Dynamic Portfolio Optimization with Liquidity Cost and Market Impact: A Simulation-and-Regression Approach

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Abstract

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Dynamic portfolio optimization with liquidity cost and market impact: a simulation-and-regression approach

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We present a simulation-and-regression method for solving dynamic portfolio optimization problems in the presence of general transaction costs, liquidity costs and market impact. This method extends the classical least squares Monte Carlo algorithm to incorporate switching costs, corresponding to transaction costs and transient liquidity costs, as well as multiple endogenous state variables, namely the portfolio value and the asset prices subject to permanent market impact. To handle endogenous state variables, we adapt a control randomization approach to portfolio optimization problems and further improve the numerical accuracy of this technique for the case of discrete controls. We validate our modified numerical method by solving a realistic cash-and-stock portfolio with a power-law liquidity model. We identify the certainty equivalent losses associated with ignoring liquidity effects, and illustrate how our dynamic optimization method protects the investor’s capital under illiquid market conditions. Lastly, we analyze, under different liquidity conditions, the sensitivities of certainty equivalent returns and optimal allocations with respect to trading volume, stock price volatility, initial investment amount, risk aversion level and investment horizon.

Keywords: Dynamic portfolio optimization; Multi-period asset allocation; Transaction cost; Liquidity cost; Permanent market impact; Least squares Monte Carlo

MSC Classification: 91G10, 93E20, 91B24, 65C05, 91G60, 91B06, 90C39, 93E24

1. Introduction

The effects of liquidity on dynamic portfolio optimization problems (a.k.a. asset allocation, portfolio allocation or portfolio management) have drawn great attention from academicians and practitioners alike. Liquidity affects portfolio allocation in two main ways: temporary liquidity cost and permanent market impact. Liquidity cost, also known as implementation shortfall, temporary market impact or transitory market impact, is the difference between the realized transaction price and the pre-transaction price. Market impact is the permanent shift in the asset price after a transaction, due to the post-transaction ‘resilience’ of the limit order book. These liquidity effects depend on several factors, such as the nature of the exchange platform, the duration of the trade execution, the transaction volume, the asset volatility and so on.

Up to now, incorporating liquidity effects into dynamic portfolio optimization is still challenging, and has been impeded by the intractability of analytical solutions and by the limited capability of numerical methods to handle endogenous stochastic prices. The purpose of the present paper is to introduce a simulation-and-regression method that is capable of handling dynamic portfolio optimization problems with general transaction costs, liquidity costs and market impact.

The original literature on dynamic portfolio optimization started with simple problems in the absence of transaction costs. The seminal papers, Mossin (1968), Samuelson (1969), Merton (1969) and Merton (1971) provide closed-form solutions of optimal asset allocation strategies for long-term investors. In reality though, every transaction incurs commission fees (or brokerage costs), and several improvements have therefore been proposed to account for transaction costs. Examples of closed-form solutions are (Davis and Norman 1990, Shreve and Soner 1994, Liu 2004, Gârleanu and Pedersen 2013). Examples of numerical methods are

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(Muthuraman and Zha 2008, Lynch and Tan 2010, Brown and Smith 2011). Transient liquidity cost, viewed as another type of transaction cost, has also been studied by many researchers in the context of dynamic portfolio optimization problems. Çetin and Rogers (2007) show the existence of optimal portfolios and how to turn the marginal price process under the optimal strategy into a martingale using the optimal terminal wealth as a change of measure. We refer to Ma et al. (2013) and Lim and Wimonkittiwat (2014) for examples of solving the Hamilton–Jacobi–Bellman (HJB) equation. Other than liquidity cost, permanent market impact is also a crucial element when dealing with large transactions. This effect has been commonly incorporated in the studies of optimal portfolio liquidation problems, see for example (Bertsimas and Lo 1998, Almgren and Chriss 2000, Obizhaeva and Wang 2013, Tsoukalas et al. 2017). These works, although restricted to either linear or linear-quadratic objective functions, provide an insightful guidance for how to handle large transactions in illiquid markets. Dynamic portfolio optimization with permanent market impact has been formulated in Ly Vath et al. (2007) as an impulse control problem under state constraints, where the authors characterize the value function as the unique constrained viscosity solution to the associated quasi-variational HJB inequality. This framework has been extended to numerical approximation in Gaigi et al. (2016). Gärleanu and Pedersen (2013) derive a closed-form optimal portfolio policy for the mean-variance framework with quadratic transaction costs such that liquidity costs and market impact are also included. Following on this framework, many extensions have been proposed, for example (Collin-Dufresne et al. 2015, Mei et al. 2016). However, due to the analytical intractability, these methods are restricted in applications especially when market impact is present.

To broaden the range of applications, the least squares Monte Carlo (LSMC) algorithm is a possible solution. The LSMC algorithm, originally developed by Carriere (1996), Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) for pricing American options, has been extended to solve dynamic portfolio optimization problems in Brandt et al. (2005), Garlappi and Skoulakis (2010) and Cong and Oosterlee (2016). Brandt et al. (2005) determine a semi-closed form by solving the first-order condition of the Taylor series expansion of the future value function. Cong and Oosterlee (2016) use a multi-stage strategy to perform forward simulation of control variables which are iteratively updated in the backward recursive program, where the admissible control sets are constructed as small neighborhoods of the solutions to the multi-stage strategy. Later, Cong and Oosterlee (2017) combine (Jain and Oosterlee 2015)’s stochastic bundling technique with (Brandt et al. 2005)’s method. To sum up, these papers have opened the way to the use of the LSMC algorithm for solving dynamic portfolio optimization problems, but are at this stage still limited and constrained in the possible formulations of transaction costs, liquidity costs and market impact.

In this paper, we make three contributions to this literature. Our first contribution is to propose a LSMC algorithm to solve dynamic portfolio optimization problems with no restriction in the formulations of transaction costs, liquidity costs and market impact, capable of handling multiple assets with general return/price dynamics in a computationally tractable way.

Our second contribution is to improve the numerical performance of Kharroubi et al. (2014)’s control randomization algorithm for the case of discrete controls. In Kharroubi et al. (2014), the randomized controls are part of the regression inputs, and the regression basis is extended accordingly. However, this approach may require finding an adequate regression basis for the randomized controls, which can be problematic in practice. To avoid this difficulty, we discretize the control space and perform one regression per control level such that the control information is included in each discrete level. This control discretization approach extends the optimal switching approach (Boogert and de Jong 2008, Aid et al. 2014) to the problems with endogenous state variables. Finally, we iterate the whole algorithm by replacing the initial randomized controls by the optimal control estimates from the previous run. In the numerical section, we show that these combined modifications improve the portfolio allocation estimates, especially when the payoff functions are highly non-linear.

Our third contribution is to present an empirical study on how dynamic portfolio allocations are affected by transient and permanent liquidity effects. We apply our LSMC method to solve a realistic cash-and-stock portfolio allocation problem, for which we adopt the power-law liquidity model of Almgren et al. (2005). We quantify the certainty equivalent losses associated with ignoring liquidity issues, and illustrate the capability of our dynamic optimization method to protect the investor’s capital in illiquid markets. Finally, based on different liquidity scenarios, we analyze the sensitivity of certainty equivalent returns (CER) and optimal portfolio allocations with respect to trading volume, stock price volatility, initial investment amount, risk aversion level and investment horizon.

The outline of the paper is as follows. Section 2 formulates the dynamic portfolio optimization problem with general transaction costs, liquidity costs and market impact. Section 3 describes the LSMC algorithm developed to solve this problem. Section 4 describes our numerical experiments based on a power-law of liquidity cost function and Section 5 concludes the paper.

2. Problem Description

In this section, we provide a detailed mathematical description of the portfolio optimization problem we aim to solve. Consider a dynamic portfolio optimization problem over a finite time horizon $T$. Suppose there are $d$ risky assets available for investment. Let $r^i$ denote the risk free rate, and let $[r^i]_{t \leq T} = \{r^i_t\}_{t \leq T}$ and $[S^i]_{t \leq T} = \{S^i_t\}_{t \leq T}$ respectively denote the asset returns and prices. Define the cumulative factors of the asset returns as $[R^i]_{t \leq T} := \{1 + r^i_t\}_{t \leq T}$. Denote by $[Z_t]_{t \leq T}$ the vector of return predictors. This vector $[Z_t]_{t \leq T}$ is used to construct the dynamics of the assets. Let $\alpha_t = (\alpha^i_t)_{1 \leq i \leq d}$ be the portfolio weights in each risky asset; the allocation in the risk free asset is then given by $\alpha^d_t = 1 - \sum_{1 \leq i \leq d} \alpha^i_t$. In a similar manner, let $[q^i]_{t \leq T}$ describe the number of units held in each risky asset and let
\( q_i^{f |_{t \leq T}} \) denote the amount allocated in the risk free cash. Define \( \Delta q_i^t := q^t_i - q^t_{i-1} \) as the transaction volume for the \( i^{th} \) risky asset at time \( t \). Let \( A \subseteq \mathbb{R}^d \) be the set of admissible portfolio strategies. These sets may include constraints defined by the investor, such as weight limits in each individual asset. Finally, let \( \{W_t\}_{t \leq T} \) denote the portfolio value (or wealth) process.

For every transaction, due to transaction costs, liquidity costs and market impact, there are immediate shifts in the endogenous asset prices and in the portfolio value. Let \( TC(\Delta q_t) = (TC((\Delta q_t^i)_{i \leq T}))_{1 \leq T} \), \( LC(\Delta q_t) = (LC((\Delta q_t^i)_{i \leq T}))_{1 \leq T} \) and \( MI(\Delta q_t) = (MI((\Delta q_t^i)_{i \leq T}))_{1 \leq T} \) respectively denote the vector of transaction costs, liquidity costs and market impact shifts generated by the transaction \( \Delta q_t \) for each risky asset \( i = 1, \ldots, d \). In general, we write these quantities as deterministic functions of transaction volume:

\[
TC : \mathbb{R} \to \mathbb{R}, \quad LC : \mathbb{R} \to \mathbb{R} \quad \text{and} \quad MI : \mathbb{R} \to \mathbb{R},
\]

and thus \( TC : \mathbb{R}^d \to \mathbb{R}^d \), \( LC : \mathbb{R}^d \to \mathbb{R}^d \) and \( MI : \mathbb{R}^d \to \mathbb{R}^d \). One key feature of the proposed portfolio optimization algorithm will be the flexibility to accommodate general transaction costs, liquidity costs and market impact. This is the reason why the problem is described with general notations of switching costs \( TC \), \( LC \) and \( MI \). We will use more specific cost functions later for our numerical tests.

Transaction costs refer to the commission fees charged by the broker, usually fixed costs or at a fixed proportional rate, and are therefore easy to account for. The paper focuses on liquidity costs and market impact which are much more challenging to deal with.

As described in Almgren et al. (2005), the following are the key observables during a transaction:

\[
\begin{align*}
S_t &= \text{market price before the transaction begins}, \\
\hat{S}_t &= \text{market price immediately after the transaction is completed}, \\
\tilde{S}_t &= \text{trading volume-weighted average price on the transaction.}
\end{align*}
\]

In our framework, the post-transaction price \( S_t \) captures the (permanent) market impact, i.e. \( MI = \hat{S}_t - S_t \) and the average price captures the (temporary) liquidity costs, i.e. \( LC = \tilde{S}_t - S_t \). Given a transaction \( \Delta q_t \) at time \( t \), the following immediate changes occur:

\[
\begin{align*}
S_t &= S_{t-} + MI(\Delta q_t), \\
W_t &= W_{t-} - TC(\Delta q_t) \cdot \tilde{I}_d - LC(\Delta q_t) \cdot \Delta q_t + MI(\Delta q_t) \cdot q_t
\end{align*}
\]

where \( \tilde{I}_d \) is a vector of size \( d \) with all the entries equal to 1. We assume that the observed price \( S_t \) immediately after a transaction is the post-reversion price such that the market impact has reverted from the temporary effect to the permanent effect.

It is important to note that there are two possible descriptions of the portfolio positions: absolute positions using the quantity (number of units) in each asset \( q_t \) and relative positions using the proportions of wealth in each asset \( \alpha_t \). We describe our portfolio allocation decisions using \( \alpha_t \), while transaction costs, liquidity costs and market impact depend on \( q_t \). Fortunately, there is a natural one-to-one correspondence between these two descriptions, namely,

\[
\alpha_t \times W_t = q_t \times S_t, \quad (2)
\]

where ‘ \( \times \) ’ denotes the element-wise multiplication and we also denote by ‘-‘ the element-wise division. Suppose that at time \( t \), one wants to rebalance the portfolio from the absolute position \( q_{t-} \) to the relative weight \( \alpha_t \in A \). Then, using the dynamics (1) and the relation (2), the following system of equations holds:

\[
\alpha_t \times \left( W_{t-} - TC(\Delta q_t) \cdot \tilde{I}_d - LC(\Delta q_t) \cdot \Delta q_t + MI(\Delta q_t) \cdot q_t \right) = q_{t-} \times (S_{t-} + MI(\Delta q_t)). \quad (3)
\]

This is a system of nonlinear equations coupled by the wealth variable. Solving these equations enables us to simultaneously update \( \alpha_t \) and \( q_t \), and avoid the potential mismatch between actual allocation and target allocation. To solve it numerically (i.e. being given \( \alpha_t \) and \( q_{t-} \), find \( q_t \)) we use a fixed-point argument as described by Algorithm 1.

**Algorithm 1 Compute Absolute Position**

1: Input: \( q_{t-}, S_{t-}, W_{t-} \) and \( \alpha_t \)
2: Result: \( q_t, q^{\alpha}_{aux}, S_t \)
3: Set tol
4: Initial guess: \( q_t = \alpha_t \times W_{t-} + S_{t-} \)
5: while dist > tol do
6: \( S_t = S_{t-} + MI(\Delta q_t) \)
7: \( W_t = W_{t-} - TC(\Delta q_t) \cdot \tilde{I}_d - LC(\Delta q_t) \cdot \Delta q_t + MI(\Delta q_t) \cdot q_t \)
8: \( q^{\alpha}_{aux} = \alpha_t \times W_t + S_t \)
9: dist = \( \sum (q^{\alpha}_{aux} - q_t)/q^{\alpha}_{aux} \)
10: \( q_t = q^{\alpha}_{aux} \)
11: end while
12: \( q_t = W_{t-} - q_t \cdot S_t \)

Sufficient conditions for the numerical convergence of Algorithm 1 are provided in Propositions 2.1 and 2.2.

**PROPOSITION 2.1 Consider the case with no market impact. If** \( |\alpha_t|/S_t(\sup_{x \in \mathbb{R}}|dT(C(x))/dx| + d(xLC(x))/dx) | < 1 \) for \( i = 1, \ldots, d \), then Algorithm 1 converges.

**Proof** Let \( q_t^{f}(n) \) be the updated position \( q_t^{f} \) after \( n \) iterations of the loop in Algorithm 1. Thus, the transaction cost and the liquidity cost after \( n \) iterations are \( TC(q_t^{f}(n) - q_t^{f}_{n-1}) \) and \( LC(q_t^{f}(n) - q_t^{f}_{n-1}) \) respectively. The iteration relation reads

\[
q_t^{f}(n + 1) = a_t^f(W_{t-} - TC(q_t^{f}(n) - q_t^{f}_{n-1}) - LC(q_t^{f}(n) - q_t^{f}_{n-1}) \cdot (q_t^{f}(n) - q_t^{f}_{n-1}))
\]

\[
S_t^{n}
\]

\[
\frac{\partial}{\partial q_t} (W_{t-} - TC(q_t^{f}(n) - q_t^{f}_{n-1}) - LC(q_t^{f}(n) - q_t^{f}_{n-1}) \cdot (q_t^{f}(n) - q_t^{f}_{n-1})) \]

\[
S_t^{n}
\]

\[
\frac{\partial}{\partial q_t} (W_{t-} - TC(q_t^{f}(n) - q_t^{f}_{n-1}) - LC(q_t^{f}(n) - q_t^{f}_{n-1}) \cdot (q_t^{f}(n) - q_t^{f}_{n-1})) \]

\[
S_t^{n}
\]

\[
\frac{\partial}{\partial q_t} (W_{t-} - TC(q_t^{f}(n) - q_t^{f}_{n-1}) - LC(q_t^{f}(n) - q_t^{f}_{n-1}) \cdot (q_t^{f}(n) - q_t^{f}_{n-1})) \]

\[
S_t^{n}
\]
Finally,
\[
|q_i'(n + 1) - q_i'(n)|
= \frac{|\alpha'_i|}{S_t} |\text{TC}'(q_i'(n - 1) - q_i'_{\tau-1}) - \text{TC}'(q_i'(n) - q_i'_{\tau-1})
+ \text{LC}'(q_i'(n - 1) - q_i'_{\tau-1})(q_i'(n - 1) - q_i'_{\tau-1})
- \text{LC}'(q_i'(n) - q_i'_{\tau-1})(q_i'(n) - q_i'_{\tau-1})|
\]
\[
= \frac{|\alpha'_i|}{S_t} \left( \left| \frac{d\text{TC}'(x)}{dx} \right|_{x=q_i'} + \left| \frac{d(x\text{LC}'(x))}{dx} \right|_{x=q_i'} \right)
\times |q_i'(n) - q_i'(n)|,
\]
where the last equality uses the Mean Value Theorem with \( q^* \) lying between \( q_i'(n-1) - q_i'(n) \) and \( q_i'(n) - q_i'_{\tau-1} \). Thus, we require that \( |\alpha'_i|/S_t (sup\{(d\text{TC}'(x)/dx) + (d(x\text{LC}'(x))/dx)) \}< 1 \) to ensure, using the Banach fixed-point theorem, that the sequence \( q_i'(n) \) converges. The proof can be easily extended for multi-dimensional portfolios. \[\square\]

**Proposition 2.2** Assume that there is no cross-asset market impact. The transaction cost, liquidity cost and market impact in the relation (3) can be jointly described by a total switching cost, denoted by SC:

\[
\text{SC}'(\Delta q_i') := W_{t-1} - S_{t-1} \left( \text{TC}'(\Delta q_i') - \text{LC}'(\Delta q_i') \cdot \Delta q_i' + \text{MI}'(\Delta q_i') \cdot q_i' \right) \cdot S_t^{-1} + \text{MI}'(\Delta q_i').
\] (4)

If \( |\alpha'_i|/S_t (sup\{d\text{SC}'(x)/dx) \)< 1 for \( i = 1, \ldots, d \), then Algorithm 1 converges.

**Proof** Consider the \( i^{th} \) risky asset. We first revert the relation (3) to have
\[
q_i' = \frac{\alpha'_i}{S_t} \left( W_{t-1} - \text{SC}'(\Delta q_i') \right).
\]
Suppose the portfolio position can also be computed via a total switching cost \( \text{SC}' \):
\[
q_i' = \frac{\alpha'_i}{S_t} \left( W_{t-1} - \text{SC}'(\Delta q_i') \right).
\]
Then the formula of \( \text{SC}' \) in (4) can be obtained by equating the above two equations. Finally, the proof can be completed by using the same reasoning as in the proof of Proposition 2.1. \[\square\]

In practice, the algorithm converges very quickly because \( (|\alpha'_i|/S_t) \cdot (d\text{SC}'(x)/dx) \) is usually very small and the initial guess of the solution is very close to the true value. Based on our numerical experiments, a stable solution can be reached within two iterations when the implementation shortfall is on average 10 basis points per unit of \( \Delta q \) (per share trade), and within three iterations when the implementation shortfall is on average 100 basis points per unit of \( \Delta q \), given a tolerance level set to be \( tol = 10^{-4} \). Note that these implementation shortfalls are artificially set high, as the main purpose here is to show that the algorithm converges quickly even if the switching costs are high.

This algorithm ensures that the post-transaction portfolio weights, accounting for immediate transaction costs, liquidity costs and market impact, match exactly the required portfolio weights \( \alpha_i \). Ignoring this in portfolio optimization could result in a large mismatch between the actual post-transaction allocation and the initial target allocation. Henceforth, we denote the transaction volume by a function of the inputs \( \alpha_i, S_t \) and \( W_t \) of Algorithm 1, i.e. \( \Delta q_i = Q(\alpha_i, S_t, W_t) \) where \( Q : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \).

In this paper, we consider a discrete-time rebalancing problem and we restrict the rebalancing times to an equally-spaced discrete grid \( 0 = t_0 < \cdots < t_l = T \). Therefore, we have \( q_i(t_n) = q_{i, t_n} \). The dynamic portfolio optimization is chosen to maximize the investor’s expected utility of final wealth \( \mathbb{E}[U(W_{t_n})] \) over all the possible strategies \( \{\alpha_{t_n} \in \mathcal{A} \}_{n \in \mathbb{N}} \). Let \( \mathcal{F}_{t_n} = \{ \mathcal{F}_{t_n} \}_{0 \leq n \leq N-1} \) be the filtration generated by all the state variables. At any time \( t_n \in [t_0, t_{n-1}] \), the objective function reads
\[
v_{t_n}(z, s, w) = \sup_{\alpha_{t_n} \in \mathcal{A}} \mathbb{E}[U(W_{t_n}) \mid Z_{t_n} = z, S_{t_n} = s, W_{t_{n-1}} = w],
\] (5)
where \( v_{t_n}(Z_{t_n}, S_{t_n}, W_{t_{n-1}}) \) and \( \alpha_{t_n} \) are \( \mathcal{F}_{t_n} \)-adapted. The state variables of the problem are:

(1) Exogenous state variables: the return predictors \( Z_{t_n} \).
(2) Endogenous state variables: the relative portfolio weights \( \alpha_{t_n} \), the absolute portfolio holdings \( q_{t_n} \), the asset prices \( S_{t_n} \) and the portfolio value \( W_{t_n} \).

The asset price processes evolve as
\[
S_{t_{n+1}} = S_{t_n} \times R_{t_{n+1}} + \text{MI}(\Delta q_{t_{n+1}}),
\] (6)
and the wealth process evolves as
\[
W_{t_{n+1}} = W_{t_n} + q_{t_n} \cdot (S_{t_n} \times R_{t_{n+1}}) - \text{TC}(\Delta q_{t_{n+1}}) \cdot \hat{1}_d
- \text{LC}(\Delta q_{t_{n+1}}) \cdot \Delta q_{t_{n+1}} + \text{MI}(\Delta q_{t_{n+1}}) \cdot q_{t_{n+1}},
\] (7)
where \( q \) and \( \Delta q \) in (6)–(7) satisfy the relation (3). The value function satisfies the following discrete dynamic programming principle
\[
v_{t_n}(z, s, w) = \sup_{\alpha_{t_n} \in \mathcal{A}} \mathbb{E}[v_{t_{n+1}}(Z_{t_{n+1}}, S_{t_{n+1}}, W_{t_{n+1}}) \mid Z_{t_n} = z, S_{t_n} = s, W_{t_{n-1}} = w],
\] (8)
and we assume that the investor begins with 100% holding in the cash account and liquidate all the risky assets at the terminal time, i.e. \( \alpha_{t_{n-1}} = \alpha_{t_n} = 0 \).
3. Solution

In this section, we describe our numerical algorithm for solving the recursive dynamic programming problem in (8). Our algorithm can be decomposed into three main parts:

1. First, a forward simulation of all the state variables of the problem, including the endogenous state variables, following the control randomization method of Kharroubi et al. (2014), is described in Section 3.1.

2. Then, a backward recursive dynamic programming where the conditional expectations are approximated by least squares regressions, and the optimal allocations obtained by exhaustive search, is described in Section 3.2.

3. Finally, an iteration procedure for updating the simulated control variables of the first step by the estimates generated from the second step, is described in Section 3.3.

3.1. Step 1: Monte Carlo simulation

The first main part consists in simulating a large sample of all the stochastic state variables. The return predictors $Z_t$, and the asset excess returns $r_t$ are exogenous risk factors, and therefore easy to simulate. By contrast, the asset prices $S_t$, and the portfolio value $W_t$ are endogenous risk factors, i.e. their dynamics depend on the portfolio weight $\alpha_t$. In order to simulate $S_t$, $W_t$, which is necessary to initiate the algorithm, we rely on the control randomization technique proposed by Kharroubi et al. (2014). In general, it is very difficult to determine the choice of random controls, as initialising a wrong guess may lead to a local optimum. Instead, we draw random controls uniformly from the admissible set $A$, which ensures that the simulation will cover the full space of controls if the sample size is large enough. We refer to Kharroubi et al. (2014) for the convergence analysis of such uniform control simulation.

In summary, we first simulate a sample of return predictors $\{Z_t\}_{1 \leq t \leq M}$, asset excess returns $\{r_t\}_{0 \leq t \leq N}$, uniformly randomized portfolio weights $\{\alpha_t\}_{0 \leq t \leq N}$, and then we compute the corresponding absolute holdings $\{q_{t,1}^m, \ldots, q_{t,m}^m\}$, asset prices $\{S_{t,1}^m, \ldots, S_{t,m}^m\}$, and portfolio values $\{W_{t,1}^m, \ldots, W_{t,m}^m\}$ according to Algorithm 1. The next subsection explains how these initial random weights will be turned into estimates of the optimal allocations.

3.2. Step 2: discretization, regression and maximization

The second part of the LSMC algorithm corresponds to the regressions and maximizations by exhaustive search. We discretize the control space as $A \approx A^{\text{disc}} := \{a_1, \ldots, a_j\}$, and the dynamic programming principle can be written as

\[
\begin{align*}
\max_{a_t \in A^{\text{disc}}} & \mathbb{E}\left[ v_{t+1}\left(Z_{t+1}, S_{t+1}^{-}, W_{t+1}^{-}\right) \right] | Z_t = z, \\
& \text{subject to } S_{t+1}^{-} = s, W_{t+1}^{-} = w
\end{align*}
\]

At time $t_N$, the policy is set to $\hat{\alpha}_N = 0$. At time $t_0$, assume that the mappings $\hat{\alpha}_0 : (z, s, w) \mapsto \hat{\alpha}_0(z, s, w)$ for $t_0, t_1, \ldots, t_N$ have been estimated. We use exhaustive grid search to estimate the conditional expectation at time $t_0$ for each decision $a_t \in A^{\text{disc}}$.

By taking the decision $\hat{a}_t = a_t$, the endogenous state variables at time $t_0$ can be updated to their post-transactions values at time $t_0$:

\[
\begin{align*}
\hat{v}_{t_0}(z, s, w) & \approx \max_{a_t \in A^{\text{disc}}} \mathbb{E}\left[ v_{t+1}\left(Z_{t+1}, S_{t+1}^{-}, W_{t+1}^{-}\right) \right] \\
& \text{subject to } S_{t+1}^{-} = s, W_{t+1}^{-} = w
\end{align*}
\]

where

\[
\begin{align*}
Z_{t_0}^{-} = z, \\
\alpha_{t_0} = a_t, \\
q_{t_0} = q_{t_0}^{-}, + \mathcal{Q}(a_t, s, w), \\
S_{t_0}^{-} = s + \text{MI} \left( \mathcal{Q}(a_s, s, w) \right), \\
W_{t_0}^{-} = w - \mathcal{T}_d (\mathcal{Q}(a_t, s, w)) - \hat{\mathcal{I}}_d - \mathcal{L}_C (\mathcal{Q}(a_t, s, w)) - \mathcal{Q}(a_t, s, w) + \text{MI} \left( \mathcal{Q}(a_t, s, w) \right) \cdot q_{t_0}^{-}.
\end{align*}
\]

Therefore, for each Monte Carlo path $m = 1, \ldots, M$, we update the decision $\hat{\alpha}_n^m$ to $a_t$ and recompute the corresponding endogenous variables at time $t_n$.

\[
\Delta \hat{q}_{t_n}^{m, (n,j)} = \mathcal{Q} \left( a_t, S_{t_n}^m, - \hat{W}_{t_n}^m \right),
\]

\[
\hat{S}_{t_n}^m = \hat{S}_{t_n}^m + \Delta \hat{q}_{t_n}^{m, (n,j)},
\]

\[
\hat{W}_{t_n}^m = \hat{W}_{t_n}^m - \mathcal{T}_d (\Delta \hat{q}_{t_n}^{m, (n,j)}) \cdot \hat{\mathcal{I}}_d - \mathcal{L}_C (\Delta \hat{q}_{t_n}^{m, (n,j)}) \cdot \Delta \hat{q}_{t_n}^{m, (n,j)} + \text{MI} \left( \Delta \hat{q}_{t_n}^{m, (n,j)} \right) \cdot \hat{q}_{t_n}^{m, (n,j)},
\]

then we can recompute the endogenous state variables one time-step forward at time $(t_{n+1})^{-}$, i.e.

\[
\hat{S}_{t_{n+1}}^m = \hat{S}_{t_n}^m \times R_{t_{n+1}},
\]

\[
\hat{W}_{t_{n+1}}^m = \hat{W}_{t_n}^m + r_t^m \hat{q}_{t_n}^{m, (n,j)} + \hat{q}_{t_n}^{m, (n,j)} \times (\hat{S}_{t_n}^m \times R_{t_{n+1}}),
\]

The augmented superscript $(n,j)$ denotes the $j$-th portfolio decision is taken at time $t_n$. We subsequently keep track of all the optimal decisions $\hat{\alpha}_{t_n}$ and update the endogenous state variables $\hat{q}_{t_n}^{m, (n,j)}$, $\hat{S}_{t_n}^m$, $\hat{W}_{t_n}^m$, until the end of the investment horizon, for the purpose of obtaining the terminal wealth $\hat{W}_{t_N}^m$. Finally, the value function at time $t_{N+1}$ can be estimated by $\hat{v}_{t_{N+1}}^{m, (n,j)} = U(\hat{W}_{t_N}^m)$ which will used for the regressand in the least squares approximation.
Set \( \{L_k(z', s', w)\}_{1 \leq k \leq K} \) to be a vector of basis functions of the state variables. We estimate the 'continuation values' (the conditional expectations in equation (9)) by least squares minimization, i.e.,

\[
\hat{\beta}_k \mid_{1 \leq k \leq K} = \arg\min_{\beta_k \in \mathbb{R}^K} \sum_{m=1}^M \left( \hat{y}_{t_{k+1}} - \sum_{k=1}^K \beta_k L_k \left( Z_{tn}^m, S_{tn}^m, \hat{W}_{tn}^m \right) \right)^2.
\]

Therefore the 'continuation value' at time \( t_n \) for \( a_j \in \mathcal{A}^{disc} \) is formulated as

\[
\hat{C}V_j(t_n, z, s, w) = \sum_{k=1}^K \hat{\beta}_k L_k(z, s, w),
\]

and the mapping \( \hat{a}_j : (z, s, w) \mapsto \hat{a}_j(z, s, w) \) is estimated by

\[
\hat{a}_j(z, s, w) = \arg\max_{a_j \in \mathcal{A}^{disc}} \hat{C}V_j(z, s, w).
\]

Here, the superscript \( j \) on \( \hat{C}V_j, \hat{\beta} \) corresponds to the \( j \)th decision in the discrete grid, \( a_j \).

Remark that the discretization of the control allows us to substitute the extended control regression of Kharroubi et al. (2014) by one regression (10) for each control level as described in this subsection.

### 3.3. Step 3: control iteration

In the forward simulation, the endogenous state variables are generated using the randomized controls \( \{\hat{a}_j^m\}_{0 \leq m \leq M} \). Although the endogenous state variables will be updated backwards in time during Step 2 of the algorithm, the value function

\[
v_{t_n}(z, s, w) = \sup_{a_j \in \mathcal{A}} \mathbb{E} \left[ v_{t_{n+1}} \left( Z_{t_{n+1}}, S_{t_{n+1}}, W_{t_{n+1}} \right) | Z_{t_n} = z, S_{t_n} = s, W_{t_n} = w \right],
\]

is evaluated by using the samples of the path-dependent variables \( \{\hat{S}_{tn}^m, \hat{W}_{tn}^m\}_{1 \leq m \leq M} \) which still depend on the 'historical' randomized controls \( \{\hat{a}_j^m\}_{0 \leq m \leq M} \). Before the time step \( t_n \).

In theory, this does not affect the optimality of the portfolio optimization, as the regression will provide an estimate of \( v_{t_n}(z, s, w) \) everywhere on the space of state variables, including the region where the optimally controlled endogenous variables will lie in. In practice, it may lead to large numerical errors if the regression is inaccurate in the optimal region, due to an insufficiently large sample size or inadequate regression basis for example. To mitigate this possibility, we propose to iterate the whole algorithm, with the initial randomized controls replaced by the estimated optimal controls produced by the previous run. Thus, the control-dependent variables \( \{\hat{S}_{tn}^m, \hat{W}_{tn}^m\}_{1 \leq m \leq M} \) that are previously computed from the randomized controls will be replaced by the new samples calculated from the optimal control estimates. This iteration procedure will bring the whole Monte Carlo sample closer to the optimal region, making the regression more accurate in the optimal region, and thus improve the overall portfolio allocation estimates. Our numerical experiments in Section 4 show that this iteration procedure does improve the numerical accuracy, especially for small sample sizes and highly nonlinear utility functions, and that most of the improvements occur after one single additional iteration.

### 3.4. Summary and remarks

Finally, this subsection provides a detailed description of the backward approximation, followed by a few additional implementation details.

**Summary of algorithm.** Being given the 'continuation values' at time \( t_{n-1} \), the detailed implementation of one backward iteration (cf. Section 3.2) is summarized in Algorithm 2, where we set \( \{\hat{a}_j^m\}_{1 \leq m \leq M} = 0 \), \( \{\hat{q}_j^m\}_{1 \leq m \leq M} = 0 \) and \( W_{t_n} = W_{t_n}^0 = \) initial investment amount. Additional implementation details are discussed below.

**VFI versus PFI.** There are two different schemes of the implementation of the LSMC algorithm: value function iteration (VFI, Carriere (1996), Tsitsiklis and Van Roy (2001), a.k.a. regression surface value iteration), and performance function iteration (PFI, Longstaff and Schwartz 2001, a.k.a. realized value iteration, or portfolio weight iteration). The difference lies in the \( t_{n+1} \)-response in the least squares regression (10); the VFI scheme regresses the estimated 'continuation value' from the previous regression, while the PFI scheme regresses the realized paths that are calculated from the estimated optimal policy. The PFI scheme produces more accurate results, as it avoids the compounding of regression errors. However, when there are endogenous state variables, the PFI scheme requires to recompute all the endogenous state variables until the end of the horizon, which increases the computational complexity from linear to quadratic in time. By contrast, the computational complexity of the VFI is linear in the number of time steps. More discussions on VFI versus PFI are available in Van Binsbergen and Brandt (2007), Garlappi and Skoulikis (2009) and Denault and Simanot (2017). In this paper, we choose to implement the PFI scheme for its greater accuracy and stability.

**Dimension reduction of state vector.** Although in theory all risk factors need to be included in the regression so as to take into account all available information when making decisions, in practice the bias-variance tradeoff suggests omitting the variables that bring little additional information. In portfolio optimization problems, the portfolio value is a linear combination of the asset prices, determined by \( W_{t_n} = S_{t_n} \cdot q_{t_n} \), such that the asset prices are also reflected in a single wealth variable. Moreover, our objective is to maximize the expected utility of final wealth, thus the wealth variable plays a much more crucial role than the price variables when approximating such objective function. After testing and comparing different subsets of regression inputs, we decide to remove the endogenous price variables in the regressions and only regress on \( (Z, W) \). Doing so improves the out-of-sample quality of regression estimates, and has the advantage that the numerical complexity of the least squares regression does not increase with the portfolio dimension.
Regressing on post- versus pre-transaction variables. The evolution of the endogenous state variables from time $t_0$ to $t_N$ can be decomposed into an immediate deterministic component depending on the switching costs $SC(\Delta \mathbf{q}_m)$ which include $TC(\Delta \mathbf{q}_m)$, $LC(\Delta \mathbf{q}_m)$ and $MI(\Delta \mathbf{q}_m)$ as defined in (4), and a stochastic component depending on the dynamics of the state variables from time $t_0$ to $t_N$, denoted by $\Delta \mathbf{q}_{m,t_k}$. For demonstration purposes, we use in this paragraph the wealth variable $W$ as one single regressor in the regression and use a simple linear utility function $U(w) = w$. Then there are two different types of regression given by

1. Regression on $W_{t_k}$:

$$ E \left[ W_{t_k} - SC(\Delta \mathbf{q}_m) + \Delta \mathbf{q}_{m,t_k} | W_{t_{k-1}} \right] \approx \beta W_{t_{k-1}}. $$

2. Regression on $W_{t_k}$:

$$ E \left[ W_{t_k} - SC(\Delta \mathbf{q}_m) + \Delta \mathbf{q}_{m,t_k} | W_{t_{k-1}} \right] \approx \beta (W_{t_{k-1}} - SC(\Delta \mathbf{q}_m)). $$

Here, the pre-transaction regression (11) accounts for both the deterministic and stochastic components, while the post-transaction regression (12) accounts for the stochastic component only. We favor the regression on post-transaction variables for several reasons. Firstly, the deterministic component $SC(\Delta \mathbf{q}_m)$ is $f_{t_k}$-adapted, and thus not necessary for the regression. Secondly, the switching costs $SC(\Delta \mathbf{q}_m)$ are, at this stage of the algorithm, computed from the randomized portfolio positions $\{q_{m,t_k}\}_{t_k \leq t_{K-1}}$ which are not smooth w.r.t the regressor $W_{t_k}$, and we have randomized switching costs (12) that are smooth w.r.t the regressor $W_{t_k}$, and thus also randomized. Consequently, the randomized switching costs $SC(\Delta \mathbf{q}_m)$ are smooth w.r.t the regressor $W_{t_k}$, and we have randomized switching costs (12) that are smooth w.r.t the regressor $W_{t_k}$. Therefore, subtracting the switching costs from the wealth regressor will avoid this problem by removing this auxiliary randomness from the regression. Finally, from a practical perspective in view of an investor would consider the known, immediate switching costs when making a portfolio rebalancing decision.

4. Numerical Experiments

In this section, we test our algorithm on a cash-and-stock portfolio optimization problem with liquidity costs and market impact. The outline of this numerical section is as follows:

1. Subsection 4.1 validates the Monte Carlo convergence of the proposed LSMC method (Algorithm 1) with different risk aversion levels, investment horizons and liquidity settings.

2. Subsection 4.2 discusses the time evolution of the distribution of portfolio value and of the percentage allocation under different liquidity settings.

3. Subsection 4.3 identifies the certainty equivalent losses associated with ignoring liquidity effects.

Algorithm 2 Backward Dynamic Programming

1: Input: $\left(Z_{m,t_k}^n, D_{m,t_k}, \bar{a}_{m,t_k}, a_{m,t_k}^n, \bar{\mathbf{w}}_{m,t_k}^n, \mathbf{r}_{m,t_k}^n \right)_{1 \leq n \leq M}$

2: $\mathbf{a}_{m,t_k}$

3: for all rebalancing time $t_k = t_N, \ldots, t_0$ do

4: for all decision $a_{m,t_k} \in A_{m,t_k}$ do

5: for all Monte Carlo path $m = 1, \ldots, M$ do

6: Compute $(\bar{q}_{m,t_k}, \bar{S}_{m,t_k}, \bar{W}_{m,t_k})$, using Algorithm 1 from $(\hat{q}_{m,t_{k-1}}, \hat{S}_{m,t_{k-1}}, \hat{W}_{m,t_{k-1}}, \alpha_{m,t_{k-1}} = a_{m,t_k})$

7: Compute asset prices one-step forward:

$$ \hat{S}_{m,(t_{k+1})} = \hat{S}_{m,(t_{k})} \times R_{m}^{t_{k+1}} $$

8: Compute wealth one-step forward:

$$ \hat{W}_{m,(t_{k+1})} = \hat{W}_{m,(t_{k})} + f_{m} \hat{q}_{m,(t_{k})} + \hat{q}_{m,(t_{k})} \cdot (\hat{S}_{m,(t_{k})} \times \mathbf{r}_{m,(t_{k})}) $$

9: for all rebalancing time $t_{k'} = t_{N+1}, \ldots, t_{k'-1}$ do

10: for all decision $a_{m,t_{k'}} \in A'_{m,t_{k'}}$ do

11: Compute $(\tilde{q}_{m,t_{k'}}, \tilde{S}_{m,t_{k'}}, \tilde{W}_{m,t_{k'}})$ using Algorithm 1 from $(\hat{q}_{m,t_{k-1}}, \hat{S}_{m,t_{k-1}}, \hat{W}_{m,t_{k-1}}, \alpha_{m,t_{k-1}} = a_{m,t_k})$

12: Compute ‘continuation value’:

$$ CV_{t_k}(Z_{m,t_k}, S_{m,t_k}, \hat{W}_{m,t_k}) = \sum_{k=1}^{K} \tilde{h}_{m,t_k} L_{k} \left(Z_{m,t_k}, S_{m,t_k}, \hat{W}_{m,t_k} \right) $$

13: end for

14: Update $(\hat{q}_{m,t_{k'}}, \hat{S}_{m,t_{k'}}, \hat{W}_{m,t_{k'}})$ with

$$ a_{m,t_{k'}} = \arg \max_{a_{m,t_{k'}}} CV_{t_k}(Z_{m,t_k}, S_{m,t_k}, \hat{W}_{m,t_k}) $$

15: Compute asset prices one-step forward:

$$ \hat{S}_{m,(t_{k+1})} = \hat{S}_{m,(t_{k})} \times R_{m}^{t_{k+1}} $$

16: Compute wealth one-step forward:

$$ \hat{W}_{m,(t_{k+1})} = \hat{W}_{m,(t_{k})} + f_{m} \hat{q}_{m,(t_{k})} + \hat{q}_{m,(t_{k})} \cdot (\hat{S}_{m,(t_{k})} \times \mathbf{r}_{m,(t_{k})}) $$

17: end for

18: Compute $\tilde{W}_{m,(t_{k})}$ from $(\tilde{q}_{m,t_{k-1}}, \tilde{S}_{m,t_{k-1}}, \tilde{W}_{m,t_{k-1}}, \tilde{a}_{m,t_{k-1}} = 0)$ using Algorithm 1

19: end for

20: if $t_k > t_0$ then

21: Least squares approximation with basis functions of state variables $\{l_{k}(z, s, w)\}_{1 \leq z, k}$:

$$ \hat{\tilde{h}_{m,t_k}} = \arg \min_{h \in K} \sum_{m=1}^{M} \left( U \left( \tilde{W}_{m,t_k} \right) - \sum_{k=1}^{K} \tilde{h}_{m,t_k} L_{k} \left(Z_{m,t_k}, S_{m,t_k}, \hat{W}_{m,t_k} \right) \right)^2 $$

22: Formulate ‘continuation value’:

$$ CV_{t_k}(z, s, w) = \sum_{k=1}^{K} \tilde{h}_{m,t_k} L_{k} (z, s, w) $$

23: else

24: Compute ‘continuation value’ at initial time:

$$ CV_{t_k} = \frac{1}{M} \sum_{m=1}^{M} U(\hat{W}_{m,t_k}) $$

25: end if

26: end for

27: end for

28: Initial optimal control: $\hat{\mathbf{a}}_{t_k} = \arg \max_{a_{m,t_k}} CV_{t_k}$
(4) Finally, Subsection 4.4 provides sensitivity analyzes of the portfolio performance and allocation with respect to liquidity settings.

But first, we detail the numerical settings that are used to perform the numerical experiments reported in this section.

Data and modeling. Table 1 summarizes the financial instruments considered for return predictors. We calibrate a first-order vector autoregressive model to monthly log-returns (i.e. \( \log S_n - \log S_{n-1} \)) from October 2007 to April 2015. We assume that the annual interest rate on the cash account is 0.012 and use the SPDR S&P500 index ETF (SPY) as a proxy for the stock return.

Switching costs. To focus on the liquidity effects, we assume for simplicity no fixed or proportional transaction costs in the numerical study. Regarding liquidity cost and market impact modeling, we adopt the calibrated power-law functions of Almgren et al. (2005) to analyze the effects of market illiquidity on the dynamic portfolio optimization problem. These power-law functions are given by

\[
\text{MI}(\Delta q) = S^{-} \cdot 0.314 \cdot \sigma_{\text{day}} \cdot \frac{\Delta q}{\text{Vol}_{\text{day}}} \cdot \left( \frac{\Theta}{\text{Vol}_{\text{day}}} \right)^{1/4},
\]

\[
\text{LC}(\Delta q) = S^{-} \cdot \left( \frac{\text{MI}(\Delta q)}{2} + 0.142 \cdot \text{sign}(\Delta q) \right)
- \sigma_{\text{day}} \cdot \left( \frac{\Delta q}{\delta \cdot \text{Vol}_{\text{day}}} \right)^{3/5},
\]

where \( S^{-} \) is the asset price before the transaction, \( \text{Vol}_{\text{day}} \) is the daily trading volume of the stock, \( \sigma_{\text{day}} \) is the daily volatility of the stock price, \( \delta \) is the time length of trade execution, \( \Theta \) is the number of outstanding shares. Note that the original market impact and liquidity cost functions given in Almgren et al. (2005) are standardized by the asset price, i.e. \( \text{MI}/S^{-} \) and \( \text{LC}/S^{-} \). In the following numerical tests, we will fix the values of \( \delta = 5 \text{ min} \) and \( \Theta = 988 \text{ m} \) and focus on the impact of \( \sigma_{\text{day}} \) and \( \text{Vol}_{\text{day}} \). We will analyze the liquidity effects characterized by different levels of \( (\sigma_{\text{day}}, \text{Vol}_{\text{day}}) \) and we follow the usual conditions of the medium- and large-cap equities in the U.S. market so that \( \sigma_{\text{day}} \in [2, 13] \) and \( \text{Vol}_{\text{day}} \in [10 \text{ m}, 120 \text{ m}] \).

We adopt these power-law parameters primarily for demonstration, but other liquidity functions can be used (for example the exponential negative square root relation calibrated by Tian et al. 2013) without affecting the convergence of the algorithm. Based on our experiments (not included in the present paper), the proposed LSMC algorithm converges well with a large variety of classical shapes of liquidity cost functions such as linear, quadratic, power and exponential.

**Certainty equivalent return.** For all the numerical tests, we report the portfolio performance in terms of the monthly adjusted CER calculated by

\[
\text{CER} = U^{-1} \left( \mathbb{E} \left[ U(W_T) \right] \right)^{1/T} - 1 \\
\approx U^{-1} \left( \frac{1}{M} \sum_{m=1}^{M} U\left(W_{T_m}^{m}\right) \right)^{1/T} - 1.
\]

The CER corresponds to the risk free return which delivers the same expected utility as the optimized dynamic portfolio. The magnitude of monthly returns is usually less than one percent, thus we display the CERs in basis points (0.01%) to make comparisons clearer.

**LSM settings.** We use \( M = 10^5 \) for Monte Carlo paths and \( N = 12 \) monthly time steps (one year horizon and 12 rebalancing periods), except when we test the numerical convergence in Subsection 4.1. After the LSMC algorithm is completed, we generate another sample of \( M = 10^6 \) and use the estimated continuation value functions to compute the out-of-sample CER. We denote by \( I \) the number of additional control iterations of the whole LSMC algorithm (Subsection 3.3), \( I = 0 \) meaning only one LSMC run and no additional iterations.

**Portfolio weight.** We denote by \( \alpha \) the percentage allocation to the stock component, and by \( 1 - \alpha \) the allocation to the cash component. We assume a discrete set of admissible controls with step size 0.01, i.e. \( \alpha \in \{0.01, 0.02, \ldots, 0.99, 1.00\} = \mathcal{A}_{\text{disc}} \). For portfolio optimization with one single risky asset, control discretization combined with grid search is simple and fast. When it comes to portfolio allocation problems with many risky assets, the grid search component will become the main computational bottleneck of the algorithm. One way to mitigate this issue is to combine a coarse control grid with local control regression, see for example (Zhang et al. 2018).

**Basis function and regression.** We first scale all the exogenous risk factors (in our case the log-returns) by dividing by their unconditional mean. For the endogenous risk factor (the portfolio wealth \( W \)), we transform it as \( U(W/W_0) \), where \( W_0 \) is the initial portfolio wealth and \( U(\cdot) \) is the CRRA utility function. These transformed quantities form the inputs of our regression basis. For the regression basis, we use a simple second-order multivariate polynomial basis. We choose this basis and the polynomial order by observing the plots of the objective function with respect to the regression bases at various intermediate times. The surface shape is close to linear and slightly curved, suggesting that polynomials of order two can be sufficient.

**4.1. Monte Carlo convergence**

We first consider a constant absolute risk aversion (CARA) utility criterion, i.e. \( U(w) = -\exp(-\gamma w) \) without switching costs and return predictability, for which the exact solution to the dynamic optimization is available, so that we can easily

<table>
<thead>
<tr>
<th>Return Predictors</th>
<th>ETF Ticker</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. stock</td>
<td>SPY</td>
</tr>
<tr>
<td>U.S. bond</td>
<td>BND</td>
</tr>
<tr>
<td>International stock</td>
<td>EFA</td>
</tr>
<tr>
<td>Emerging market stock</td>
<td>EEM</td>
</tr>
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<td>Gold</td>
<td>GLD</td>
</tr>
<tr>
<td>International bond</td>
<td>BWX</td>
</tr>
<tr>
<td>Silver</td>
<td>SLV</td>
</tr>
<tr>
<td>Crude oil</td>
<td>USO</td>
</tr>
<tr>
<td>U.S. dollar</td>
<td>UUP</td>
</tr>
</tbody>
</table>
identify the accuracy of a numerical method. In particular, we consider a portfolio of a risk free asset with \( \sigma_f = 0.012 \) and a risky asset with mean return 0.03 and volatility 0.15. Table 2 shows the Monte Carlo convergence of the proposed numerical method (Algorithm 2). These convergence results are compared to the original control randomization approach developed by Kharroubi et al. (2014) (KLP) for which we use the same second-order global polynomial basis. The main observation is that the presented LSMC method improves the accuracy of the KLP algorithm. The improvement is more substantial with a long maturity (\( N = 15 \)), a large risk aversion (\( \gamma = 15 \)) or a small sample size (\( M = 10^3 \)). In these three cases, the benefit of using control iteration (Subsection 3.3) is noticeable, and most of the improvements are achieved after one single additional control iteration (\( I = 1 \)).

A similar result can be observed in table 3 where the Monte Carlo convergence of the CER is reported for different liquidities settings characterized by daily volatility \( \sigma_{day} \) and daily trading volume \( V_{olday} \). Once again, Algorithm 2 with one additional control iteration (\( I = 1 \)) is superior to both KLP and Algorithm 2 with no additional iteration (\( I = 0 \)). Adding further control iterations (\( I = 2 \) and more) does not lead to significant improvements over \( I = 1 \). When market liquidity effects are small, e.g. small \( \sigma_{day} \) or large \( V_{olday} \), a small Monte Carlo sample size is sufficient for convergence, while a large sample size is needed for large market liquidity effects, e.g. large \( \sigma_{day} \) or small \( V_{olday} \). For the rest of this numerical section, we will use \( M = 10^3 \) with \( I = 1 \) so as to ensure convergence and accuracy.

A final remark is that, for a large enough sample size (\( M \geq 10^4 \)), our LSMC method with \( I = 0 \) greatly outperforms the KLP algorithm for large risk aversion levels (table 2), but the difference is insignificant for large switching costs and low risk aversion levels (table 3), indicating that the shape of the final payoff function plays a more crucial role than the magnitude of switching costs in the numerical accuracy of using the simulation-and-regression approach to approximate dynamic programming schemes.

Table 2. Monte Carlo convergence with respect to risk aversion level and time horizon. This table compares the Monte Carlo convergence of Algorithm 1 with the control regression algorithm of Kharroubi et al. (2014) (KLP) with second-order polynomial basis. \( I = 0, 1, 2, 3 \) denotes the number of additional control iterations, and \( CER^* \) stands for the exact solution. We assume a CARA utility investor, i.e. \( U(w) = -\exp(-\gamma w) \), a cash-and-stock portfolio with no switching costs, risk free rate 0.012, and stock annual return with mean 0.03 and volatility 0.15. The annually adjusted CERs (in percentage points) are reported for different Monte Carlo sample sizes (\( M = 10^3, 10^4, 10^5, 10^6 \)), different investment horizons (\( N = 5yrs, 15yrs \)) and different risk aversion parameters (\( \gamma = 5, 15 \)).

<table>
<thead>
<tr>
<th>( \gamma = 5 )</th>
<th>KLP</th>
<th>( I = 0 )</th>
<th>( I = 1 )</th>
<th>( I = 2 )</th>
<th>( I = 3 )</th>
</tr>
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<tr>
<td>( N = 5 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>( M = 10^3 )</td>
<td>1.45</td>
<td>1.54</td>
<td>1.55</td>
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<td>1.58</td>
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<tr>
<td>( M = 10^4 )</td>
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<td>1.58</td>
<td>1.58</td>
<td>1.58</td>
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<tr>
<td>( M = 10^5 )</td>
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<td>1.58</td>
<td>1.58</td>
<td>1.58</td>
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<td>1.55</td>
<td>1.58</td>
<td>1.58</td>
<td>1.58</td>
<td>1.58</td>
</tr>
<tr>
<td>( CER^* = 1.58 )</td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>( \gamma = 15 )</th>
<th>KLP</th>
<th>( I = 0 )</th>
<th>( I = 1 )</th>
<th>( I = 2 )</th>
<th>( I = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 15 )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( M = 10^3 )</td>
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<td>1.40</td>
<td>1.42</td>
<td>1.43</td>
</tr>
<tr>
<td>( M = 10^4 )</td>
<td>1.33</td>
<td>1.49</td>
<td>1.51</td>
<td>1.52</td>
<td>1.53</td>
</tr>
<tr>
<td>( M = 10^5 )</td>
<td>1.47</td>
<td>1.51</td>
<td>1.52</td>
<td>1.53</td>
<td>1.53</td>
</tr>
<tr>
<td>( M = 10^6 )</td>
<td>1.48</td>
<td>1.52</td>
<td>1.53</td>
<td>1.53</td>
<td>1.53</td>
</tr>
<tr>
<td>( CER^* = 1.53 )</td>
<td></td>
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<td></td>
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</tr>
</tbody>
</table>

Table 3. Monte Carlo convergence with liquidity costs and market impact. This table shows the Monte Carlo convergence of the monthly adjusted CER (in basis points) for a CRRA utility investor with \( \gamma = 5 \) and investment horizon \( N = 12 \) months, under different daily volatility (\( \sigma_{day} = 2.5, 7.5, 12.5 \)) and different daily trading volumes (\( V_{olday} = 120 \), \( 55 \), \( 12 \) months), where the Monte Carlo procedure is performed with different sizes of Monte Carlo sample (\( M = 10^3, 10^4, 10^5 \)) and different iterations (\( I = 0, 1, 2, 3 \)). A portfolio of cash and the SPDR S&P 500 ETF (SPY) is investigated, with annual risk free rate \( r^f = 0.012 \), portfolio weight increment 0.01 and initial investment amount \( W_0 = 100 \) $.

<table>
<thead>
<tr>
<th>( \sigma_{day} )</th>
<th>Vol_{day} = 120m</th>
<th>KLP</th>
<th>( I = 0 )</th>
<th>( I = 1 )</th>
<th>( I = 2 )</th>
<th>( I = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2.5 )</td>
<td>( M = 10^3 )</td>
<td>33.3</td>
<td>55.0</td>
<td>56.9</td>
<td>57.7</td>
<td>57.8</td>
</tr>
<tr>
<td></td>
<td>( M = 10^4 )</td>
<td>55.1</td>
<td>56.5</td>
<td>57.5</td>
<td>57.7</td>
<td>57.8</td>
</tr>
<tr>
<td></td>
<td>( M = 10^5 )</td>
<td>57.0</td>
<td>57.5</td>
<td>57.8</td>
<td>57.8</td>
<td>57.9</td>
</tr>
<tr>
<td></td>
<td>( M = 10^6 )</td>
<td>57.5</td>
<td>57.5</td>
<td>57.9</td>
<td>57.9</td>
<td>57.9</td>
</tr>
<tr>
<td>( 7.5 )</td>
<td>( M = 10^3 )</td>
<td>21.3</td>
<td>47.0</td>
<td>47.6</td>
<td>47.6</td>
<td>47.6</td>
</tr>
<tr>
<td></td>
<td>( M = 10^4 )</td>
<td>45.4</td>
<td>47.2</td>
<td>47.6</td>
<td>47.6</td>
<td>47.6</td>
</tr>
<tr>
<td></td>
<td>( M = 10^5 )</td>
<td>47.1</td>
<td>47.4</td>
<td>47.6</td>
<td>47.6</td>
<td>47.6</td>
</tr>
<tr>
<td></td>
<td>( M = 10^6 )</td>
<td>47.4</td>
<td>47.5</td>
<td>47.9</td>
<td>47.9</td>
<td>47.9</td>
</tr>
<tr>
<td>( 12.5 )</td>
<td>( M = 10^3 )</td>
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<td>39.9</td>
<td>40.4</td>
<td>40.9</td>
<td>41.2</td>
</tr>
<tr>
<td></td>
<td>( M = 10^4 )</td>
<td>39.2</td>
<td>40.4</td>
<td>41.1</td>
<td>41.1</td>
<td>41.2</td>
</tr>
<tr>
<td></td>
<td>( M = 10^5 )</td>
<td>40.5</td>
<td>40.8</td>
<td>41.2</td>
<td>41.2</td>
<td>41.2</td>
</tr>
<tr>
<td></td>
<td>( M = 10^6 )</td>
<td>41.0</td>
<td>41.1</td>
<td>41.3</td>
<td>41.3</td>
<td>41.3</td>
</tr>
</tbody>
</table>
4.2. Time evolution of distribution of control and wealth

Figure 1 shows the time evolution of the portfolio allocation distribution and the wealth distribution. When market liquidity effects are small ($\sigma_{\text{day}} = 2.5$, Vol$_{\text{day}} = 120$ m, left-hand side column), both the portfolio allocation and the wealth are widely spread, along with large portfolio turnovers at the beginning and at the end of the investment horizon. By contrast, for large market liquidity effects ($\sigma_{\text{day}} = 12.5$, Vol$_{\text{day}} = 12$ m, right-hand side column) transactions become very costly and the algorithm disallows large portfolio turnovers. As a consequence, the portfolio allocation

![Figure 1](image_url)

Figure 1. Time evolution of the distribution of the control and wealth. This figure shows the time evolution of the distribution of the portfolio allocation (top panel) and wealth (bottom panel) for a CRRA utility investor with $\gamma = 5$ and investment horizon $N = 12$ months with different liquidity effects ($\sigma_{\text{day}}, \text{Vol}_{\text{day}}$) = (2.5, 120 m), (12.5, 12 m), using $M = 10^5$ Monte Carlo paths with one control iteration $I = 1$. A portfolio of cash and the SPDR S&P 500 ETF (SPY) is investigated, with annual risk free rate $r_f = 0.012$, portfolio weight increment 0.01 and initial investment amount $W_0 = $100 m.

Table 4. Certainty equivalent losses with ignoring liquidity effects. This table compares the monthly adjusted CER (in basis points) for two CRRA investors with $\gamma = 5, 10$ and investment horizon $N = 12$ months: the first one takes heed of liquidity effects (liquidity-aware) while the second one ignores liquidity effects (liquidity-blind). The results are compared under different daily volatility ($\sigma_{\text{day}} = 2.5, 7.5, 12.5$), different daily trading volumes (Vol$_{\text{day}} = 120$ m, 55 m, 12 m), and different initial investment amount ($W_0 = $100 m, 500 m, 1b), using $M = 10^5$ Monte Carlo paths with one control iteration $I = 1$. A portfolio of cash and the SPDR S&P 500 ETF (SPY) is investigated, with annual risk free rate $r_f = 0.012$, portfolio weight increment 0.01.

<table>
<thead>
<tr>
<th>Vol$_{\text{day}}$</th>
<th>$W_0$</th>
<th>$\sigma_{\text{day}} = 2.5$</th>
<th>$\sigma_{\text{day}} = 7.5$</th>
<th>$\sigma_{\text{day}} = 12.5$</th>
<th>$\sigma_{\text{day}} = 2.5$</th>
<th>$\sigma_{\text{day}} = 7.5$</th>
<th>$\sigma_{\text{day}} = 12.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>120 m</td>
<td>0.1b</td>
<td>57.8</td>
<td>47.6</td>
<td>41.2</td>
<td>56.6</td>
<td>38.5</td>
<td>21.2</td>
</tr>
<tr>
<td></td>
<td>0.5b</td>
<td>49.2</td>
<td>35.3</td>
<td>28.5</td>
<td>41.8</td>
<td>24.8</td>
<td>14.1</td>
</tr>
<tr>
<td></td>
<td>1.0b</td>
<td>44.2</td>
<td>29.9</td>
<td>24.0</td>
<td>29.9</td>
<td>23.7</td>
<td>14.5</td>
</tr>
<tr>
<td>55 m</td>
<td>0.1b</td>
<td>54.2</td>
<td>41.8</td>
<td>35.1</td>
<td>51.1</td>
<td>23.0</td>
<td>3.2</td>
</tr>
<tr>
<td></td>
<td>0.5b</td>
<td>43.5</td>
<td>29.3</td>
<td>23.5</td>
<td>28.2</td>
<td>17.2</td>
<td>8.6</td>
</tr>
<tr>
<td></td>
<td>1.0b</td>
<td>38.1</td>
<td>24.5</td>
<td>19.5</td>
<td>27.3</td>
<td>11.7</td>
<td>6.4</td>
</tr>
<tr>
<td>12 m</td>
<td>0.1b</td>
<td>44.2</td>
<td>29.9</td>
<td>24.0</td>
<td>29.9</td>
<td>32.7</td>
<td>82.6</td>
</tr>
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<td>0.5b</td>
<td>31.6</td>
<td>19.6</td>
<td>14.1</td>
<td>22.0</td>
<td>135.6</td>
<td>197.3</td>
</tr>
<tr>
<td></td>
<td>1.0b</td>
<td>26.5</td>
<td>15.6</td>
<td>7.3</td>
<td>58.7</td>
<td>186.0</td>
<td>239.8</td>
</tr>
</tbody>
</table>

$\gamma = 5$

<table>
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<tr>
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<th>$W_0$</th>
<th>$\sigma_{\text{day}} = 2.5$</th>
<th>$\sigma_{\text{day}} = 7.5$</th>
<th>$\sigma_{\text{day}} = 12.5$</th>
<th>$\sigma_{\text{day}} = 2.5$</th>
<th>$\sigma_{\text{day}} = 7.5$</th>
<th>$\sigma_{\text{day}} = 12.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>120 m</td>
<td>0.1b</td>
<td>37.7</td>
<td>31.6</td>
<td>27.8</td>
<td>36.2</td>
<td>25.4</td>
<td>15.3</td>
</tr>
<tr>
<td></td>
<td>0.5b</td>
<td>32.6</td>
<td>24.2</td>
<td>20.1</td>
<td>27.4</td>
<td>1.7</td>
<td>19.9</td>
</tr>
<tr>
<td></td>
<td>1.0b</td>
<td>29.6</td>
<td>20.9</td>
<td>16.4</td>
<td>20.4</td>
<td>15.3</td>
<td>42.9</td>
</tr>
<tr>
<td>55 m</td>
<td>0.1b</td>
<td>35.5</td>
<td>28.2</td>
<td>24.1</td>
<td>32.9</td>
<td>16.3</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>0.5b</td>
<td>29.2</td>
<td>20.4</td>
<td>16.0</td>
<td>19.3</td>
<td>17.8</td>
<td>46.1</td>
</tr>
<tr>
<td></td>
<td>1.0b</td>
<td>25.9</td>
<td>16.8</td>
<td>5.7</td>
<td>8.7</td>
<td>40.3</td>
<td>73.9</td>
</tr>
<tr>
<td>12 m</td>
<td>0.1b</td>
<td>29.6</td>
<td>20.9</td>
<td>16.4</td>
<td>20.4</td>
<td>15.3</td>
<td>42.9</td>
</tr>
<tr>
<td></td>
<td>0.5b</td>
<td>21.9</td>
<td>12.6</td>
<td>9.1</td>
<td>9.3</td>
<td>72.9</td>
<td>112.1</td>
</tr>
<tr>
<td></td>
<td>1.0b</td>
<td>18.5</td>
<td>7.4</td>
<td>2.1</td>
<td>29.7</td>
<td>104.3</td>
<td>150.9</td>
</tr>
</tbody>
</table>

$\gamma = 10$
distribution is tightened at a relatively low level ($\alpha \approx 0.2$) and its time evolution is smooth. Regarding the wealth distribution, as expected, the less liquid the market, the lower the CER (as was shown in table 3) and the lower the dispersion of the wealth distribution.

### 4.3. Certainty equivalent losses associated with ignoring liquidity effects

Table 4 compares the CER of an investor who takes heed of liquidity effects when making portfolio decisions, to the CER of an investor who ignores liquidity effects. For the investor who ignores liquidity effects, we set $LC = MI = 0$ in the LSMC algorithm, then reset $LC = LC(\Delta q)$ and $MI = MI(\Delta q)$ for calculating the CER. Unsurprisingly, the liquidity-aware investor always has a positive CER, while the CER of the liquidity-blind investor can reach negative territory in illiquid markets. The massive gain in the CER of liquidity-aware portfolio allocation over liquidity-blind portfolio allocation illustrates how taking these costs into account is vital for reaching one’s performance target in real life situations where these liquidity effects do occur. The results illustrate the ability of the proposed algorithm to handle switching costs.

### 4.4. Sensitivity analysis

Table 5 reports the sensitivity of the CER and the initial stock allocation $\alpha_0$ with respect to the daily trading volume. The effect of changing the daily trading volume (and therefore increasing liquidity effects, cf. equations (13)–(14)) on the CER and $\alpha_0$ is consistent under different levels of daily volatility: the CER and $\alpha_0$ increase with the daily trading volume at diminishing rates. Similarly, table 6 shows that an increase in daily volatility (and therefore will increase...

<table>
<thead>
<tr>
<th>Vol$_{\text{day}}$</th>
<th>$\sigma_{\text{day}} = 2.5$</th>
<th>$\sigma_{\text{day}} = 7.5$</th>
<th>$\sigma_{\text{day}} = 12.5$</th>
<th>Initial allocation $\alpha_0$</th>
<th>$\sigma_{\text{day}} = 2.5$</th>
<th>$\sigma_{\text{day}} = 7.5$</th>
<th>$\sigma_{\text{day}} = 12.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 m</td>
<td>65.9</td>
<td>42.8</td>
<td>28.6</td>
<td>22.9</td>
<td>0.76</td>
<td>0.31</td>
<td>0.18</td>
</tr>
<tr>
<td>20 m</td>
<td>65.9</td>
<td>47.9</td>
<td>33.8</td>
<td>27.4</td>
<td>0.76</td>
<td>0.36</td>
<td>0.23</td>
</tr>
<tr>
<td>30 m</td>
<td>65.9</td>
<td>50.6</td>
<td>37.0</td>
<td>30.4</td>
<td>0.76</td>
<td>0.41</td>
<td>0.26</td>
</tr>
<tr>
<td>40 m</td>
<td>65.9</td>
<td>52.4</td>
<td>39.3</td>
<td>32.6</td>
<td>0.76</td>
<td>0.42</td>
<td>0.27</td>
</tr>
<tr>
<td>50 m</td>
<td>65.9</td>
<td>53.7</td>
<td>41.1</td>
<td>34.3</td>
<td>0.76</td>
<td>0.45</td>
<td>0.30</td>
</tr>
<tr>
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<td>65.9</td>
<td>54.7</td>
<td>42.5</td>
<td>35.8</td>
<td>0.76</td>
<td>0.45</td>
<td>0.31</td>
</tr>
<tr>
<td>70 m</td>
<td>65.9</td>
<td>55.4</td>
<td>43.7</td>
<td>37.0</td>
<td>0.76</td>
<td>0.46</td>
<td>0.32</td>
</tr>
<tr>
<td>80 m</td>
<td>65.9</td>
<td>56.1</td>
<td>44.7</td>
<td>38.0</td>
<td>0.76</td>
<td>0.48</td>
<td>0.33</td>
</tr>
<tr>
<td>90 m</td>
<td>65.9</td>
<td>56.6</td>
<td>45.5</td>
<td>39.0</td>
<td>0.76</td>
<td>0.50</td>
<td>0.34</td>
</tr>
<tr>
<td>100 m</td>
<td>65.9</td>
<td>57.1</td>
<td>46.3</td>
<td>39.0</td>
<td>0.76</td>
<td>0.51</td>
<td>0.34</td>
</tr>
<tr>
<td>110 m</td>
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<td>57.5</td>
<td>47.0</td>
<td>40.6</td>
<td>0.76</td>
<td>0.52</td>
<td>0.35</td>
</tr>
<tr>
<td>120 m</td>
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<td>57.8</td>
<td>47.6</td>
<td>41.1</td>
<td>0.76</td>
<td>0.53</td>
<td>0.37</td>
</tr>
</tbody>
</table>

Table 6. Sensitivity w.r.t daily volatility. This table reports the sensitivity of the monthly adjusted CER (in basis points) and the initial stock allocation with respect to the daily volatility $\sigma_{\text{day}} = 2, 3, \ldots, 13$, for a CRRA utility investor with $\gamma = 5$ and investment horizon $N = 12$ months under different daily volatility ($\sigma_{\text{day}} = 2.5, 7.5, 12.5$), using $M = 10^5$ Monte Carlo paths with one control iteration $I = 1$. A portfolio of cash and the SPDR S&P 500 ETF (SPY) is investigated, with annual risk free rate $r_f = 0.012$, portfolio weight increment 0.01 and initial investment amount $W_0 = \$100$ m. ‘liquid’ stands for a perfectly liquid market without liquidity costs and market impact.

<table>
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<th>liquid</th>
<th>$V_{\text{day}} = 120$ m</th>
<th>$V_{\text{day}} = 55$ m</th>
<th>$V_{\text{day}} = 12$ m</th>
<th>Initial allocation $\alpha_0$</th>
<th>liquid</th>
<th>$V_{\text{day}} = 120$ m</th>
<th>$V_{\text{day}} = 55$ m</th>
<th>$V_{\text{day}} = 12$ m</th>
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<tbody>
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<td>2</td>
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<td>59.2</td>
<td>56.1</td>
<td>46.9</td>
<td>0.76</td>
<td>0.55</td>
<td>0.48</td>
<td>0.35</td>
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</tr>
<tr>
<td>3</td>
<td>65.9</td>
<td>56.5</td>
<td>52.5</td>
<td>41.8</td>
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<td>0.50</td>
<td>0.44</td>
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</tr>
<tr>
<td>4</td>
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<td>54.2</td>
<td>49.5</td>
<td>38.1</td>
<td>0.76</td>
<td>0.45</td>
<td>0.39</td>
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<td>35.1</td>
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<td>0.42</td>
<td>0.36</td>
<td>0.24</td>
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</tr>
<tr>
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<td>0.39</td>
<td>0.31</td>
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<td>46.9</td>
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<td>29.1</td>
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<td>0.29</td>
<td>0.18</td>
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<td>65.9</td>
<td>45.4</td>
<td>39.4</td>
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<td>0.76</td>
<td>0.34</td>
<td>0.28</td>
<td>0.17</td>
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<td>38.0</td>
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<td>0.33</td>
<td>0.27</td>
<td>0.16</td>
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<td>11</td>
<td>65.9</td>
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<td>0.26</td>
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</table>
liquidity effects, cf. equations (13)–(14)) will decrease the CER and \( \alpha_0 \), but at diminishing rates.

Table 7 reports the sensitivity of the CER and the initial stock allocation \( \alpha_0 \) with respect to the initial investment amount \( W_0 \). As expected, the CER and \( \alpha_0 \) decrease with respect to \( W_0 \) due to the increased liquidity effects on larger portfolio turnovers. Under extreme liquidity effects (\( \sigma_{\text{day}}, \text{Vol}_{\text{day}} = (12.5, 120 \text{ m}) \), \( \alpha_0 \) remains zero for \( W_0 \geq 600 \text{ m} \).

Table 8 reports the sensitivity of the CER and \( \alpha_0 \) with respect to the investment horizon \( N \). When \( N \) is small, the CER and \( \alpha_0 \) are very sensitive to the choice of \( N \). In particular, the CER increases with \( N \) when \( N \) is small, for which \( \alpha_0 \) quickly converges to a certain level as \( N \) increases from \( N = 2 \). As the time horizon \( N \) further increases, the CER decreases when liquidity effects are small (perfectly liquid, \( \sigma_{\text{day}} = 2.5, \text{Vol}_{\text{day}} = 120 \text{ m} \) and \( \sigma_{\text{day}} = 7.5, \text{Vol}_{\text{day}} = 55 \text{ m} \)) and increases when liquidity effects are large (\( \sigma_{\text{day}} = 12.5, \text{Vol}_{\text{day}} = 12 \text{ m} \)).

Finally, table 9 reports the sensitivity of the CER and \( \alpha_0 \) with respect to the risk aversion level \( \gamma \) of the CRRA utility. As expected, the CER and \( \alpha_0 \) both decrease under every liquidity condition when risk aversion increases.

To conclude this numerical section, we emphasize that a key feature of the proposed optimization algorithm is the ability to measure and account for the effects of imperfect liquidity for dynamic portfolio optimization. The numerical convergence of the algorithm has been validated, so that we have been able to evaluate the sensitivities of portfolio allocation and portfolio performance with respect to various parameters. Such analyzes can be adapted to different models, markets and investment styles, and bring advantageous insights to how to dynamically adjust portfolio allocations in illiquid markets.

**Table 7.** Sensitivity w.r.t initial investment amount. This table reports the sensitivity of the monthly adjusted CER (in basis points) and the initial stock allocation with respect to the initial investment amount \( W_0 = \$100, 200, \ldots , 1b \), for a CRRA utility investor with \( \gamma = 5 \) and investment horizon \( N = 12 \) months under different liquidity settings (\( \sigma_{\text{day}}, \text{Vol}_{\text{day}} \) (denoted \( \text{Vol}_{\text{day}} \)) = (2.5, 120 m), (7.5, 55 m), (12.5, 12 m)), using \( M = 10^5 \) Monte Carlo paths with one control iteration \( I = 1 \). A portfolio of cash and the SPDR S&P 500 ETF (SPY) is investigated, with annual risk free rate \( r^f = 0.012 \), portfolio weight increment 0.01. ‘liquid’ stands for a perfectly liquid market without liquidity costs and market impact.

<table>
<thead>
<tr>
<th>( W_0 ) ($)</th>
<th>( \sigma_{\text{day}} = 2.5 )</th>
<th>( \sigma_{\text{day}} = 7.5 )</th>
<th>( \sigma_{\text{day}} = 12.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>liquid</td>
<td>( V_{\text{day}} = 120 \text{ m} )</td>
<td>( V_{\text{day}} = 55 \text{ m} )</td>
<td>( V_{\text{day}} = 12 \text{ m} )</td>
</tr>
<tr>
<td>100 m</td>
<td>0.76</td>
<td>0.52</td>
<td>0.31</td>
</tr>
<tr>
<td>200 m</td>
<td>0.76</td>
<td>0.47</td>
<td>0.25</td>
</tr>
<tr>
<td>300 m</td>
<td>0.76</td>
<td>0.43</td>
<td>0.22</td>
</tr>
<tr>
<td>400 m</td>
<td>0.76</td>
<td>0.40</td>
<td>0.20</td>
</tr>
<tr>
<td>500 m</td>
<td>0.76</td>
<td>0.38</td>
<td>0.19</td>
</tr>
<tr>
<td>600 m</td>
<td>0.76</td>
<td>0.36</td>
<td>0.18</td>
</tr>
<tr>
<td>700 m</td>
<td>0.76</td>
<td>0.35</td>
<td>0.16</td>
</tr>
<tr>
<td>800 m</td>
<td>0.76</td>
<td>0.34</td>
<td>0.15</td>
</tr>
<tr>
<td>900 m</td>
<td>0.76</td>
<td>0.33</td>
<td>0.14</td>
</tr>
<tr>
<td>1b</td>
<td>0.76</td>
<td>0.32</td>
<td>0.13</td>
</tr>
</tbody>
</table>

**Table 8.** Sensitivity w.r.t investment horizon. This table reports the sensitivity of the monthly adjusted CER (in basis points) and the initial stock allocation with respect to the investment horizon \( N = 2, 3, \ldots , 12 \) months, for a CRRA utility investor with \( \gamma = 5 \) under different liquidity settings (\( \sigma_{\text{day}}, \text{Vol}_{\text{day}} \) (denoted \( \text{Vol}_{\text{day}} \)) = (2.5, 120 m), (7.5, 55 m), (12.5, 12 m)), using \( M = 10^5 \) Monte Carlo paths with one control iteration \( I = 1 \). A portfolio of cash and the SPDR S&P 500 ETF (SPY) is investigated, with annual risk free rate \( r^f = 0.012 \), portfolio weight increment 0.01 and initial investment amount \( W_0 = \$100 \text{ m} \). ‘liquid’ stands for a perfectly liquid market without liquidity costs and market impact.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \sigma_{\text{day}} = 2.5 )</th>
<th>( \sigma_{\text{day}} = 7.5 )</th>
<th>( \sigma_{\text{day}} = 12.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>liquid</td>
<td>( V_{\text{day}} = 120 \text{ m} )</td>
<td>( V_{\text{day}} = 55 \text{ m} )</td>
<td>( V_{\text{day}} = 12 \text{ m} )</td>
</tr>
<tr>
<td>2</td>
<td>0.89</td>
<td>0.60</td>
<td>0.24</td>
</tr>
<tr>
<td>3</td>
<td>0.83</td>
<td>0.58</td>
<td>0.30</td>
</tr>
<tr>
<td>4</td>
<td>0.81</td>
<td>0.57</td>
<td>0.31</td>
</tr>
<tr>
<td>5</td>
<td>0.79</td>
<td>0.55</td>
<td>0.31</td>
</tr>
<tr>
<td>6</td>
<td>0.79</td>
<td>0.55</td>
<td>0.31</td>
</tr>
<tr>
<td>7</td>
<td>0.78</td>
<td>0.54</td>
<td>0.31</td>
</tr>
<tr>
<td>8</td>
<td>0.78</td>
<td>0.54</td>
<td>0.31</td>
</tr>
<tr>
<td>9</td>
<td>0.77</td>
<td>0.53</td>
<td>0.31</td>
</tr>
<tr>
<td>10</td>
<td>0.76</td>
<td>0.53</td>
<td>0.31</td>
</tr>
<tr>
<td>11</td>
<td>0.76</td>
<td>0.53</td>
<td>0.31</td>
</tr>
<tr>
<td>12</td>
<td>0.76</td>
<td>0.53</td>
<td>0.31</td>
</tr>
</tbody>
</table>
Dynamic portfolio optimization with liquidity cost and market impact

Table 9. Sensitivity w.r.t. risk aversion level. This table reports the sensitivity of the monthly adjusted CER (in basis points) and the initial stock allocation with respect to the risk aversion level $\gamma = 2, 3, \ldots, 10$ for a CRRA utility investor for investment horizon $N = 12$ months, under different liquidity settings $(\sigma_{\text{day}}, \text{Vol}_{\text{day}}$ (denoted $V_{\text{day}}$)) $= (2.5, 120)$, $(7.5, 55)$, $(12.5, 12)$, using $M = 10^5$ Monte Carlo paths with one control iteration $I = 1$. A portfolio of cash and the SPDR S&P 500 ETF (SPY) is investigated, with annual risk free rate $r = 0.012$, portfolio weight increment 0.01 and initial investment amount $W_0 = $100$ m. ‘liquid’ stands for a perfectly liquid market without liquidity costs and market impact.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>liquid</th>
<th>$\sigma_{\text{day}} = 2.5$</th>
<th>$\sigma_{\text{day}} = 7.5$</th>
<th>$\sigma_{\text{day}} = 12.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$V_{\text{day}} = 120$ m</td>
<td>$V_{\text{day}} = 55$ m</td>
<td>$V_{\text{day}} = 12$ m</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>91.3</td>
<td>82.2</td>
<td>61.8</td>
<td>34.0</td>
</tr>
<tr>
<td>3</td>
<td>81.6</td>
<td>72.6</td>
<td>53.4</td>
<td>29.5</td>
</tr>
<tr>
<td>4</td>
<td>73.1</td>
<td>64.5</td>
<td>46.8</td>
<td>26.4</td>
</tr>
<tr>
<td>5</td>
<td>65.9</td>
<td>57.8</td>
<td>41.8</td>
<td>24.0</td>
</tr>
<tr>
<td>6</td>
<td>59.7</td>
<td>52.3</td>
<td>37.9</td>
<td>22.1</td>
</tr>
<tr>
<td>7</td>
<td>54.4</td>
<td>47.6</td>
<td>34.7</td>
<td>20.4</td>
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<tr>
<td>8</td>
<td>49.7</td>
<td>43.8</td>
<td>32.1</td>
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</tr>
<tr>
<td>9</td>
<td>45.6</td>
<td>40.5</td>
<td>30.0</td>
<td>17.8</td>
</tr>
<tr>
<td>10</td>
<td>41.9</td>
<td>37.7</td>
<td>28.2</td>
<td>16.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Initial allocation $\alpha_0$</th>
<th>liquid</th>
<th>$\sigma_{\text{day}} = 2.5$</th>
<th>$\sigma_{\text{day}} = 7.5$</th>
<th>$\sigma_{\text{day}} = 12.5$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$V_{\text{day}} = 120$ m</td>
<td>$V_{\text{day}} = 55$ m</td>
<td>$V_{\text{day}} = 12$ m</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
<td>0.52</td>
<td>0.23</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.95</td>
<td>0.66</td>
<td>0.35</td>
<td>0.16</td>
</tr>
<tr>
<td>0.76</td>
<td>0.53</td>
<td>0.31</td>
<td>0.13</td>
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<td>0.62</td>
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<td></td>
</tr>
<tr>
<td>0.54</td>
<td>0.38</td>
<td>0.23</td>
<td>0.10</td>
<td></td>
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<tr>
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<td>0.33</td>
<td>0.22</td>
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<td></td>
</tr>
<tr>
<td>0.41</td>
<td>0.29</td>
<td>0.20</td>
<td>0.08</td>
<td></td>
</tr>
<tr>
<td>0.34</td>
<td>0.27</td>
<td>0.18</td>
<td>0.08</td>
<td></td>
</tr>
</tbody>
</table>

5. Conclusion

This paper describes a simulation-and-regression method for solving portfolio optimization problems with general transaction costs, temporary liquidity costs and permanent market impact. To deal with permanent market impact without substantially increasing the computational complexity, we treat the price dynamics as the endogenous state variables that complement the exogenous return dynamics. The simulation nature of the algorithm makes it suitable for multivariate portfolio optimization with realistic asset dynamics and realistic liquidity effects. The presented algorithm adapts (Kharroubi et al. 2014)'s control randomization approach for the problems of discrete portfolio allocations. For each allocation level, the endogenous state variables are correspondingly updated for each path and the updated variables are used to approximate the value function by least squares regressions. We iterate the whole algorithm by using the optimal control estimates of the first run as the initial controls of the second run. Our numerical tests show that, with second-order polynomial basis, the proposed control discretization combined with global control iteration outperforms a direct control regression approach, all the more so for highly nonlinear utility functions (high risk aversion).

We apply the proposed method to solve a realistic cash-and-stock portfolio optimization problem with the power-law liquidity model of Almgren et al. (2005). We show that the losses associated with ignoring liquidity effects can be substantial, indicating the necessity to account for liquidity effects when making portfolio decisions in real markets. Most importantly, our algorithm is able to protect the portfolio value under illiquid markets. Going further, we analyze the sensitivities of CERs and optimal allocations with respect to trading volume, stock price volatility, initial capital, risk aversion level and investment horizon.

At this stage, one limitation of the proposed algorithm is that the control discretization approach will suffer from the curse of dimensionality with respect to the number of risky assets. Mitigation techniques are currently being investigated. For example, Zhang et al. (2018) recently investigated local control regression techniques on optimizing a portfolio with multiple risky assets.

The flexibility of the algorithm motivates further investigations on alternative portfolio performance measures, alternative liquidity models, and on additional realistic features such as cross-asset price impact. It could also be easily adapted to optimal portfolio liquidation problems as well as more general optimal switching problems with endogenous uncertainty.

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Disclosure statement

No potential conflict of interest was reported by the authors.

References


