

From Theorem 3.1, Assumption 4.1 and (4.11)–(4.12), it follows that in order to prove the lower bound, i.e. the inequality
\[ \lim_{n \to \infty} \inf_{V_n} \sup_{Q \in \mathcal{Q}_n} \mathbb{E}_Q[F(S)] \]
it remains to establish the following lemma.

**Lemma 4.4.** For any \( \alpha \in \Gamma_c \) there exists a sequence of probability measures \( \tilde{Q}_n \in \mathcal{Q}_n^H \) such that we have the weak convergence \( \left\{ \tilde{S}_t^{(n)} \right\}_{t=0}^1 \Rightarrow \tilde{S}_t^{(\alpha)} \). \( \Rightarrow \{\tilde{S}_t^{(\alpha)}\}_{t=0}^1. \)

**Proof.** Choose \( \alpha \in \Gamma_c \) and let \( 0 = t_0 < t_1 < \cdots < t_J = 1, \epsilon > 0, \rho_j : \mathbb{R}^J \to [\epsilon, \sigma \sqrt{\mathbb{H} + 1}], J = 0, 1, \ldots, J-1 \) such that (4.13) holds true. The proof will be done in three steps.

**Step 1:** In this step we construct the probability measures \( \tilde{Q}_n \), \( n \in \mathbb{N} \). Fix \( n \in \mathbb{N} \). For any \( j = 0, 1, \ldots, J-1 \) consider the interval \( I_j := ([nt_j], [nt_{j+1}]) \) and let \( U_1^{(j)}, U_2^{(j)}, U_3^{(j)} \subset I_j \) be disjoint sets such that \( U_1^{(j)} \cup U_2^{(j)} \cup U_3^{(j)} = I_j \).

We allow \( U_1^{(j)}, U_2^{(j)}, U_3^{(j)} \) to be random sets in the sense that they can depend on \( S_1^{(n)}, \ldots, S_j^{(n)} \) (i.e. can depend on the stock prices up to the moment \([nt_j])\).

From the simple estimate \( \tilde{S}_k^{(n)} = \tilde{S}_k^{(n)} \left( 1 + \frac{\sigma \sqrt{\mathbb{H} + 1}}{\sqrt{n}} + O(n^{-1}) \right) \) it follows that for sufficiently large \( n \) we can find a probability measure \( \tilde{Q}_n \) such that
\[ \mathbb{E}_{\tilde{Q}_n} \left( \tilde{S}_k^{(n)} - \tilde{S}_k^{(n)} | F_k \right) = 0 \ \forall k \in U_1^{(j)} \]
\[ \mathbb{E}_{\tilde{Q}_n} \left( \tilde{S}_k^{(n)} - \tilde{S}_k^{(n)} | F_k \right) = \left( 1 - \frac{1}{\sqrt{n}} \right) \left( \tilde{S}_k^{(n)} - \tilde{S}_{k-1}^{(n)} \right) \ \forall k \in U_2^{(j)} \]
\[ \mathbb{E}_{\tilde{Q}_n} \left( \tilde{S}_k^{(n)} - \tilde{S}_k^{(n)} | F_k \right) = -\left( 1 - \frac{1}{\sqrt{n}} \right) \left( \tilde{S}_k^{(n)} - \tilde{S}_{k-1}^{(n)} \right) \ \forall k \in U_3^{(j)}. \]
Now, we explain how to choose the sets $U_1^{(j)}, U_2^{(j)}, U_3^{(j)}$, $j = 0, 1, ..., J - 1$. For any $j$ divide the interval $I_j$ into $[\sqrt{n}(t_{j+1} - t_j)]$ blocks of the same number $[\sqrt{n}]$ of successive time points. Because we restrict ourselves to integer blocks it might happen that these blocks cover the interval $I_j$, besides of $O(n^{-1/2})$ successive time points which lie in the right end of the interval. We define all the not covered time points to be elements of the set $U_1^{(j)}$.

Next, on each of the $[\sqrt{n}(t_{j+1} - t_j)]$ blocks of $[\sqrt{n}]$ successive time points we apply the following procedure. First, each block is divided into $[\sqrt{n}/(2H + 2)]$ blocks of $(2H + 2)$ points, and again the missing points are defined to be elements of the set $U_1^{(j)}$. Introduce the random variable

\begin{equation}
A_j^{(n)} = \frac{\rho_j^2(\tilde{S}_{[nt])}, ..., \tilde{S}_{[nt])})}{2\sigma^2(H + 1)^2}.
\end{equation}

In the first $A_j^{(n)}$ blocks, for each block $\{k, k+1, ..., k+2H+1\}$, the points $k, k+H+1$ are sent to the set $U_1^{(j)}$ and the rest of the points are sent to the set $U_2^{(j)}$. In the remaining $[\sqrt{n}/(2H + 2)] - A_j^{(n)}$ blocks (notice that $A_j^{(n)} \leq [\sqrt{n}/(2H + 2)]$), for each block $\{k, k+1, ..., k+2H+1\}$, the points $k, k+2, k+4, ...$ are sent to the set $U_1^{(j)}$ and the rest of the points are sent to $U_3^{(j)}$.

**Step II:** In this step we derive essential properties of the construction. For any $n \in \mathbb{N}$ define the stochastic processes $\tilde{M}^{(n)} = \{\tilde{M}_k^{(n)}\}_{k=0}^n$ and $\tilde{N}^{(n)} = \{\tilde{N}_k^{(n)}\}_{k=0}^n$ by

\begin{align*}
\tilde{M}_k^{(n)} &= \tilde{S}_{\max\{m \leq k, m \in U_1^{(j)}\}}, \quad [nt_j] \leq k < [nt_j] + 1, \quad j = 0, 1, ..., J - 1, \\
\tilde{N}_k^{(n)} &= \sum_{i=0}^{k-1} \frac{\tilde{M}_i^{(n)} - \tilde{M}_i^{(n)}}{\Theta_i^{(n)}}, \quad k = 0, 1, ..., n
\end{align*}

where $\Theta_i^{(n)} = \rho_j(\tilde{S}_{[nt])}, ..., \tilde{S}_{[nt])})$ for $[nt_j] \leq i < [nt_j+1]$ and we set $\tilde{M}_k^{(n)} = \tilde{M}_k^{(n)}$. Introduce the filtration $\mathcal{G}^{(n)} = \{\mathcal{G}_k^{(n)}\}_{k=0}^n$ by

\begin{align*}
\mathcal{G}_k^{(n)} &= \sigma\{\tilde{M}_1^{(n)}, ..., \tilde{M}_k^{(n)}, \tilde{S}_{[nt])}, ..., \tilde{S}_{[nt])}\}, \quad [nt_j] \leq k < [nt_j+1], \quad j = 0, 1, ..., J - 1.
\end{align*}

From the definition of of the sets $U_1^{(j)}, U_2^{(j)}, U_3^{(j)}$ it follows that $\tilde{M}^{(n)}$ is a $\tilde{Q}_\alpha$ martingale with respect to the filtration $\mathcal{G}^{(n)}$. Hence $\tilde{N}^{(n)}$ is also a martingale.

Next, we observe that for any sequence of $H + 1$ successive time points there is at least one point which belongs to $U_1^{(j)}$ for some $j$. This together with the definition of the sets $U_2^{(j)}, U_3^{(j)}$ implies that (3.1) holds true. Thus $\tilde{Q}_n \in \mathcal{Q}_n$ and

\begin{equation}
|\tilde{M}_k^{(n)} - \tilde{S}_k^{(n)}| \leq \frac{C}{\sqrt{n}} \tilde{S}_k^{(n)} \quad \forall k \leq n
\end{equation}

for some constant $C > 0$.

Lemma 3.3 in [2] and (4.16) implies that the sequence $\{\tilde{S}_{[nt])}^{(n)}\}_{t=0}^\infty, \tilde{Q}_\alpha\}$, $n \in \mathbb{N}$ is tight and any cluster point is a strictly positive, continuous martingale. Moreover, the martingales $\tilde{M}_n^{(n)}$, $n \in \mathbb{N}$ satisfy (4.4). We need to prove that any cluster point satisfies (4.14) (recall the uniqueness property of (4.14)). Thus, choose a subsequence (we still index it by $n$) $\{\tilde{S}_{[nt])}^{(n)}\}_{t=0}^\infty, \tilde{Q}_n\}$ which converges to a martingale
$M$. We will apply the stability results from [4]. First, (4.4) implies that the sequence $M^{(n)}$, $n \in \mathbb{N}$ satisfies Condition A in [4]. Thus, from Theorem 4.1 in [4] we conclude (recall that $\rho_j \geq \epsilon > 0$ for all $j$).

\begin{equation}
\left( \{ \tilde{N}^{(n)}_{[nt]} \}_{t=0}^1, \tilde{Q}_n \right) \Rightarrow \left\{ \sum_{j=0}^{t-1} \int_{j\Delta t}^{j+1\Delta t} \frac{dM_u}{\rho_j(M_{j+1},...,M_{j})M_u} \right\}^1_{t=0}.
\end{equation}

**Step III:** In view of (4.17) in order to complete the proof we need to show that the sequence $\left( \{ \tilde{N}^{(n)}_{[nt]} \}_{t=0}^1, \tilde{Q}_n \right)$, $n \in \mathbb{N}$ converges to the standard Brownian motion. We will apply the stability results from [4]. First, (4.4) implies that the sequence $\tilde{N}^{(n)}_k - \tilde{N}^{(n)}_{k-1} = O(n^{-1/2})$ for all $k \leq n$. Thus, it remains to show that the predictable variation of $\tilde{N}^{(n)}_k$, $n \in \mathbb{N}$ satisfies

\begin{equation}
\left( \{ \langle \tilde{N}^{(n)} \rangle_{[nt]} \}_{t=0}^1, \tilde{Q}_n \right) \Rightarrow \{ t \}_{t=0}^1.
\end{equation}

Fix $n \in \mathbb{N}$ and consider an interval $[nt_j, nt_{j+1}]$ for some $j$. Recall the construction in Step I and choose a block of $[\sqrt{n}]$ successive time points. We notice that in the first $A_j^{(n)}$ blocks, for any block $\{k, k+1, ..., k+2H+1\}$ we have

$$S^{(n)}_k = M^{(n)}_k = \bar{M}^{(n)}_k = ... = \bar{M}^{(n)}_{k+H},$$

and

$$S^{(n)}_{k+H+1} = \bar{M}^{(n)}_{k+H+1} = ... = \bar{M}^{(n)}_{k+2H+1}.$$

Moreover, from the definition of the set $U^{(j)}_2$ we get

$$\tilde{Q}_n \left( S^{(n)}_{k+H+1} = S^{(n)}_k \exp \left( \pm \sigma(H+1)n^{-1/2} \right) \right) = 1 - O(n^{-1/2})$$

and

$$\tilde{Q}_n \left( S^{(n)}_{k+2H+2} = S^{(n)}_{k+H+1} \exp \left( \pm \sigma(H+1)n^{-1/2} \right) \right) = 1 - O(n^{-1/2}).$$

We conclude that for any block $\{k, k+1, ..., k+2H+1\}$ (of the first $A_j^{(n)}$ blocks)

\begin{equation}
E_{\tilde{Q}_n} \left( (\tilde{N}^{(n)}_{k+1} - \tilde{N}^{(n)}_k)^2 | G^{(n)}_t \right) = \frac{\sigma^2(H+1)^2}{n\rho_k^2(S^{(n)}_k)^2} + O(n^{-3/2})
\end{equation}

if $i \in \{k + H, k + 2H + 1\}$ and

$$E_{\tilde{Q}_n} \left( (\tilde{N}^{(n)}_{k+1} - \tilde{N}^{(n)}_k)^2 | G^{(n)}_t \right) = 0.$$  

On the other hand for any block $\{k, k+1, ..., k+2H+1\}$ of the remaining $[\sqrt{n}/(2H + 2)] - A_j^{(n)}$ blocks we have $M^{(n)}_i = S^{(n)}_i$ for $i = k, k+2, ...$ and $M^{(n)}_i = S^{(n)}_{i-1}$ for $i = k+1, k+3, ...$. Furthermore, from the definition of the set $U^{(j)}_3$, it follows that for any $i \in \{k, k+2, ...\}$

$$\tilde{Q}_n \left( S^{(n)}_{i+2} = S^{(n)}_{i+1} \right) = 1 - O(n^{-1/2}).$$

We conclude that for any block $\{k, k+1, ..., k+2H+1\}$ (of the first remaining $[\sqrt{n}/(2H + 2)] - A_j^{(n)}$ blocks)

\begin{equation}
E_{\tilde{Q}_n} \left( (\tilde{N}^{(n)}_{k+1} - \tilde{N}^{(n)}_k)^2 | G^{(n)}_t \right) = O(n^{-3/2})
\end{equation}  \forall i.
Finally, choose $j$ and $t_j < t < t' < t_{j+1}$. From (4.15) and (4.19)–(4.20) we obtain
\[
\langle \hat{X}^{(n)} \rangle_{[nt]} - \langle \hat{X}^{(n)} \rangle_{[nt']} \leq \frac{2A_{n}^{(n)}(t'-t)}{\sqrt{n}} \sigma_{(n)}^{2} + O(n^{-1/2}) = (t' - t) + O(n^{-1/2})
\]
and (4.18) follows. □

Acknowledgments
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References