

Dynamic trading strategies for storage.

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Abstract

We consider a dynamic model of an oligopolistic market with demand shocks, in which a storage unit buys and sells over time subject to a capacity constraint. To make progress in this stochastic game with constraints, we restrict attention to simple heuristics, and we can characterise the optimal policy of a storage unit in this restricted class of strategies. The heuristics, the exogenous stochastic process and the capacity constraint interact to induce rich dynamics. The optimal policy is sensitive to the nature of demand shocks and to storage capacity. For a fixed capacity, the storage unit internalises its unilateral market power; it acts like a monopolist on its arbitrage spread. We uncover a new phenomenon that we call *continuation risk*. It is a corollary of market power and induces the optimal capacity to be interior even absent investment cost. We discuss some implications.

This work applies to any storable commodity such as crops, raw materials or fuels, and more recently, electricity.

Key words: *stochastic game, dynamic trading, storage*

JEL: *C73, D43, D47*

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1 Introduction

Storage is an important step in any value chain in the economy. For any commodity, it delivers the intertemporal smoothing of consumption and production that so much contributes to enhancing welfare. In electricity markets, storage is the innovation that may render the energy transition feasible, and the essential complement to renewable energy generation. Yet we know very little of the economics of *merchant storage*, that is, the activity of buying and selling commodities to take advantage of price differences over time. This paper addresses this gap – in part.

In electricity, storage can be the device that soaks up energy produced in the middle of the day and releases it during the high-demand period. Closer to our purposes, for some commodities, prices can vary widely and unexpectedly in very short intervals. The Australian electricity market illustrates the point in Figure 1, but sizeable price swings are also not rare in California, Texas or the Nordic Market of Europe.¹ Like a financial traders in securities markets, storage can take advantage of these unexpected and uncertain price differences. This is the focus of this paper.

Time	Queensland (\$/MWh)	NSW (\$/MWh)
6.05 pm	15,500	12,465
6.10 pm	15,500	14,218
6.15 pm	3,569	3,164
6.20 pm	15,500	14,120
6.25 pm	15,500	14,507
6.30 pm	399	358
6.35 pm	15,500	13,519
6.40 pm	355	308
6.45 pm	9,999	9,026
6.50 pm	9,797	8,738
6.55 pm	370	323
7 pm	304	265

Figure 1: Sample price sequence 16 March 2024 in the NEM. Source: RenewEconomy using AEMO data.

To tackle this problem we study a stylised model of merchant storage over a long horizon and rooted in an oligopolistic market. A finite number of sellers play a quantity game. Each period they produce for immediate sale, as in electricity markets, while demand is subject to aggregate shocks. These shocks induce a sequence of high and low prices over time. A storage operator can step in to exploit these price differences by implementing the simple idea

¹Denmark, Norway, Sweden and Finland.

of “buying low and selling high” (intertemporal arbitrage), the details of which are in fact quite complicated and rich because of market power.

A storage operator faces two essential trade-offs: internalising its unilateral market power (current trades) and continuation (the value of future trades). The novelty lies in how these well-known trade-offs materialise. On the first account, a storage unit with large enough a capacity internalises its own market power and thus withholds quantities – bought and sold – in any period. On the second one, it must have enough capacity to have the flexibility to fully exploit arbitrage opportunities but not too much so as to not face large costs induced by excessive buying, which happens with positive probability. The cost of excessive buying increases in capacity; so capacity entails a trade-off between flexibility and costs. We call this the *continuation risk*; with too small a capacity there is insufficient continuation trade, with too large a capacity that continuation trade can be too costly. The continuation risk never disappears, except when shocks are governed by a low-persistence Markov chain; then the continuation risk is negated. This new phenomenon is unique to *merchant* storage because it must *buy* before selling, and is a corollary of market power.² Indeed, if capacity is small enough, it is used in full at every opportunity and there is no risk of buying too much. We explore some extensions, including a richer, Markovian shock structure with more or less persistence in shocks; all confirm our results.

Our model gives rise to a stochastic game first introduced by [Shapley \(1953\)](#), which typically admits a very large number of equilibria. To make progress, we must limit ourselves to studying for a restricted class of admissible strategies that we call heuristics rather than the more desirable unconstrained strategies that remain out of reach. In this restricted class of admissible strategies we characterize equilibrium and study some comparative statics. Even with this restriction on the space of admissible strategies for the storage unit, we must also fix an equilibrium (a behavior) of the sellers; we argue this is a small concession to make. For two intuitive heuristics, we uncover a recursive structure that is tractable and allows us to reduce the corresponding value function to a polynomial, which can then be optimised. These heuristics are not only intuitive, they are also reasonable – as we discuss in [Section 3.2.5](#). As in any stochastic game, the strategies interact with the exogenous shocks to induce endoge-

²*Merchant* storage differs from consumer storage or producers holding inventories: the merchant store must both buy and sell. Consumers only buy while producers only sell, hence they do not face the continuation risk.

nous transitions between states. Here, these transitions are further enriched by *constraints* on capacity and on the initial condition to generate novel dynamics. These dynamics can only emerge because of market power, both in production and in storage, as we show in Section 3.1.

Although stylised, the model we present in this paper can be applied to many commodities such as fuels, raw materials or electricity by adjusting primitives such as the discount factor or storage efficiency, as well as the underlying stochastic process. A higher discount factor may apply to frequent trading, such as electricity; a lower one may be appropriate for crops. It can also apply to market making in securities. Indeed, this intermediation activity shares many characteristics with trading through storage: assets are bought and sold, a revenue is generated by *intertemporal* arbitrage, holding inventory is necessary, the market maker provides insurance against aggregate risk and price impact matters a great deal.³ We confine ourselves to a positive analysis; yet that analysis does lead to some tentative implications for competition policy and market design. The market is better served by a large fleet of small units; in a centralised market (such as electricity), a market operator may improve outcomes by offering a measure of insurance. An important question of market design remains open: how to best determine the bidding space and clear a market when bidders play dynamic strategies?

This paper is one of very few on the economics of *merchant storage*, even though the practice is as old as trading itself. Our work departs from the extant literature because it seeks to characterise behaviour in a stochastic environment with market power. Samuelson (1971) first suggests a model of speculative arbitrage with uncertainty, however under perfect competition. That assumption renders the dynamics moot, whence uncertainty has little importance. Wright and Williams (1984) suggest the storage of crops increases welfare but do not solve the dynamic problem—even under perfect competition. Deaton and Laroque (1992) rationalise the (price) behavior of 13 essential crops by allowing for speculative storage, however again under perfect competition.

More recently storage has piqued the interest of economists because of the promise it presents in electricity markets. Karaduman (2020) is the first to study grid scale storage, using Australian data. Producers and the storage unit play an infinite horizon game and

³The works of Vayanos (1999) and Glebkin et al. (Forthcoming) differ from ours in that traders seek to diversify idiosyncratic risk, whereas here the arbitrageur takes advantage of aggregate risk, and so *supplies* insurance.

market power is internalised. However, [Karaduman \(2020\)](#) does not compute the best reply; rather he simulates it from the data. [Andres-Cerezo and Fabra \(2023b\)](#) study the question of market structure with storage but leave aside how storage actually behaves. A producer enhances its market power by also owning storage: when demand is the highest, the substitutability between storage and generation should be exploited to its fullest, but their joint ownership induces more quantity withholding. [Butters et al. \(Working Paper\)](#) use California data to estimate the equilibrium effect of large-scale storage. As the storage fleet expands, arbitrage revenue decreases, which hinders adoption. In that model, however, storage is assumed to behave competitively. We study the details of buying and selling with market power. [Schmalensee \(2022\)](#) studies storage investment, which we take as exogenous; he models the intra-day behaviour of storage rather than short-term arbitrage opportunities. Energy generation and storage are competitive.⁴ [Williams and Green \(2022\)](#) compute the welfare effects of storage on the current British market using simulations and so without characterising any equilibrium, with time-varying demand and no uncertainty. [Geske and Green \(2020\)](#) do study arbitrage in a model of imperfect competition with demand uncertainty and diurnal, weekly and seasonal patterns. In such a complicated environment they must limit themselves to numerical (approximate) solutions to the welfare maximization problem. We show that market power and uncertainty are critical aspects of the problem.

This paper also bears connection to the rich literature on financial arbitrage – see [Shleifer and Vishny \(1997\)](#), [Oehmke \(2009\)](#) or [Dávila et al. \(2024\)](#) among many others. In these works, arbitrage is contemporaneous, risk-free and between segmented markets. Our model is one of intertemporal arbitrage of aggregate risk in a single market.

After introducing the model, we turn to the main analysis of two intuitive heuristics when shocks are symmetric. As a robustness check, we introduce a richer Markovian structure to allow for persistence. We draw some policy implications and conclude.

2 Model

Consider a market with one storage unit, n producers labelled $j = 1, 2, \dots, n$, and a pool of consumers. We simplify institutional details so that retailers and consumers are confounded

⁴[Schmalensee \(2022\)](#) also assumes that storage is fully discharged after the “night-time”, while we let the storage operator make that decision in equilibrium.

and retailing has no cost; another way of saying this is that retailers perfectly reflect the behavior of consumers. That behavior is described by the demand function $D(p_t, \varepsilon_t)$ for each period t , where ε_t is a shock distributed according to some commonly known distribution F . Stochastic demand is isomorphic to stochastic supply. Each of the sellers j produces a quantity q_t^j for each period t , and may or may not be subject to capacity constraints. The storage unit has finite capacity k . In each period, it can either buy up to its capacity, or sell any available amount.⁵ This process can be described formally by a simple equation of motion:

$$c_t = c_{t-1} + b_t - \frac{s_t}{\delta}, \quad t \in \mathbb{N}, \quad c_0 = 0. \quad (1)$$

Here, c_t is a current level of inventory ($0 \leq c_t \leq k$), δ is a round-trip efficiency parameter ($0 < \delta \leq 1$), and $b_t \geq 0, s_t \geq 0$. A storage operator can only either buy or sell in each period, so $b_t \cdot s_t = 0$ – this is a technical characteristic, but also optimal. The market clears if

$$D(p_t, \varepsilon_t) = \sum_{j=1}^n q_t^j - b_t + s_t$$

for any t , where we suppose that players engage in quantity competition.⁶ Since the nature of competition is not the primary object of interest, throughout the rest of the paper we consider a linear demand function:

$$D(p_t, \varepsilon_t) = 1 - p_t + \varepsilon_t, \quad \varepsilon_t \in \{-a, a\}.$$

Rather, the goal is to find optimal policies $\{b_t, s_t\}_{t=0}^{\infty}$ which are the part of a dynamic Nash equilibrium. We suppose the storage unit has a discount factor $\beta < 1$; it is exposed to a strictly positive interest rate. Depending on the decisions of the storage operator, in each round there may be either

- n (symmetric) competitors; or

⁵When it comes to electricity we make no distinction between power and energy; it is as if a quantity were either energy or power for a prescribed duration (e.g. for the trading interval).

⁶Even through the norm in electricity markets is to use the more elegant supply-function equilibrium (SFE), this model remains relevant. First, the Cournot outcome is a possible equilibrium outcome of the SFE and constitutes an upper bound for the payoffs to suppliers (Klemperer and Meyer (1989)). Second, Cournot competition is used as a successful proxy in many papers (Acemoglu et al. (2017), Willems et al. (2009), Lundin and Tangerås (2020)); much of this work relies on the estimations of Borenstein and Bushnell (1999), Borenstein et al. (1999) or Bushnell et al. (2008).

- $n + 1$ competitors, with the storage unit having a limited capacity.

This model assembles all the essential characteristics of any storage operation: a long horizon, buying and selling, uncertain demand and market power. Its interpretation can vary by adjusting the parameters β, δ or the shock ε and its distribution F . Let $\sum_{j=1}^n q_t^j = Q_t^{-i}$, the objective of the storage operator is

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t p_t (Q_t^{-i} + b_t - s_t) \cdot (s_t - b_t) \right], \quad (2)$$

subject to the law of motion (1) and the important capacity constraint

$$0 \leq c \leq k, \quad (3)$$

with corresponding value function

$$V(c) = \sup_{b,s} \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t p_t (Q_t^{-i} + b_t - s_t) \cdot (s_t - b_t) \right]^7, \quad (4)$$

in which the inventory c is the state variable of interest.

3 Trading over the long horizon

This section begins with a competitive benchmark that is useful to illustrate the salience of market power in this problem. The main analysis, with market power, relies on independent shocks. We also present some robustness checks.

3.1 Competitive storage

Suppose storage is a competitive industry; the objective function of any such operator is a simplified version of (2):

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t p_t \cdot (s_t - b_t) \right], \quad (5)$$

⁷We dispense proving that the Dynamic Programming Principle holds in this environment, which is quite standard.

subject to the same law of motion and the same capacity constraint, and where the price p_t is not a function of (s_t, b_t) . One can write a value function too, as in (4), but it is not really necessary to arrive at the conclusion that the problem is easily solved. With a payoff function that is linear in quantities, a competitive storage unit sells all it can, so $q^s \in \{0, c\}$; from the demand function $D(p_t, \varepsilon)$, q^s differs from 0 only under a positive shock. To maximise that payoff, it must buy all it can: $q^b \in \{0, k\}$ and differs from 0 only under a negative demand shock, so that $c \in \{0, k\}$. The corresponding buying and selling prices are, respectively,

$$\hat{p}^b = \frac{1-a}{n+1} < \frac{1+a}{n+1} = \hat{p}^s.$$

A storage unit that is not strategic simply repeats that static optimum. Even if the game has a long horizon, there are no intertemporal linkages because competitive storage units do not withhold quantities, therefore there is no “carry-over” between periods. Absent intertemporal linkages, there is no other solution to this problem. Quantity-withholding (or the infra-marginal effect) is rooted in market power and is the source of all dynamics. This is the focus of the ensuing analysis.⁸

3.2 Independent binary shocks

Now we turn to the problem with market power – solving (2) rather than (5). Consider the simple independent shock structure

$$\Pr(\varepsilon = -a) = \Pr(\varepsilon = a) = 1/2,$$

which affords us some tractability. Even then, short of constructing equilibria that exhibit features the analyst seeks, it is impossible to characterise an optimal strategy. But we wish to make progress to answer a practical question. To overcome this problem, we reduce the space of admissible strategies in two ways. First, we must describe the equilibrium behavior of the other n players. We elect to restrict attention to the repetition of the Cournot equilibrium stage

⁸If producers are competitive, they sell at marginal cost (absent in our model); when producers have convex marginal costs, a storage operator buys in period of low demand at a price equal to the low marginal cost and sells in period of high demand at a price equal to the high marginal cost. Because of competitive pressure that is all it can do and there is no economic incentive to withhold quantities. This is a simpler stochastic control problem, where prices are exogenous to the problem of the storage operator.

game. This is a genuine equilibrium of the dynamic game between producers (see [Tirole \(1988\)](#), Chapter 6, for example) and it is simple to describe, unlike any of the more sophisticated equilibrium strategies one can construct. A further justification is the work of [Bonatti et al. \(2017\)](#), who study a dynamic Cournot model under incomplete information with learning. The equilibrium converges to the repeated static Nash equilibrium; we start from this point. Second, we restrict the space of strategies to intuitive heuristics, which allows us to reduce the value function of the storage operator to a polynomial function. While these heuristics commit the storage operator to a fixed behaviour over time, this property is not essential to our results in a sense we make precise in Section [3.2.3](#).

The recursive equations characterising the solution to [\(4\)](#) may be written as:

$$\begin{cases} V(c) &= \frac{1}{2} \left(-\frac{1-a+b(c)}{n+1} \cdot b(c) + \beta V(c + b(c)) \right) \\ &+ \frac{1}{2} \left(\frac{1+a-\delta s(c)}{n+1} \cdot \delta s(c) + \beta V(c - s(c)) \right), \\ V(0) &= \frac{1}{2-\beta} \left(-\frac{1-a+b(0)}{n+1} \cdot b(0) + \beta V(b(0)) \right), \\ V(k) &= \frac{1}{2-\beta} \left(\frac{1+a-\delta s(k)}{n+1} \cdot s(k) + \beta V(k - s(k)) \right), \end{cases} \quad (6)$$

where [\(1\)](#) and [\(3\)](#) imply $0 \leq b(c) \leq k - c$ and $0 \leq s(c) \leq c$ for any c . The function $b(c)$ is the quantity storage would like to buy if its inventory is c ; it is different from 0 only if the shock is negative. Likewise, $s(c)$ is the quantity storage with inventory c would like to sell, which is relevant only under the positive shock. We aim to find functions $b(c)$ and $s(c)$ that maximize $V(0)$ from time 0 and subject to [\(1\)](#) and [\(3\)](#). Given the nature of the stochastic process, it is immediate that V is time-invariant.

3.2.1 Heuristic 1: proportional bids

Our first heuristic calls for constant fractions in buying and selling. For example, starting from empty, if the chosen fraction is $3/4$, the storage operator buys $3k/4 = c$, which is now the new inventory. If facing another negative shock, she buys $3(k - c)/4 = 3k/16$, now the inventory is $c = 15k/16$. If facing a positive shock instead, she sells $3c/4 = 9k/16$, so the inventory is $c = 3k/16$. And so on. More formally, $b(c) = r(k - c)$ or $s(c) = rc$, respectively ($0 \leq r \leq 1$). This problem is rendered complicated for two reasons. First, the constraint $0 \leq c \leq k$ implies that, while the action (buy or sell) is governed by the stochastic shock,

its quantum depends on the state c . Second, the grid of the state space grows exponentially (if $r < 1$); boundaries are never reached. In addition, a storage unit cannot start from an arbitrary state, but it must commence at $c = 0$. Hence it can be stuck at that level for some period before being able to buy and start trading. Nonetheless there exists a recursive structure that can be exploited. Let $b(c) = r(k - c)$ and $s(c) = rc$ and

$$B(rk) = \frac{1 - a + rk}{n + 1} \cdot rk, \quad A(rk) = \frac{1 + a - \delta rk}{n + 1} \cdot \delta rk.$$

denote the purchasing cost and sale revenue, respectively. In each of these, rk (δrk) is a quantity and the rest of the expression is the clearing price. Clearly, it cannot be optimal to buy when the shock ε is positive, nor can it be optimal to sell when it is negative. Then we show the value function $V(c)$ reduces to a polynomial.

Proposition 1. *The overall expected profit of storage is*

$$U_s^P = \frac{1}{2[1 - (1 - r)\beta]} \left(-B(rk) + \frac{\beta r}{2(1 - \beta)} (-B(rk) + A(rk)) \right) + \frac{\beta}{1 - \beta} \frac{k^2 r^3 (1 - r)(1 + \delta^2)}{4(n + 1)(1 - (1 - r)^2 \beta)}. \quad (7)$$

The proof of this Proposition, as all others, is relegated to the Appendix.

Expression (7) entails three elements, abstracting from the multiplier that is a modified discount factor. The first term in the bracket is the cost of the initial purchase. The second term is the discounted arbitrage profit from the first trade onward; this is the simple “buy low, sell high” mantra. A round trip of buying and selling takes two periods minimum, multiplied by the fraction r . The last term is strictly positive and not at all connected to arbitrage since it is independent of a . It represents the benefit of flexibility in the face of uncertainty. Indeed, when $r = 1$ this last term is zero, and all the weight is assigned to the arbitrage revenue. For $r < 1$, facing a series of negative shock, for example, the storage unit buys less and less. But upon a reversal, it sells a lot. This term is largest for some interior value, which captures this notion of flexibility that arises from the asymptotic behaviour of this heuristic. We call this the “continuation risk”, and come back to this point later.

The payoff given in (7) is expressed in terms of the capacity k of the storage unit and its choice of heuristic r . This allows us not only to find the optimal proportion r to maximise

U_s^P , but also to engage in comparative statics with respect to k . The first-order condition of (7) does not lend itself to easy manipulation nor interpretation, but Condition (7) can be graphed. To this end, let $n = 2$, $\delta = 0.95$, and $\beta = 0.95$. Consider payoffs values for different capacities and shock magnitudes. Figure 2 corresponds to high shocks $a = 0.6$ with capacity k moving from 0.15 to 1.15, and Figure 3 corresponds to low shocks $a = 0.2$ where k changes from 0.05 to 0.3.

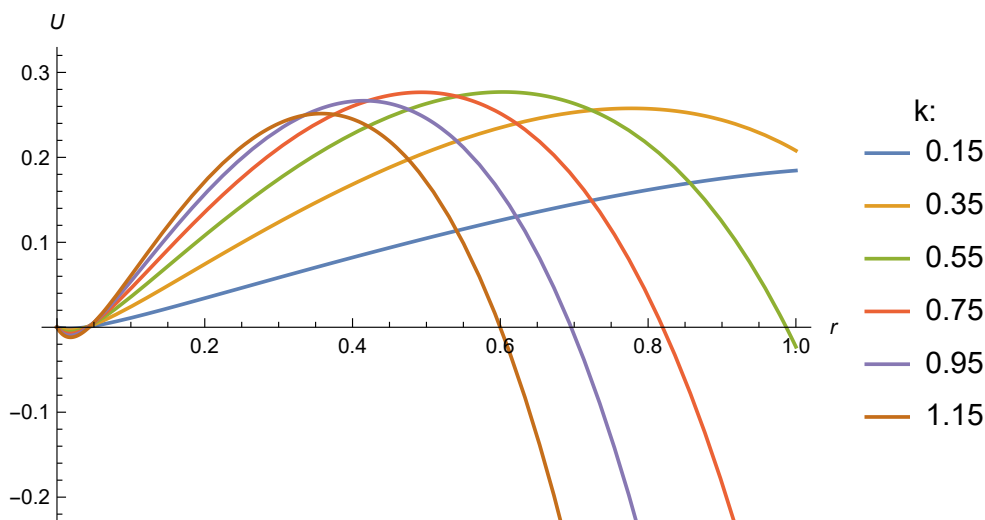


Figure 2: Payoff functions $U_s^P(r)$ for different capacities k when the magnitude of the shock is high enough: $a = 0.6$.

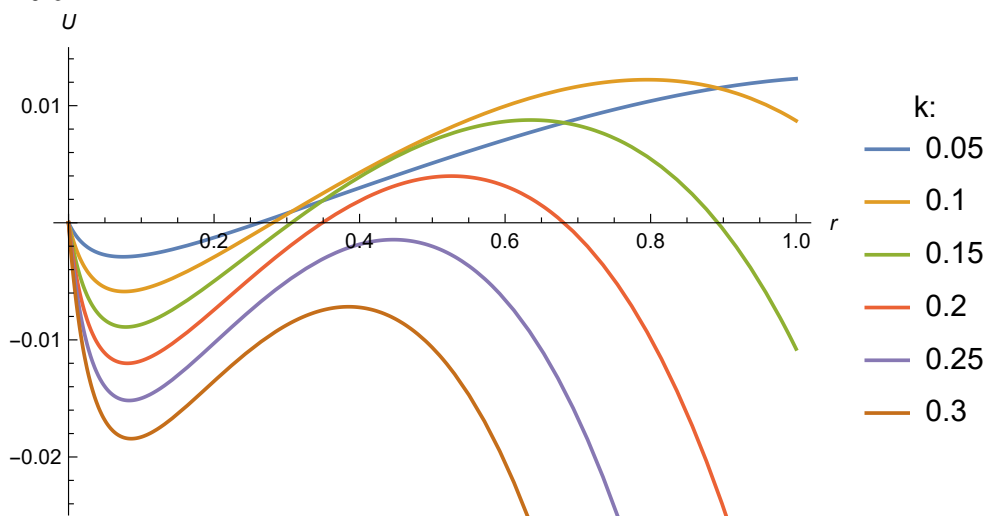


Figure 3: Payoff functions $U_s^P(r)$ for different capacities k when the magnitude of the shock is low: $a = 0.2$.

First we see that for small relative capacity k/a , it is optimal to buy and sell in full at each opportunity. It is easy to understand: with a small capacity, the storage unit cannot wield much market power, so the arbitrage spread is not eroded. Second, from the lowest relative capacity, the maximum of the payoff function U_s^P increases as capacity expands, however

only to a point. Third, concurrently, as relative capacity increases, the optimal proportion r decreases: large storage units use less of their capacity in any single trade. This is apparent from $A(rk)$ and $B(rk)$: the arbitrage term $(A - B)$ rapidly decreases in k .

We also see that very small values of r deliver no surplus at all even though there are no fixed costs; this is apparent from (7): at r close to zero, there is neither arbitrage ($A - B \approx 0$) nor flexibility. These “frictions” can be explained too: when r is very small, the quantities traded keep decreasing rapidly and become negligible in finite time. This nullifies the arbitrage spread $A(rk) - B(rk)$, so the continuation value *after* the first purchase rapidly becomes negligible, but that first purchase is a cost.

Finally, very small shocks cannot sustain the operations of a storage unit, as we see from Figure 3. This is easy to understand: with small shocks ε , the spread can only be small. Any quantity bought or sold by the storage unit erodes it, and a (relatively) large capacity easily nullifies that spread for any choice of r .⁹ This can also be seen from the profit function (7), which is linear in a but quadratic (negative) in k .

There are considerable subtleties to these findings, the discussion of which we postpone until after presenting our second heuristic.

3.2.2 Heuristic 2: constant quantities

Here the storage operator buys or sells a constant quantity X each period, starting from empty as well. To avoid having to deal with partial fills at the boundaries, we let $X := k/m$, so $m \in \mathbb{N}$ is the number of steps to move from empty to full. In the set of admissible strategies, restricting m to be an integer may not be fully optimal, but we expect the corresponding loss to be small – if it exists.¹⁰

The constraints at 0 and at k matter even more here than in the proportional case, where the storage unit can never be completely full nor completely empty in finite time. Here it becomes completely empty or completely full with positive probability. This induces rich dynamics that we label “waves”. Handling these waves is the main challenge in this otherwise

⁹Even though the relative capacity k/a is almost constant for each choice k across Figures 2 and 3.

¹⁰This is actually duly verified in Section 3.2.4.

simple environment. First let

$$B\left(\frac{k}{m}\right) = \frac{1 - a + k/m}{n + 1} \cdot \frac{k}{m}, \quad A\left(\frac{k}{m}\right) = \frac{1 + a - \delta k/m}{n + 1} \cdot \delta \frac{k}{m}.$$

It is worth investing a little time to describe the stochastic process under this heuristic.

Consider a standard binomial tree representing the state space as drawn in Figure 4.

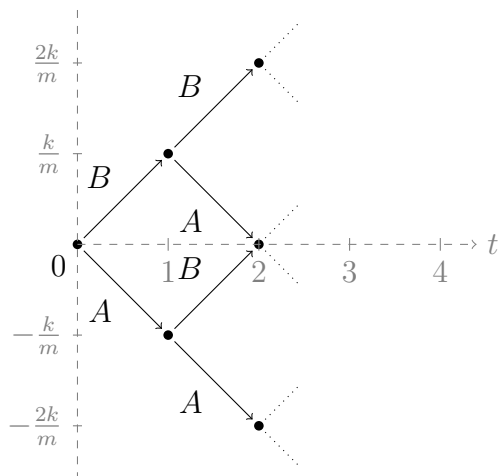


Figure 4: No boundaries. A and B available in any point.

Because an empty unit cannot sell (aka, no short-selling), starting from zero, truncate this binomial tree from below – this leaves the top half of the tree, as in Figure 5. In this Figure, the light gray area is the region where the probability weights that cannot go down start going up instead. This changes the probabilities of reaching any node; for example, the point with coordinates $(1, 0)$ can be reached from the preceding node $(0, 0)$, whereas in the unconstrained binomial tree (Figure 4), it can *never* be reached. Likewise for the point $(2, 1)$, which can be reached from $(1, 0)$ in the truncated tree but never in the unconstrained tree. In turn this affects the probability of reaching $(3, 1)$, which is accessible in both cases. The states with affected probabilities are marked with a thicker dot.

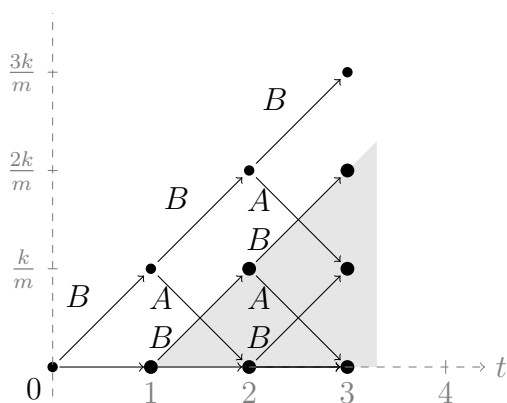


Figure 5: One bound – 0. There is no A in state $c = 0$.

A full unit cannot buy. So truncate this tree further from above at the capacity level k . This is depicted in Figure 6. The admissible state space is limited to that tunnel. Hitherto unreachable states can be reached, and the probability mass on already reachable states can be changed. In Figure 6 the darker gray area represents a second region, in which the upper bound k becomes active and forces the probability mass back down; hence the waves. This process continues on over the infinite horizon, and the successive reflections at the boundaries perpetuate these waves.

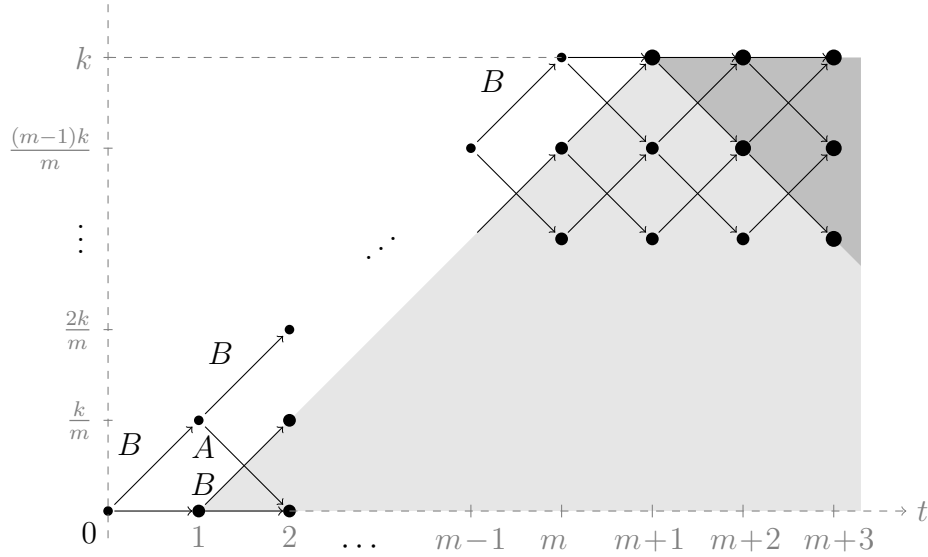


Figure 6: Two bounds – 0 and k . Here there are three types of thickness of points (states), depending on how many waves affect the corresponding probability (here, 0, 1, or 2). There is no A in state $c = 0$ and no B in state $c = k$.

These waves are periodic, which suggests a recursive structure can be uncovered. We are able to exploit this and compute the value function of the storage operator for this heuristic too.

Proposition 2. *The overall expected profit of the storage operator is*

$$\begin{aligned}
 U_s^C &= \frac{-B(k/m) + \beta A(k/m)}{2(1-\beta)} - \frac{A(k/m)}{\beta} \sum_{i=1}^{\infty} \left(\frac{\beta}{2}\right)^{2i} \left(C_{2i-1}^{i-1} + \frac{\beta}{2} C_{2i}^i \right) \\
 &+ \frac{1+\beta}{\beta} \left(B \left(\frac{k}{m}\right) \sum_{j=0}^{\infty} \left(\frac{\beta}{2}\right)^{(m+1)(2j+1)} \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^{2i} C_{2i+(m+1)(2j+1)}^i \right. \\
 &\left. - A \left(\frac{k}{m}\right) \sum_{j=1}^{\infty} \left(\frac{\beta}{2}\right)^{2(m+1)j} \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^{2i} C_{2i+2(m+1)j}^i \right). \tag{8}
 \end{aligned}$$

Using the formula

$$\sum_{i=0}^{\infty} \beta^i C_{2i+r}^i = \frac{2^r}{\sqrt{1-4\beta} \left(1 + \sqrt{1-4\beta}\right)^r}$$

from [Graham et al. \(1994\)](#) (p. 203) and introducing the new discounting coefficient

$$\tilde{\beta} = \frac{\beta}{1 + \sqrt{1-\beta^2}},$$

(8) rewrites more compactly as

$$\begin{aligned} U_s^C &= \frac{1}{2(1-\beta)} \left(-B \left(\frac{k}{m} \right) + \tilde{\beta} A \left(\frac{k}{m} \right) \right. \\ &\quad \left. - \frac{2\sqrt{1-\beta^2}}{\beta} \frac{\tilde{\beta}^{m+1}}{1-\tilde{\beta}^{2(m+1)}} \left(-B \left(\frac{k}{m} \right) + \tilde{\beta}^{m+1} A \left(\frac{k}{m} \right) \right) \right). \end{aligned} \quad (9)$$

In (8), the C_n^l terms are binomial coefficients; (9) is a lot easier to understand and can be decomposed in three parts (rearranging some terms). The first part, rewritten $\frac{1}{2(1-\beta)} \left(-B \left(\frac{k}{m} \right) + A \left(\frac{k}{m} \right) \right)$, is the discounted sum of the risk-free arbitrage spread $-B + A$ (again, the mantra “buy low, sell high”), which results from the unconstrained process represented in Figure 4. Now subtract the “short-selling” revenue $(1-\tilde{\beta})A$ because of the constraint $0 \leq c$ shown in Figure 5. The last part is the cost of observing the capacity constraint $c \leq k$ – the object of Figure 6 – and all the subsequent repetitions of the constraints $0 \leq c \leq k$ that follow because of the waves.

This last terms embeds what we call the “continuation risk”: the storage operator may keep buying and fail to sell for a long time, which renders the operation unprofitable.¹¹ It takes a different form from (7) as a enters directly; more importantly, so does m (as r enters the last part of (7)). Perhaps counter-intuitively, the constraint $c \leq k$ actually helps: it caps losses from buying “forever” – the term $B(k/m)$ enters positively. This last term also showcases a trade-off in the number of steps m to buy and sell: for large enough m , $-B + \tilde{\beta}^{m+1}A$ turns negative, but then the multiplier in front of that bracket converges to zero. Too few steps are bad, and so are too many; indeed in Figure 7, m is clearly interior—just as the optimal r is interior. The payoff function (9) thus differs from (7) for with proportional bids, the boundaries are *never* reached (once the lower boundary has been exited).

¹¹We further explain and discuss the continuation risk in Section 3.

As with the proportional case, we graph this function using the same parameters as before. The dots on the curves correspond to the actual choices that are possible – a fixed quantity, e.g. 0.5. Let $n = 2$, $\delta = 0.95$, and $\beta = 0.95$. Figure 7 shows high shocks $a = 0.6$ with capacity k moving from 0.15 to 1.15, and Figure 8 depicts low shocks $a = 0.2$ where k changes from 0.05 to 0.35.

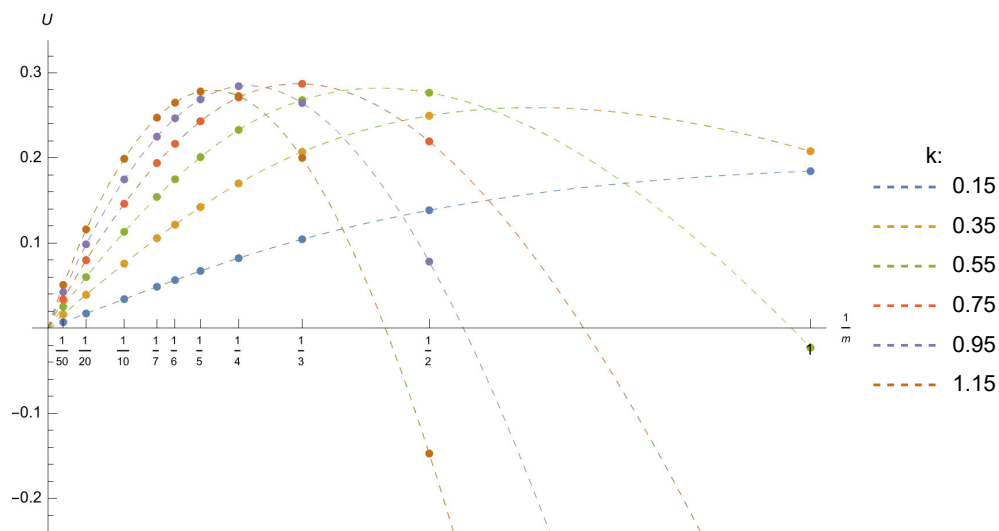


Figure 7: Payoff functions $U_s^C(r)$ for different capacities k when the magnitude of the shock is high enough: $a = 0.6$.

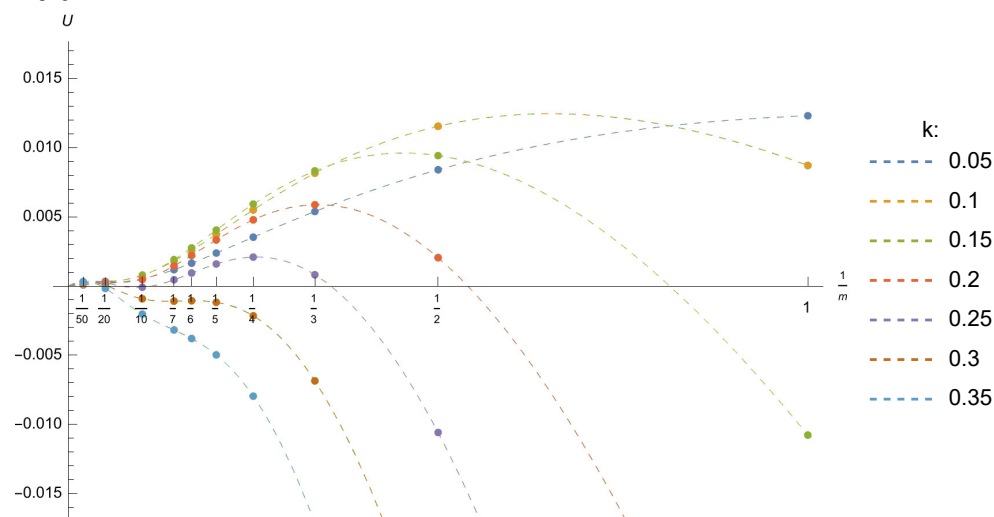


Figure 8: Payoff functions $U_s^C(r)$ for different capacities k when the magnitude of the shock is low: $a = 0.2$.

Overall Figures 7 and 8 complement nicely Figures 2 and 3; they simultaneously enrich them, and confirm the overall message. That is, the optimal quantity choice is interior except for a very small capacity, and rapidly much less than k as capacity increases. In turn the optimal capacity choice is also interior so as to not wipe out the arbitrage spread with the continuation

risk. Finally, too small a shock induces too small a spread, which cannot sustain storage.¹²

3.2.3 The main point

Up to some details, the two heuristics we study lead to the same conclusions. Fix a capacity, the storage operator is a monopolist on its arbitrage opportunities. Its profit function is concave in the variable of interest (be it r or m) and it internalises the well-known trade-off between the extensive margin and the intensive margin. This is the usual market power effect. Note now for the next point that the quantities traded in equilibrium are approximately the same for the pairs (k, r) and (k, m) – see Section A.1 of the Appendix.¹³

Capacity also entails a trade-off – even if not quite a choice variable but rather a parameter in this model. With a small capacity, the storage operator moves rapidly from one boundary to another (e.g $m = 1$), where she gets stuck with positive probability in a sequence of unfavourable shocks. For example, a sequence of negative shocks when already full ($c = k$). A larger capacity allows the storage unit to delay hitting (or approaching, for $r < 1$) a boundary; it provides the flexibility necessary to fully exploit all trading opportunities – see Figure 6. So, starting from a low level, increasing capacity increases payoffs. But this is not monotone; too large a capacity is costly too. Facing a string of negative shocks, a large-scale unit keeps purchasing and keeps delaying selling. This delay compounds, and does so all the more with a large capacity. To see this, consider the following deterministic example: buy and sell in one step, which yields the knife-edge payoff $-B(k_1) + \beta A(k_1) = 0$, and in two steps, yielding $-B(k_2/2) - \beta B(k_2/2) + \beta^2 A(k_2/2) + \beta^3 A(k_2/2) < 0$ (since the actual quantities are constant). In other words, there exists a trade off between flexibility to exploit trading opportunities – for example, $m = 5$, $m = 6$ in Figures 6 and 7 – and the compounding cost of delay in selling. We call this phenomenon the “continuation risk”; it is solved by an interior optimal capacity.

The notion of *continuation risk* is important and our labeling of it careful. For a small capacity, there is too little continuation trade when stuck at the boundary; for a large capacity, there is too much continuation (buying). The cost of too much continuation is rooted in the geometry of the constrained Pascal triangle shown in Figure 6: it is asymmetric as a storage

¹²In the context of electricity, there is an implicit relationship between m and storage duration: when m is large, storage can (dis)charge for longer. But (i) this also implies a lower power output and (ii) it is determined optimally by the operator rather than being a technical characteristic.

¹³We compute the quantities corresponding to each m (and r); they increase slightly for the proportional heuristic, but are almost constant for the second one.

buys up to X as soon as it can and that step is large. Then either it sells that quantity or buys $k - X$ and thereby exhausts its capacity; here that step is small. The same thing works in the other direction: when the storage unit only holds $k - X$ and faces one more consecutive positive shock, it can sell only the remaining capacity $k - X$ to become completely empty.

A finer structure allowing for more steps has $k/(m + 1) < X < k/m$, $m \geq 1$, and we immediately see from Figure 9 that for any $m \geq 1$, only the last step in either direction may be curtailed – just as for two steps. We are able to obtain expressions for the storage payoffs for $k/3 < X < k/2$, $k/4 < X < k/3$, and $k/5 < X < k/4$. Analytical results for smaller X (that is, $m \geq 5$) are out of reach because of the expanding size of the transition matrix.

Proposition 3. *Assume the storage unit sets constant bids X ($k/(m + 1) < X < k/m$) when it buys under any state different from mX or when it sells under any state different from $k - mX$. Otherwise, the storage unit bids $k - mX$ when it either buys under state mX or sells under state $k - mX$. The expected profit of storage reads:*

1. For $1/2 < X < 1$ ($m = 1$):

$$U_s^X = -\frac{1}{2} \left(B(X) + \frac{\beta}{2 - \beta} B(k - X) \right) + \frac{\beta}{4} \left(D(X) + \frac{\beta^2}{4 - \beta^2} D(k - X) \right); \quad (10)$$

2. For $1/3 < X < 1/2$ ($m = 2$):

$$\begin{aligned} U_s^X &= -\frac{1}{2 - \beta} \left(B(X) + \frac{\beta^2}{2(2 - \beta^2)} B(k - 2X) \right) \\ &+ \frac{\beta}{4 - \beta^2} \left(D(X) + \frac{\beta^4}{8(2 - \beta^2)} D(k - 2X) \right); \end{aligned} \quad (11)$$

3. For $1/4 < X < 1/3$ ($m = 3$):

$$\begin{aligned} U_s^X &= -\frac{1}{2(2 - \beta^2)} \left((2 + \beta)B(X) + \frac{\beta^3}{4 - 2\beta - \beta^2} B(k - 3X) \right) \\ &+ \frac{\beta}{8(2 - \beta^2)} \left((4 - \beta^2)D(X) + \frac{\beta^6}{(4 - \beta^2)^2 - 4\beta^2} D(k - 3X) \right); \end{aligned} \quad (12)$$

4. For $1/5 < X < 1/4$ ($m = 4$):

$$\begin{aligned}
U_s^X &= -\frac{1}{4 - 2\beta - \beta^2} \left(2B(X) + \frac{\beta^4}{2(4 - 3\beta^2)} B(k - 4X) \right) \\
&+ \frac{\beta}{(4 - \beta^2)^2 - 4\beta^2} \left(2(2 - \beta^2)D(X) + \frac{\beta^8}{4(4 - \beta^2)(4 - 3\beta^2)} D(k - 4X) \right);
\end{aligned} \tag{13}$$

where $D(X)$ is the NPV of the arbitrage spread:

$$D(X) = \frac{A(X) - B(X)}{1 - \beta}.$$

Echoing the introductory example, the first term in all these formulae is the initial purchase cost, and the second one is the arbitrage spread. Both are suitably discounted.

3.2.5 Discussion: comparing heuristics, capacity constraints and demand specification

Comparing heuristics. In Figures 10 and 11 we directly compare the performance of the heuristics we study and of the more flexible approach (still restricted to $1/5 < X < 1$) of the preceding section. Red represents proportional bids, blue stands for constant bids with integer m , and purple shows the payoffs from bids for any $X \in (k/5, k)$. The graphs of these payoff functions feature kinks at each of the integer m because altering m amounts to altering the regime under which the storage unit operates. The first series is concerned with relatively large shocks ($a = 0.6$), and the second one with small shocks ($a = 0.2$).

First, it is difficult to globally rank the two main heuristics. While constant quantities do not systematically dominate proportional bidding, the maximum always exceeds – at least weakly – the maximum achieved under the proportional rule, but this may be sensitive to the parameter values we use. We can also see that as capacity k increases, the proportional-bid heuristic performs better. That is, the proportional heuristic better internalises the *continuation risk* – without completely negating it. This improvement stems from the more flexible nature of this heuristic, which (i) never reaches the boundaries for any $r < 1$, (ii) commits the storage operator to trade ever smaller quantities (in a sequence of identical shocks) and (iii) allows for the event tree of the binomial process to grow without bounds. Further, for r large enough, as soon as reversal does occur, the quantity traded at the first favourable

shock, is large. This is an approach that is both more prudent and more flexible and performs comparatively better when capacity is large—that is, precisely when the *continuation risk* is more acute.

Indeed, for large shocks and large capacity (the four bottom panels of Figure 10), the maximiser under the proportional rule lies to the right of the maximiser under constant bids, and the constant-bid heuristic clearly dominates for (many) small steps. Second, under constant quantities, a storage unit is immune to the (small) losses that accrue under the proportional heuristic when the quantities are very small. Under constant bids, the quantities never become vanishingly small, so the arbitrage revenue is never negligible. Finally we can see that large shocks are completely essential to profitable trading, as seen in Figure 11.

Of course these differences in behaviour stem from the difference in the stochastic process that is induced by the choice of heuristic. That is, the heuristics interact with the exogenous stochastic process to define an *endogenous* process; this is what defines a stochastic game Shapley (1953). Under the proportional heuristic, the capacity constraint induces an asymptotic path but the binomial tree itself is never truncated. This is what makes it more flexible than the constant-quantity approach, which modifies the binomial tree outright and generates reflections – see Figures 4 to 6.

From Figure 10 we see that, except for very small capacity k (in which case $m = 1$ is optimal), relaxing the integer constraint and allowing $X \in (k/5, k)$ (in purple) delivers a modest improvement compared to $X \in \{k/m, m \in \mathbb{N}\}$. For the majority of cases it finds a new optimum that beats the other two heuristics – for example, $2 < m < 3$ for $k = 0.65$. We also remark that this more flexible heuristic approximately behaves like some kind of combination of the constant bid and proportional bid approaches. This robustness check gives us comfort in thinking that the heuristics we study operate “not far” from the strictly optimal strategy.

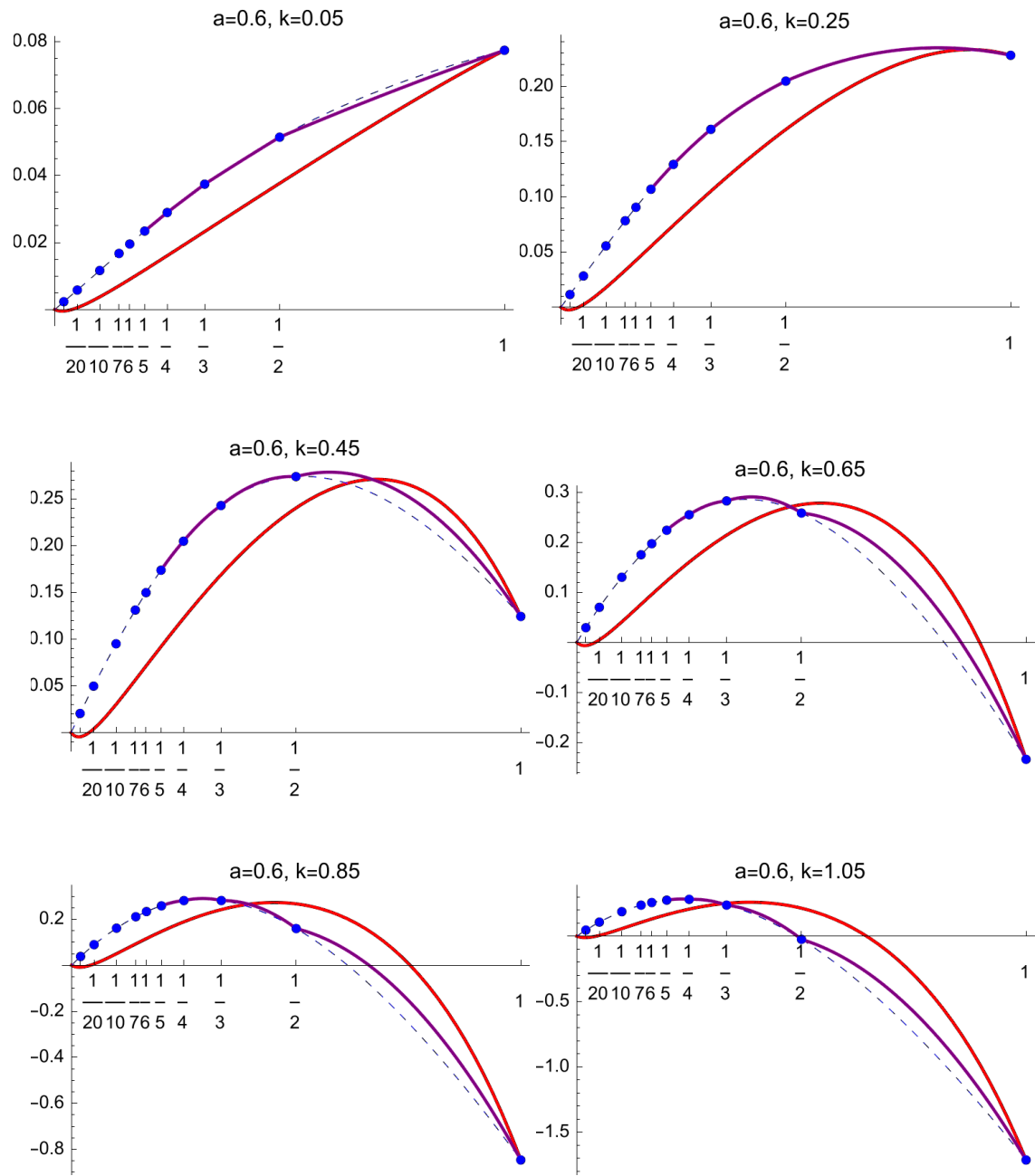


Figure 10: Both linear and constant bids payoffs in the same graphs for different k and $a = 0.6$.

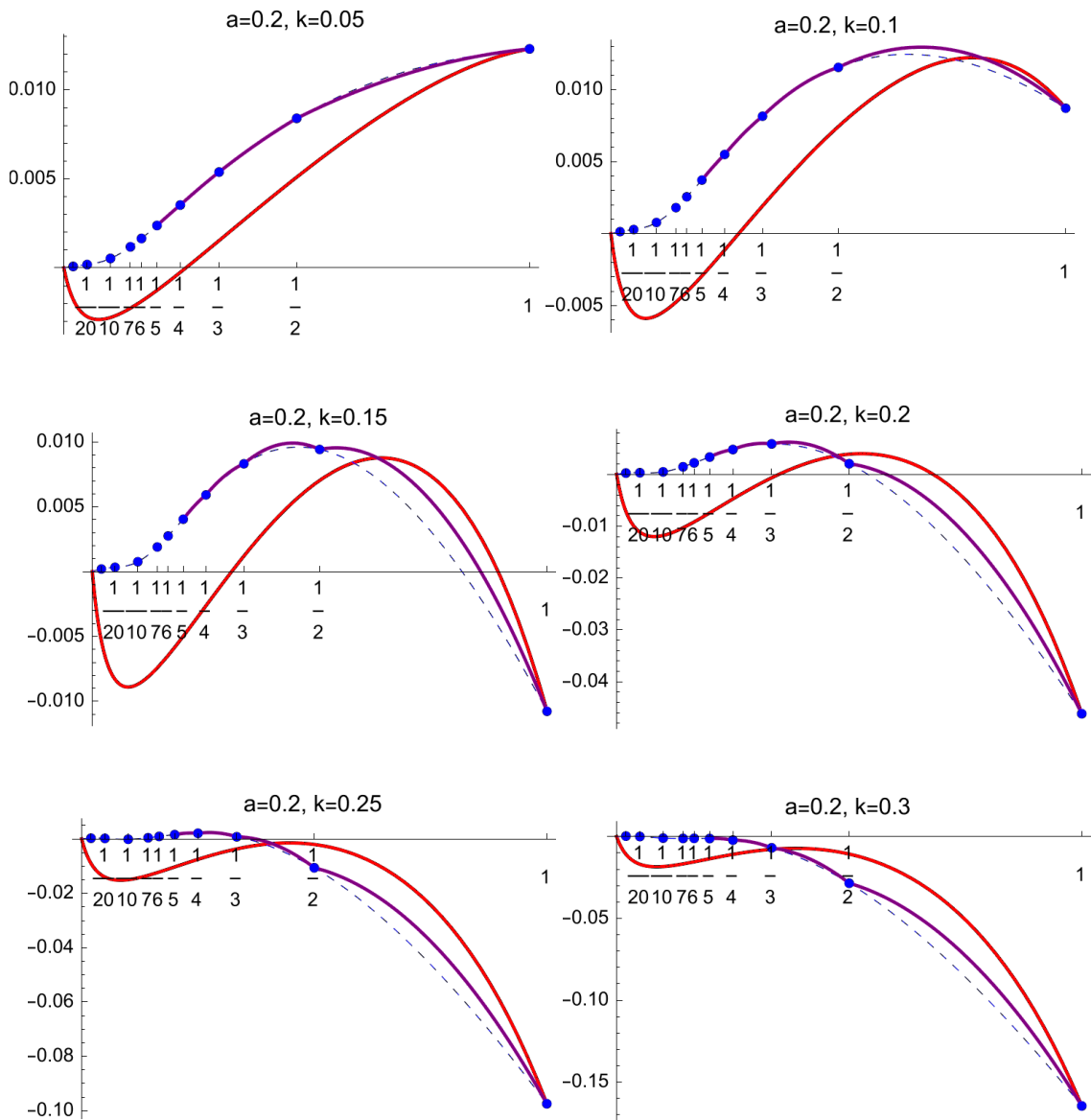


Figure 11: Both linear and constant bids payoffs in the same graphs for different k and $a = 0.2$.

Capacity constraints. In the model we present, producers are unconstrained in their ability to supply; in consequence they always supply according to their best response and clearing prices are standard “Cournot prices” with bounded markups that reflect the relative competitiveness of the market. In some markets, such as electricity, binding capacity constraints are a major concern – for then aggregate supply may not meet demand – and they are reflected in widely fluctuating prices. Large price differences invite arbitrage.

In an extension (available upon request) we study this problem when a fraction of producers are also constrained by a finite capacity. As that fraction increases, the market power of the unconstrained producers increases, and so do their payoffs. From the expressions for the

quantities $A(rk)$, $B(rk)$, $A(k/m)$, and $B(k/m)$, it is immediate that the payoffs to the storage unit also increase. There is more capacity withholding; r decreases and m increases.

Demand specification. Both our demand specification and the structure of the demand shocks are stark (and simple). Altering the slope of demand, rather than the intercept, has predictable effects: a (uniformly) less elastic demand induces both larger mark ups and more withholding, and conversely for a more elastic demand. Equivalently, the optimal fraction r decreases and the optimal number of steps m increases. The impact on the continuation risk is ambiguous: a lower r and higher m imply more flexibility, which decreases the continuation risk. But a less elastic demand implies higher prices, including higher purchase prices, which increase the cost of inaction at (or near) a boundary.

3.3 A richer Markovian structure

The payoff functions we can compute in Section 3.2 feature the continuation risk, that is, a cost of uncertainty – see equations (7) and (8). This cost stems for the risk of facing a sequence of shocks in the same direction, which eventually cripples storage. A glance at the payoff functions (7) and (8) suggests that a sequence of perfectly negatively correlated shocks $-a, a, -a, a, \dots$ would deliver the highest (and certain) payoff.

In this Section we relax the strict independence assumption and investigate the constant-quantity heuristics when shocks follow a non-degenerate Markov chain, the persistence of which can vary. The goal is to better understand the impact of uncertainty on the behavior of the storage operator, and the salience of the continuation risk. To this end, we let storage buy and sell in any of one step ($m = 1$), two steps ($m = 2$) or three steps ($m = 3$); richer heuristics, with $m > 3$ are no less interesting but beyond what we can manage. With $m = 2$, for example, there are four possible states that are payoff relevant: an empty unit, a half-full unit after the negative shock, a half-full unit after the positive shock, and finally a full storage unit. There is no need to distinguish the nature of the shock at the boundaries because states 0 and k are accessible only after positive and negative shocks, respectively. Assume now that

shocks ε_t form a discrete-time Markov chain:

$$\begin{aligned}
Pr\{\varepsilon_0 = a\} &= x, & Pr\{\varepsilon_0 = -a\} &= 1 - x, \\
Pr\{\varepsilon_{t+1} = a|\varepsilon_t = a\} &= x, & Pr\{\varepsilon_{t+1} = -a|\varepsilon_t = a\} &= 1 - x, \\
Pr\{\varepsilon_{t+1} = a|\varepsilon_t = -a\} &= 1 - y, & Pr\{\varepsilon_{t+1} = -a|\varepsilon_t = -a\} &= y
\end{aligned} \tag{14}$$

for any $t \geq 0$. We denote the transition matrix by Q and its determinant by d :

$$Q = \begin{pmatrix} x & 1-x \\ 1-y & y \end{pmatrix}, \quad d = \det Q = x + y - 1,$$

and we let the functions $A(k/m)$ and $B(k/m)$ be defined as before.

Proposition 4. *Under conditions laid out below, there exists a dynamic equilibrium, such that*

- *the storage unit buys k/m under the negative shock until it reaches capacity k and sells $\delta k/m$ under the positive shock until it becomes empty;*
- *in each period, the producers set quantities q^* according to static Cournot competition and based on the current shock and the state of the storage (full or empty). Namely,*

$$\begin{aligned}
q^* &= \frac{1-a}{n+1} \quad \text{if storage is full and } \varepsilon = -a, \\
q^* &= \frac{1-a+k/m}{n+1} \quad \text{if storage is not full and } \varepsilon = -a, \\
q^* &= \frac{1+a}{n+1} \quad \text{if storage is empty and } \varepsilon = a, \\
q^* &= \frac{1+a-\delta k/m}{n+1} \quad \text{if storage is not empty and } \varepsilon = a.
\end{aligned}$$

1. $m = 1$; this equilibrium exists if

$$B < \frac{\beta(1-y)}{1-\beta y} A. \tag{15}$$

and the expected payoff of the storage operator U_s^1 takes the following form:

$$U_s^1 = \frac{1-x}{(1-\beta)(1-\beta d)} (-B + \beta(1-y)A + \beta y B).$$

2. $m = 2$; this equilibrium exists if

$$B \left(\frac{k}{2} \right) < \frac{\beta(1-y)(1+\beta^2 d)}{1-\beta^2(1-x+yd)} A \left(\frac{k}{2} \right), \quad (16)$$

and the expected payoff of the storage operator U_s^2 takes the following form:

$$U_s^2 = \frac{1-x}{(1-\beta)(1-\beta d)} \left(-B \left(\frac{k}{2} \right) + \beta(1-y)A \left(\frac{k}{2} \right) + \beta^2 y \frac{yB \left(\frac{k}{2} \right) + \beta x(1-y)A \left(\frac{k}{2} \right)}{1-\beta^2(1-x)(1-y)} \right).$$

3. $m = 3$; this equilibrium exists if

$$B \left(\frac{k}{3} \right) < \beta(1-y) \frac{(1-\beta^2(1-x)(1-y))^2 + \beta^2 xy(1+\beta^2(xy-2(1-x)(1-y)))}{(1-\beta^2(1-x)(1-y))^2 - \beta^3 y^3 - \beta^4 xy(1-x)(1-y)} A \left(\frac{k}{3} \right), \quad (17)$$

and the expected payoff of the storage operator U_s^3 reads

$$U_s^3 = \frac{1-x}{(1-\beta)(1-\beta d)} \left[-B \left(\frac{k}{3} \right) + \beta(1-y)A \left(\frac{k}{3} \right) + \beta^3 y \frac{y^2 B \left(\frac{k}{3} \right) + x(1-y)(1+\beta^2 d)A \left(\frac{k}{3} \right)}{(1-\beta^2(1-x)(1-y))^2 - \beta^4 xy(1-x)(1-y)} \right].$$

When there is little persistence, the probability of being stuck at either boundary 0 or k is small; that is, the storage operator is almost guaranteed to reverse direction – for example, to sell after buying. Compared to Section 3.2.2, the “waves” of Figure 6 unfold faster. In turn, this stokes the incentives to buy in the first place, and so on. These rapid cycles reduce the *uncertainty*, but not the *volatility*; in fact, certain volatility is best for the storage operator.

Low persistence effectively negates the continuation risk, as can be seen from the payoffs U_s^m , $m = 1, 2, 3$, taking y and x as small. Indeed, the payoffs U_s^m , $m = 1, 2, 3$ include two terms: the first one is $-B(k/m) + \beta(1-y)A(k/m)$, which is, modulo a multiplier, the discounted payoff from buying and selling every other period. This is a storage unit operating under perfect foresight. The second term captures the cost of uncertainty, as best-responded

to by the storage operator. That cost is negligible if x and y are small.

In Figures 12 and 13 we plot the payoff functions of the storage operator projected on the dimensions x and y , which denote persistence, and with $x = y$. The red stands for the payoff function when $m = 1$, the blue for $m = 2$ and the green for $m = 3$, all for the constant-quantity heuristic. All other parameters remain as in the other plots. Low persistence is clearly better in this environment, but we note that in some cases a very high persistence seems to improve payoffs over a moderate persistence. With higher persistence there are fewer cycles, each of which induces some losses; high persistence delays the onset of each these cycles.

When capacity is relatively small (compared to the shock), it is best to buy and sell in full ($m = 1$) for almost any persistence. The reason is that the storage operator has no (significant) market power, so there is no (significant) price impact. But when capacity increases, we observe more mixed results. First, as persistence increases, flexibility becomes valuable: buying and selling in two steps and three steps starts dominating. It is better able to cope with the probability of shocks repeating themselves. Second, with a large(r) capacity, restraint also pays off: buying and selling in two steps (blue) dominates one step (red) for any persistence; and three steps (green) dominates both. It is best to not use the capacity in full at any point in time because of the market power effect, and this is exactly what $m = 3$ delivers. These conclusions are replicated, but even starker, when shocks and capacity are even smaller. Then, in some cases, the three-step strategy is the only one that can deliver any positive surplus.

With this simple structure it is difficult to speak of the impact of high persistence in greater detail. With a lot of persistence in shock ($x \rightarrow 1$ or $y \rightarrow 1$), the storage operator can spend a lot of time at either boundary (0 or k); if that is the case, one can conjecture she would like to buy or sell over many periods (so, m be large). We cannot handle this case, and aside from a strict numerical treatment, there is no hope of doing so because even a computer cannot find the eigenvalues of the matrices of interest.

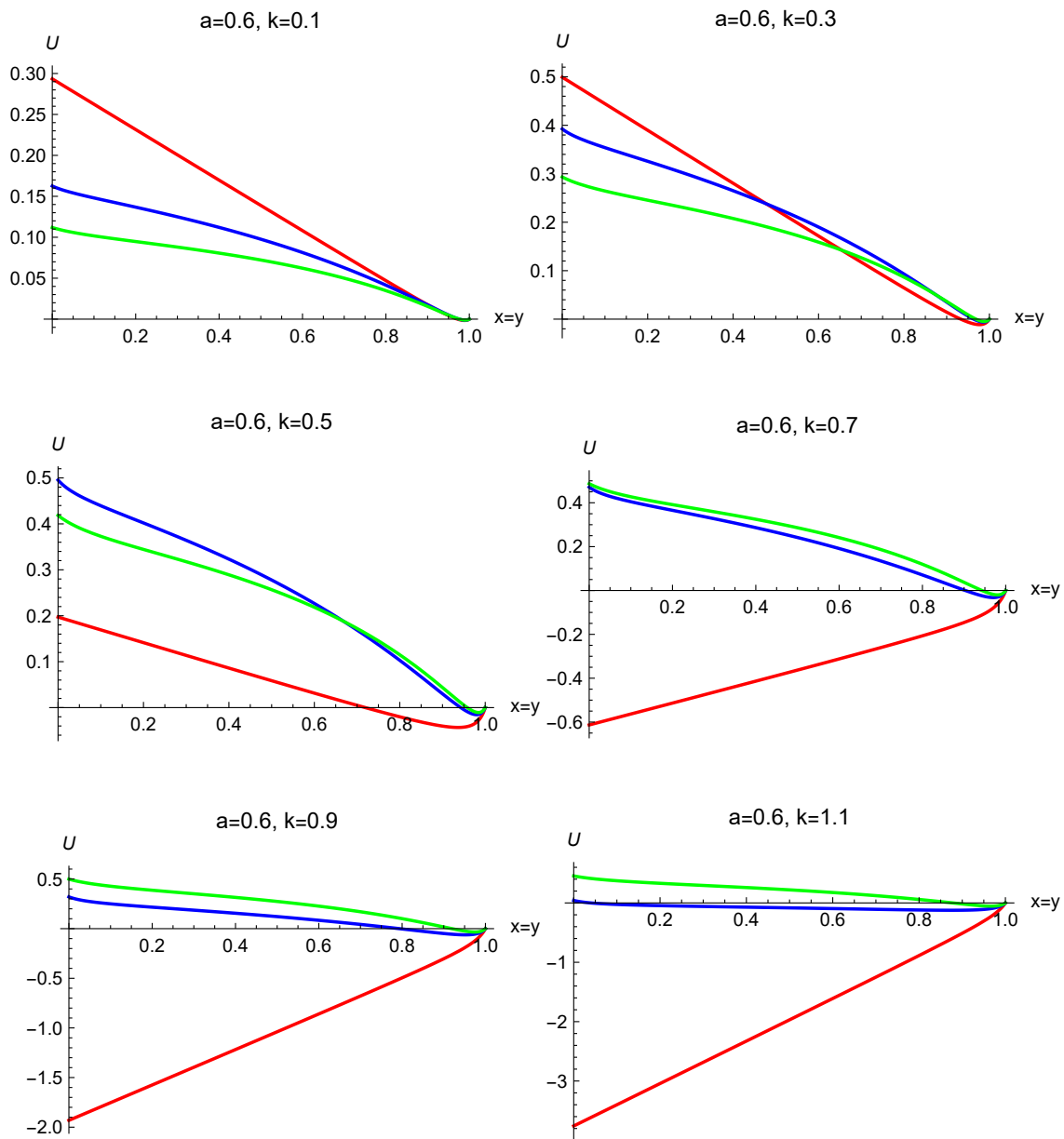


Figure 12: Payoffs under symmetric Markov shocks for divisible (green for $k/3$ and blue for $k/2$) and indivisible (red) capacities in the same graphs for different k and $a = 0.6$.

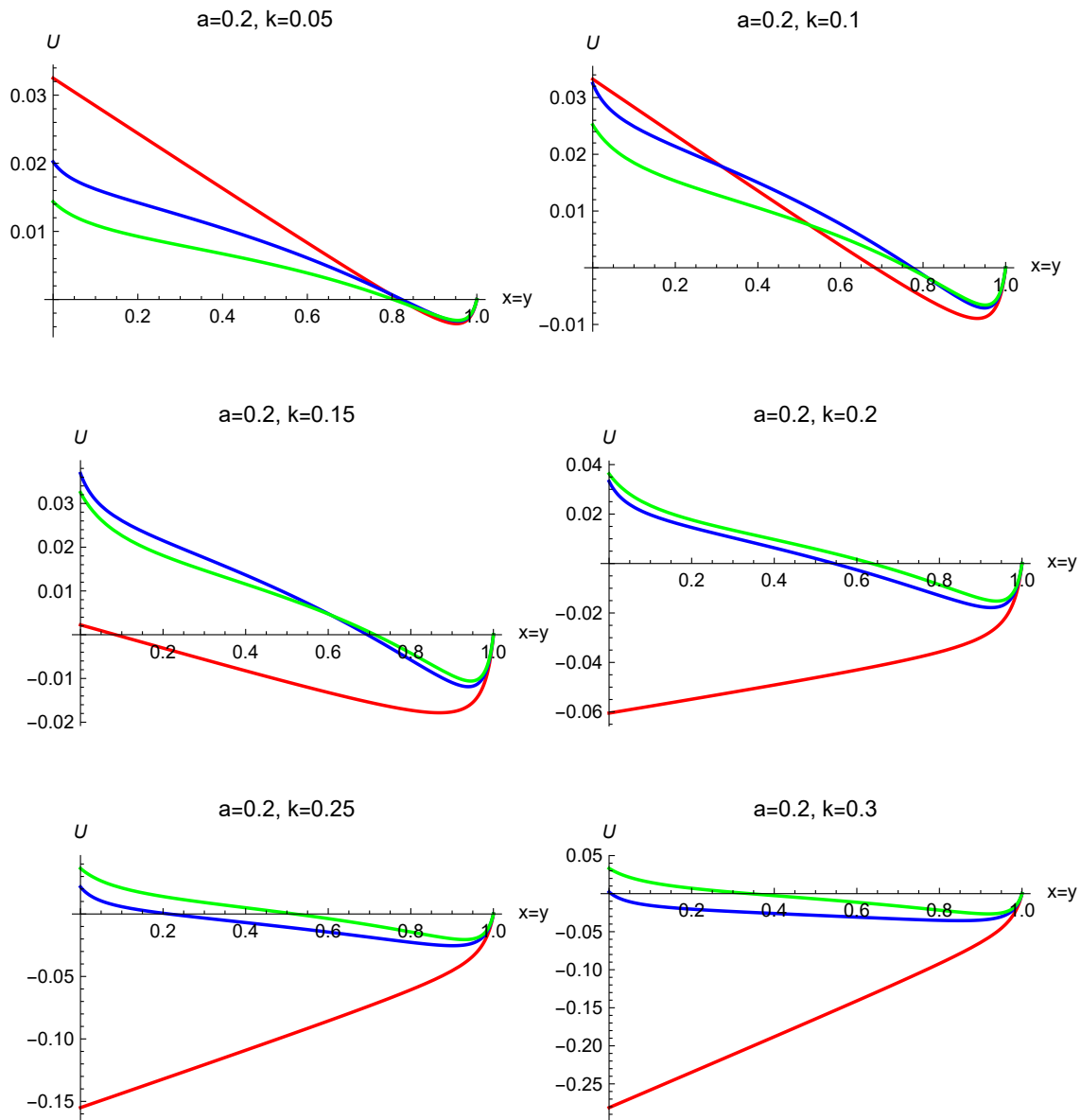


Figure 13: Payoffs under symmetric Markov shocks for divisible (green for $k/3$ and blue for $k/2$) and indivisible (red) capacities in the same graphs for different k and $a = 0.2$.

3.4 Implications for competition policy and market design

Our results reveal a strategic storage operator withholds quantities for two essential reasons. One, they want to exercise their market power; second, they seek to actively manage the continuation risk, for which they need flexibility in their trading decisions. Preserving that flexibility implies withholding as well.

While our model features a single storage operator, one can speculate that on both accounts it is socially best for the storage units to remain small. It is tautological but nonetheless useful to recall that it is easier to mitigate the exercise of market power if storage has no market

power. A small unit (compared to the magnitude of the shocks) uses its full capacity every time it trades. The reason is that the arbitrage gains dominate the continuation risk. Whether any of this is implementable in practice depends in part on the exact storage technology. Economies of scale favour a large size, but most storage operations need space rather than costly investment, which suggests constant returns to scale. In electricity, battery capacity as supplied by Tesla or Siemens, increases linearly in the number of “packs”. This accords with constant returns to scale.

On a different register, [Andres-Cerezo and Fabra \(2023b\)](#) show storage and conventional electricity generation should not be integrated, for integration facilitates coordination and so enhances market power. This happens to be true of any commodity. The corollary to [Andres-Cerezo and Fabra \(2023b\)](#) is that some qualified integration of storage and production, for selected production technologies (e.g. solar electricity), may facilitate the emergence and operation of merchant storage. This is particularly pertinent in electricity markets. The reason is that storage is a substitute for thermal generation (at critical times) but it is (typically) a complement for renewable generation.¹⁵ This is not a consideration in [Butters et al. \(Working Paper\)](#), where storage is competitive by assumption. The same logic applies to cyclical commodities such as crops or to extractive industries subject to weather disruptions, but not market making in securities. There an issuer, or wholesaler, may sway too much market power if also acting as a market maker.

In centralised markets like electricity, there may also be a role for a market operator to play insuring the storage unit against the continuation risk—being stuck either full or empty. Recall that the continuation risk contributes to quantity withholding: flexibility in trading requires the capacity k to be divided in many steps m (equivalently, r to be small), hence the traded quantity k/m (or rk) is small. All things otherwise equal, a welfare-maximizing operator would like to see m decrease (r increase). To do so, the operator could tax transactions and disburse these premia to encourage a full unit to sell at a low price (when the shock is negative), or an empty one to buy at a high price (conversely). To be clear, the incentives of the market operator are not to alter prices in a particular period or a particular state, but to provide systematic insurance against the continuation risk. In response, the storage operator can behave more aggressively every period. We note this counters the insurance

¹⁵See [Andres-Cerezo and Fabra \(2023a\)](#) for some qualifications.

service implicit in the activities of the storage unit, and increases price volatility. However this is not very different from a reinsurance contract.

Still in the context of centralised trading, we finally ask, but are yet unable to answer, how should a bidding space be defined and a market be cleared when (at least some) sellers use dynamic strategies. Indeed, if storage units are forward-looking, so should be the market operator. This suggests that the myopic clearing of simple quantity offers may not be the optimal mechanism for an operator seeking to minimize procurement costs.

4 Conclusion

In this paper we study a model of dynamic trading based on storage. This is an important step to understand the economics of storage, which have long been understudied, and to tackle the ambitious question of market design with storage. We limit ourselves to a Cournot environment with stochastic shocks that follow a Markov chain, and allow for capacity constraints; there is no change in the mean demand. This environment features market power and strategic behaviour, in departure the rest of the literature on (other forms of) storage. This work applies to any storable commodity, including now electricity.

Even then, the analysis of such a simple problem is very demanding. To make progress, we must confine ourselves to studying simple but reasonable heuristics, which allows us to derive explicit forms for the long-horizon payoffs of the storage unit. For high enough a discount factor, we know these payoffs are close approximations. We uncover two competing forces that a storage operator must balance: market power, which is quite standard, and the continuation risk, which is completely new. The continuation risk embodies the need for flexibility in trading that a large capacity can afford and the expected cost of buying too much energy and delaying sale that a large capacity induces. It is only relevant when storage has market power and can be construed as a corollary of that market power. The full implications of these behaviours on market design are yet to be understood.

There is still a tremendous amount of work to do to really understand the economics of storage. In this model there is no change in the mean demand over time. This is a central feature of most markets, and especially of electricity markets, but it is difficult to incorporate in a model of dynamic trading. There is also no competition between storage operators, and

therefore no opportunity to collude either. With some simplifications, this model can be adapted to allow for oligopolistic storage units. We tackle this problem in a separate paper.

A Appendix – for online publication

Proof of Proposition 1. Since under proportional bids, we never reach upper limit k and never reach lower limit 0 again after starting there, the system of equations (6) takes the following form:

$$\begin{cases} V(0) &= \frac{1}{2-\beta} \left(-\frac{1-a+rk}{n+1} \cdot rk + \beta V(rk) \right), \\ V(c) &= \frac{1}{2} \left(-\frac{1-a+r(k-c)}{n+1} \cdot r(k-c) + \beta V(c+r(k-c)) \right) \\ &+ \frac{1}{2} \left(\frac{1+a-\delta rc}{n+1} \cdot \delta rc + \beta V((1-r)c) \right). \end{cases}$$

We need to find $V(0) = U_s^P$. Let's enumerate all c , $b(c)$, and $a(c)$ in order of their appearance when we expand our equation for $V(0)$. Namely, in the first period we have

$$V(0) = \frac{1}{2}\beta V(0) + \frac{1}{2} \left(-\frac{1-a+rk}{n+1} \cdot rk + \beta V(rk) \right) = -\frac{1}{2}B(rk) + \frac{\beta}{2} (V(0) + V(rk)),$$

so we define $c^0 = 0$, $b^0 = r(k - c^0) = rk$, $c^1 = c^0 + b^0 = rk$. In the second round, we get

$$\begin{aligned} V(0) &= -\frac{1}{2}B(rk) + \frac{\beta}{2} \left(\frac{1}{2}\beta V(0) + \frac{1}{2} (-B(rk) + \beta V(rk)) \right. \\ &\quad \left. + \frac{1}{2} (-B(r(k-rk)) + \beta V(rk + r(k-rk))) + \frac{1}{2} (A(r \cdot rk) + \beta V((1-r)rk)) \right) \\ &= -\frac{1}{2}B(rk) + \frac{\beta}{4} \left(-B(rk) - B(r(1-r)k) + A(r^2k) \right) \\ &\quad + \frac{\beta^2}{4} \left(V(0) + V(rk) + V(r(2-r)k) + V(r(1-r)k) \right). \end{aligned}$$

Thus,

$$\begin{aligned} b^1 &= r(k - c^1) = r(1-r)k, & a^1 &= rc^1 = r^2k, \\ c^2 &= c^1 + b^1 = r(2-r)k, & c^3 &= c^1 - a^1 = r(1-r)k. \end{aligned}$$

In the third round, we get

$$V(0) = -\frac{1}{2}B(b^0) + \frac{\beta}{4}(-B(b^0) - B(b^1) + A(a^1)) \\ + \frac{\beta^2}{8}(-B(b^0) - B(b^1) - B(b^2) - B(b^3) + A(a^1) + A(a^2) + A(a^3)) + \frac{\beta^3}{8} \sum_{i=0}^7 V(c^i),$$

where

$$\begin{aligned} b^2 &= r(k - c^2) = r(1 - 2r - r^2)k, & b^3 &= r(k - c^3) = r(1 - r + r^2)k, \\ a^2 &= rc^2 = r^2(2 - r)k, & a^3 &= rc^3 = r^2(1 - r)k, \\ c^4 &= c^2 + b^2 = r(3 - 3r - r^2)k, & c^5 &= c^3 + b^3 = r(2 - 2r + r^2)k, \\ c^6 &= c^2 - a^2 = r(1 - r)(2 - r)k, & c^7 &= c^3 - a^3 = r(1 - r)^2k. \end{aligned}$$

Finally, in round t ,

$$V(0) = -\frac{1}{2}B(b^0) + \frac{\beta}{4}(-B(b^0) - B(b^1) + A(a^1)) + \frac{\beta^2}{8} \left(-\sum_{i=0}^3 B(b^i) + \sum_{i=1}^3 A(a^i) \right) + \dots \\ + \frac{\beta^{t-1}}{2^t} \left(-\sum_{i=0}^{2^{t-1}-1} B(b^i) + \sum_{i=1}^{2^{t-1}-1} A(a^i) \right) + \frac{\beta^t}{2^t} \sum_{i=0}^{2^t-1} V(c^i),$$

where b^i , a^i , and c^i can be found recursively. Continuing this process infinitely and noticing that

$$\frac{\beta^t}{2^t} \sum_{i=0}^{2^t-1} V(c^i) \leq \beta^t V(\max_i c^i) \xrightarrow{t \rightarrow \infty} 0,$$

we obtain:

$$V(0) \xrightarrow{t \rightarrow \infty} -\frac{1}{2}B(b^0) + \frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{\beta}{2} \right)^j \left(-\sum_{i=0}^{2^j-1} B(b^i) + \sum_{i=1}^{2^j-1} A(a^i) \right). \quad (18)$$

Let

$$G(t) = \sum_{i=0}^{2^t-1} c^i, \quad H(t) = \sum_{i=0}^{2^t-1} (c^i)^2.$$

We can get a recursive equation for $G(t)$:

$$\begin{aligned}
G(t) &= \sum_{i=0}^{2^{t-1}-1} (c^i + b^i) + \sum_{i=1}^{2^{t-1}-1} (c^i - a^i) \\
&= \sum_{i=0}^{2^{t-1}-1} (c^i + r(k - c^i)) + \sum_{i=0}^{2^{t-1}-1} (c^i - rc^i) = rk2^{t-1} + 2(1-r)G(t-1),
\end{aligned}$$

which implies

$$\begin{aligned}
G(t) &= rk2^{t-1} + 2(1-r)(rk2^{t-2} + 2(1-r)G(t-2)) = \dots \\
&= rk \sum_{i=0}^{t-1} 2^i (1-r)^i 2^{t-1-i} + 2^t (1-r)^t G(0) = k2^{t-1} (1 - (1-r)^t).
\end{aligned} \tag{19}$$

The same technique works for $H(t)$:

$$\begin{aligned}
H(t) &= \sum_{i=0}^{2^{t-1}-1} (c^i + b^i)^2 + \sum_{i=1}^{2^{t-1}-1} (c^i - a^i)^2 = \sum_{i=0}^{2^{t-1}-1} (rk + (1-r)c^i)^2 + \sum_{i=0}^{2^{t-1}-1} ((1-r)c^i)^2 \\
&= r^2 k^2 2^{t-1} + 2r(1-r)kG(t-1) + (1-r)^2 H(t-1) + (1-r)^2 H(t-1) \\
&= rk^2 2^{t-1} (1 - (1-r)^t) + 2(1-r)^2 H(t-1),
\end{aligned}$$

and

$$\begin{aligned}
H(t) &= rk^2 2^{t-1} (1 - (1-r)^t) + 2(1-r)^2 (rk^2 2^{t-2} (1 - (1-r)^{t-1}) + 2(1-r)^2 H(t-2)) \\
&= \dots = rk^2 \sum_{i=0}^{t-1} 2^i (1-r)^{2i} 2^{t-i-1} (1 - (1-r)^{t-i}) + 2^t (1-r)^{2t} H(0) \\
&= k^2 2^{t-1} \frac{(1 - (1-r)^t) (1 - (1-r)^{t+1})}{2-r}.
\end{aligned} \tag{20}$$

From (19), we can find expressions for sums of b^i and a^i :

$$\begin{aligned}
\sum_{i=0}^{2^t-1} b^i &= \sum_{i=0}^{2^t-1} r(k - c^i) = rk2^t - rG(t) = rk2^{t-1} (1 + (1-r)^t), \\
\sum_{i=0}^{2^t-1} a^i &= \sum_{i=0}^{2^t-1} rc^i = rG(t) = rk2^{t-1} (1 - (1-r)^t).
\end{aligned}$$

From (19) and (20), we can find expressions for sums of squares of b^i and a^i :

$$\begin{aligned}\sum_{i=0}^{2^t-1} (b^i)^2 &= \sum_{i=0}^{2^t-1} r^2(k - c^i)^2 \\ &= r^2k^22^t - 2r^2kG(t) + r^2H(t) = r^2k^22^{t-1} \left(\frac{1 + (1-r)^{2t+1}}{2-r} + (1-r)^t \right), \\ \sum_{i=0}^{2^t-1} (a^i)^2 &= \sum_{i=0}^{2^t-1} r^2(c^i)^2 = r^2H(t) = r^2k^22^{t-1} \left(\frac{1 + (1-r)^{2t+1}}{2-r} - (1-r)^t \right).\end{aligned}$$

Now we are ready to calculate sums of $B(b(i))$ and $A(a(i))$ and get the final formula for $V(0)$. From (18), we have

$$\begin{aligned}V(0) &= -\frac{1}{2}B(rk) + \frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{\beta}{2} \right)^j \left(-\sum_{i=0}^{2^j-1} \frac{1-a+b^i}{n+1} b^i + \sum_{i=1}^{2^j-1} \frac{1+a-\delta a^i}{n+1} \delta a^i \right) \\ &= -\frac{1}{2} \frac{1-a+rk}{n+1} rk + \frac{1}{2(n+1)} \sum_{j=1}^{\infty} \left(\frac{\beta}{2} \right)^j \left(-(1-a)rk2^{j-1} (1+(1-r)^j) \right. \\ &\quad \left. - r^2k^22^{j-1} \left(\frac{1+(1-r)^{2j+1}}{2-r} + (1-r)^j \right) + (1+a)\delta rk2^{j-1} (1-(1-r)^j) \right. \\ &\quad \left. - \delta^2r^2k^22^{j-1} \left(\frac{1+(1-r)^{2j+1}}{2-r} - (1-r)^j \right) \right) \\ &= \frac{rk}{4(n+1)(1-\beta)(1-(1-r)\beta)} \left(r\beta((1+a)\delta - (1-a)) - 2(1-\beta)(1-a+rk) \right. \\ &\quad \left. - rk \frac{\beta r^2(1+\delta^2)}{1-(1-r)^2\beta} \right).\end{aligned}$$

The last expression is exactly formula (7). □

Proof of Proposition 2. The system of equations (6) takes the following form:

$$\left\{ \begin{array}{l} V_t\left(\frac{ik}{m}\right) = \frac{1}{2} \left(A\left(\frac{k}{m}\right) + \beta V_{t+1}\left(\frac{(i-1)k}{m}\right) - B\left(\frac{k}{m}\right) + \beta V_{t+1}\left(\frac{(i+1)k}{m}\right) \right), \\ V_t(0) = \frac{1}{2} \left(\beta V_{t+1}(0) - B\left(\frac{k}{m}\right) + \beta V_{t+1}\left(\frac{k}{m}\right) \right), \\ V_t(k) = \frac{1}{2} \left(A\left(\frac{k}{m}\right) + \beta V_{t+1}\left(\frac{(m-1)k}{m}\right) + \beta V_{t+1}(k) \right) \end{array} \right. \quad (21)$$

for all $1 \leq i \leq m-1$. We are interested in coefficients c_t^i in front of value functions $V_t(ik/m)$

for each particular $t \geq 0$ and $0 \leq i \leq m$, such that

$$V_0(0) = F_{t-1} \left(\beta, A \left(\frac{k}{m} \right), B \left(\frac{k}{m} \right) \right) + \left(\frac{\beta}{2} \right)^t \sum_{i=0}^m c_t^i V_t \left(\frac{ik}{m} \right). \quad (22)$$

Note that $\sum_i c_t^i = 2^t$ for any t .

In each period t , the storage unit buys energy with probability $1/2$. This cannot be done only if the unit has reached its full capacity. Also, in period t the storage unit sells energy with probability $1/2$ if it's not empty. Thus, the overall expected earnings of storage up to period t can be described by the following expression:

$$F_t = \sum_{i=0}^t \left(\frac{\beta}{2} \right)^i \cdot \frac{1}{2} \cdot \left((2^i - c_i^0) A \left(\frac{k}{m} \right) - (2^i - c_i^m) B \left(\frac{k}{m} \right) \right).$$

Indeed, $c_t^0/2^t$ and $c_t^m/2^t$ are the probabilities of the storage to be correspondingly empty or full at t , according to (22).

Since $\sum_i c_t^i = 2^t$, $\beta < 1$, and V_t is a nondecreasing function, the last term in (22) goes to zero if $t \rightarrow \infty$:

$$\left(\frac{\beta}{2} \right)^t \sum_{i=0}^m c_t^i V_t \left(\frac{ik}{m} \right) \leq \left(\frac{\beta}{2} \right)^t \sum_{i=0}^m c_t^i V_t(k) = \beta^t V_t(k) \xrightarrow{t \rightarrow \infty} 0.$$

Then the expected payoff function takes the following form:

$$U_s^C = V_0(0) = \sum_{i=0}^{\infty} \left(\frac{\beta}{2} \right)^i \cdot \frac{1}{2} \cdot \left((2^i - c_i^0) A \left(\frac{k}{m} \right) - (2^i - c_i^m) B \left(\frac{k}{m} \right) \right). \quad (23)$$

We need to find c_i^0 and c_i^m . From (21), we can see that

$$c_{t+1}^0 = c_t^0 + c_t^1, \quad c_{t+1}^i = c_t^{i-1} + c_t^{i+1} \quad (1 \leq i \leq m-1), \quad c_{t+1}^m = c_t^{m-1} + c_t^m.$$

Thus, we have a modified version of Pascal's triangle. Let's see if we can express c_t^i in terms of binomial coefficients $C_n^k = n!/(k!(n-k)!)$. The two main properties of C_n^k we are going to use are

$$C_n^k = C_n^{m-k}, \quad C_n^k + C_n^{k+1} = C_{n+1}^{k+1}.$$

We start with $c_0^0 = 1 = C_0^0$ (and all other $c_0^i = 0, i \geq 1$). In period 1, we have $c_1^1 = c_1^0 = 1 = C_1^0$ with all other c_1^i equal to zero. Period 2 delivers $c_2^2 = c_2^1 = 1 = C_2^0$ and $c_2^0 = c_1^1 + c_1^0 = C_1^0 + C_1^0 = C_1^1 + C_1^0 = C_2^1$, with all remaining c_1^i equal to zero. In period 3, we have $c_3^3 = c_3^2 = 1 = C_3^0, c_3^1 = c_2^2 + c_2^0 = C_2^0 + C_2^1 = C_3^1$, and $c_3^0 = c_2^1 + c_2^0 = C_2^0 + C_2^1 = C_3^1$, with all other c_1^i equal to zero. We get rid of all the zeroes by period m with $c_m^i = C_m^{\lfloor (m-i)/2 \rfloor}$ (here, $\lfloor x \rfloor$ is the largest integer which is less than or equal to x). This process is summarized in Table 1.

state i	m	0	0	0	0	C_m^0
	m-1	0	0	0	C_{m-1}^0	C_m^0
	m-2	0	0	C_{m-2}^0	C_{m-1}^0	C_m^1
	...	\vdots										\vdots
	2	0	0	C_2^0	C_3^0	C_4^1	.	.	.	$C_{m-2}^{\lfloor (m-4)/2 \rfloor}$	$C_{m-1}^{\lfloor (m-3)/2 \rfloor}$	$C_m^{\lfloor (m-2)/2 \rfloor}$
	1	0	C_1^0	C_2^0	C_3^1	C_4^1	.	.	.	$C_{m-2}^{\lfloor (m-3)/2 \rfloor}$	$C_{m-1}^{\lfloor (m-2)/2 \rfloor}$	$C_m^{\lfloor (m-1)/2 \rfloor}$
0	C_0^0	C_1^0	C_2^1	C_3^1	C_4^2	.	.	.	$C_{m-2}^{\lfloor (m-2)/2 \rfloor}$	$C_{m-1}^{\lfloor (m-1)/2 \rfloor}$	$C_m^{\lfloor m/2 \rfloor}$	
		0	1	2	3	4	.	.	.	m-2	m-1	m
c_t^i		time t										

Table 1: The first $m+1$ steps of evolving c_t^i

However, after period m we cannot go up anymore. Instead, all the extra mass we accumulate goes down step by step. Namely, in period $m+1$ we still have $c_{m+1}^i = C_{m+1}^{\lfloor (m+1-i)/2 \rfloor}$ for all $0 \leq i \leq m-1$, but for $i = m$ we now have $c_{m+1}^m = C_{m+1}^0 + C_{m+1}^0$. In period $m+2$, we still have $c_{m+2}^i = C_{m+2}^{\lfloor (m+2-i)/2 \rfloor}$, but only for $0 \leq i \leq m-2$. For $i = m$ and $i = m-1$, we have $c_{m+2}^m = c_{m+2}^{m-1} = C_{m+2}^1 + C_{m+2}^0$, etc. Finally, in period $2m+1$, we have

$$c_{2m+1}^i = C_{2m+1}^{\lfloor \frac{2m+1-i}{2} \rfloor} + C_{2m+1}^{\lfloor \frac{i}{2} \rfloor} \quad 0 \leq i \leq m.$$

See Table 2 for the entire picture.

m	C_m^0	$C_{m+1}^0 + C_{m+1}^0$	$C_{m+2}^1 + C_{m+2}^0$	\dots	$C_{2m-1}^{\lfloor \frac{m-1}{2} \rfloor} + C_{2m-1}^{\lfloor \frac{m-2}{2} \rfloor}$	$C_{2m}^{\lfloor \frac{m}{2} \rfloor} + C_{2m}^{\lfloor \frac{m-1}{2} \rfloor}$	$C_{2m+1}^{\lfloor \frac{m+1}{2} \rfloor} + C_{2m+1}^{\lfloor \frac{m}{2} \rfloor}$
m-1	C_m^0	C_m^1	$C_{m+2}^1 + C_{m+2}^0$	\dots	$C_{2m-1}^{\lfloor \frac{m}{2} \rfloor} + C_{2m-1}^{\lfloor \frac{m-3}{2} \rfloor}$	$C_{2m}^{\lfloor \frac{m+1}{2} \rfloor} + C_{2m}^{\lfloor \frac{m-2}{2} \rfloor}$	$C_{2m+1}^{\lfloor \frac{m+2}{2} \rfloor} + C_{2m+1}^{\lfloor \frac{m-1}{2} \rfloor}$
m-2	C_m^1	C_{m+1}^1	C_{m+2}^2	\dots	$C_{2m-1}^{\lfloor \frac{m+1}{2} \rfloor} + C_{2m-1}^{\lfloor \frac{m-4}{2} \rfloor}$	$C_{2m}^{\lfloor \frac{m+2}{2} \rfloor} + C_{2m}^{\lfloor \frac{m-3}{2} \rfloor}$	$C_{2m+1}^{\lfloor \frac{m+3}{2} \rfloor} + C_{2m+1}^{\lfloor \frac{m-2}{2} \rfloor}$
\dots	\vdots			\vdots			\vdots
2	$C_m^{\lfloor \frac{m-2}{2} \rfloor}$	$C_{m+1}^{\lfloor \frac{m-1}{2} \rfloor}$	$C_{m+2}^{\lfloor \frac{m}{2} \rfloor}$	\dots	$C_{2m-1}^{m-2} + C_{2m-1}^0$	$C_{2m}^{m-1} + C_{2m}^0$	$C_{2m+1}^{m-1} + C_{2m+1}^1$
1	$C_m^{\lfloor \frac{m-1}{2} \rfloor}$	$C_{m+1}^{\lfloor \frac{m}{2} \rfloor}$	$C_{m+2}^{\lfloor \frac{m+1}{2} \rfloor}$	\dots	C_{2m-1}^{m-1}	$C_{2m}^{m-1} + C_{2m}^0$	$C_{2m+1}^m + C_{2m+1}^0$
0	$C_m^{\lfloor \frac{m}{2} \rfloor}$	$C_{m+1}^{\lfloor \frac{m+1}{2} \rfloor}$	$C_{m+2}^{\lfloor \frac{m+2}{2} \rfloor}$	\dots	C_{2m-1}^{m-1}	C_{2m}^m	$C_{2m+1}^m + C_{2m+1}^0$
t/t	m	m+1	m+2	\dots	2m-1	2m	2m+1

Table 2: The second $m+1$ steps of evolving c_t^i

Now the excess mass reached the lower boundary again. Since we cannot go down anymore, this mass has to spread up again and add one more binomial coefficient as a summand to all c_t^i starting from $t = 2m + 2$ and until $t = 3m + 2$. This process continues infinitely. We can

now derive c_i^0 and c_i^m :

$$c_i^0 = \begin{cases} C_i^{\lfloor \frac{i}{2} \rfloor} & \text{if } 0 \leq i \leq 2m, \\ C_i^{\lfloor \frac{i}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} & \text{if } i = 2m + 1, \\ C_i^{\lfloor \frac{i}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-2}{2} \rfloor} & \text{if } 2m + 2 \leq i \leq 2m + 2 + 2m, \\ C_i^{\lfloor \frac{i}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-2}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1-2(m+1)}{2} \rfloor} & \text{if } i = 2(m+1) + 2m + 1, \\ C_i^{\lfloor \frac{i}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-2}{2} \rfloor} + C_i^{\lfloor \frac{i-2m-1-2(m+1)}{2} \rfloor} + C_i^{\lfloor \frac{i-4(m+1)-2m}{2} \rfloor} & \text{if } 4(m+1) \leq i \leq 4(m+1) + 2m, \\ \dots & \end{cases}$$

$$c_i^m = \begin{cases} 0 & \text{if } 0 \leq i \leq m - 1, \\ C_i^{\lfloor \frac{i-m}{2} \rfloor} & \text{if } i = m, \\ C_i^{\lfloor \frac{i-m}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1}{2} \rfloor} & \text{if } m + 1 \leq i \leq m + 1 + 2m, \\ C_i^{\lfloor \frac{i-m}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1-2m-1}{2} \rfloor} & \text{if } i = m + 1 + 2m + 1, \\ C_i^{\lfloor \frac{i-m}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-m-1-2m-1}{2} \rfloor} + C_i^{\lfloor \frac{i-3(m+1)}{2} \rfloor} & \text{if } 3(m+1) \leq i \leq 3(m+1) + 2m, \\ \dots & \end{cases}$$

The coefficient in front of $A(k/m)$ in (23) takes the following form:

$$\begin{aligned} \frac{1}{2} \cdot \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^i (2^i - c_i^0) &= \frac{1}{2} \cdot \sum_{i=0}^{\infty} \beta^i - \sum_{i=0}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i}{2} \rfloor} - \sum_{i=2m+1}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-1}{2} \rfloor} - \\ &- \sum_{i=2m+2}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-2}{2} \rfloor} - \sum_{i=4m+3}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-4m-3}{2} \rfloor} - \dots = \frac{1}{2(1-\beta)} - \sum_{i=0}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i}{2} \rfloor} - \\ &- \sum_{j=0}^{\infty} \left(\sum_{i=2m+1+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-1-2j(m+1)}{2} \rfloor} + \sum_{i=2m+2+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-2-2j(m+1)}{2} \rfloor} \right). \end{aligned}$$

We can rewrite the second term:

$$\sum_{i=0}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i}{2} \rfloor} = \sum_{i=0}^{\infty} \frac{\beta^{2i}}{2^{2i+1}} C_{2i}^i + \sum_{i=0}^{\infty} \frac{\beta^{2i+1}}{2^{2i+2}} C_{2i+1}^i = \frac{1}{2} \cdot \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^{2i} \left(C_{2i}^i + \frac{\beta}{2} C_{2i+1}^i\right).$$

The term in parenthesis can also be simplified by considering even and odd indices separately:

$$\begin{aligned}
& \sum_{i=2m+1+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-1-2j(m+1)}{2} \rfloor} + \sum_{i=2m+2+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-2m-2-2j(m+1)}{2} \rfloor} = \\
& = \frac{\beta^{2(j+1)(m+1)-1}}{2^{2(j+1)(m+1)}} + \sum_{i=2(j+1)(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} \left(C_i^{\lfloor \frac{i+1-2(j+1)(m+1)}{2} \rfloor} + C_i^{\lfloor \frac{i-2(j+1)(m+1)}{2} \rfloor} \right) = \\
& = \frac{\beta^{2(j+1)(m+1)-1}}{2^{2(j+1)(m+1)}} + \sum_{i=0}^{\infty} \frac{\beta^{2(j+1)(m+1)+2i}}{2^{2(j+1)(m+1)+2i}} C_{2(j+1)(m+1)+2i}^i + \\
& + \sum_{i=0}^{\infty} \frac{\beta^{2(j+1)(m+1)+2i+1}}{2^{2(j+1)(m+1)+2i+2}} C_{2(j+1)(m+1)+2i+2}^{i+1} = \frac{1+\beta}{\beta} \sum_{i=0}^{\infty} \left(\frac{\beta}{2} \right)^{2(j+1)(m+1)+2i} C_{2(j+1)(m+1)+2i}^i.
\end{aligned}$$

The coefficient in front of $B(k/m)$ in (23) takes the following form:

$$\begin{aligned}
-\frac{1}{2} \cdot \sum_{i=0}^{\infty} \left(\frac{\beta}{2} \right)^i (2^i - c_i^m) &= -\frac{1}{2} \cdot \sum_{i=0}^{\infty} \beta^i + \sum_{i=m}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m}{2} \rfloor} + \sum_{i=m+1}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-1}{2} \rfloor} \\
&+ \sum_{i=3m+2}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-3m-2}{2} \rfloor} + \sum_{i=3m+3}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-3m-3}{2} \rfloor} + \dots \\
&= -\frac{1}{2(1-\beta)} + \sum_{j=0}^{\infty} \left(\sum_{i=m+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-2j(m+1)}{2} \rfloor} \right. \\
&+ \left. \sum_{i=m+1+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-2j(m+1)-1}{2} \rfloor} \right).
\end{aligned}$$

The term in parenthesis can be simplified the same way as for $A(k/m)$:

$$\begin{aligned}
& \sum_{i=m+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-2j(m+1)}{2} \rfloor} + \sum_{i=m+1+2j(m+1)}^{\infty} \frac{\beta^i}{2^{i+1}} C_i^{\lfloor \frac{i-m-2j(m+1)-1}{2} \rfloor} = \\
& = \frac{1+\beta}{\beta} \sum_{i=0}^{\infty} \left(\frac{\beta}{2} \right)^{(2j+1)(m+1)+2i} C_{(2j+1)(m+1)+2i}^i.
\end{aligned}$$

Summing everything up, we obtain the overall expected profit of the storage unit from

(23):

$$U_s^C = \frac{A(k/m) - B(k/m)}{2(1 - \beta)} + \frac{1 + \beta}{\beta} \left(B \left(\frac{k}{m} \right) \sum_{j=0}^{\infty} \left(\frac{\beta}{2} \right)^{(m+1)(2j+1)} \sum_{i=0}^{\infty} \left(\frac{\beta}{2} \right)^{2i} C_{2i+(m+1)(2j+1)}^i - A \left(\frac{k}{m} \right) \sum_{j=1}^{\infty} \left(\frac{\beta}{2} \right)^{2(m+1)j} \sum_{i=0}^{\infty} \left(\frac{\beta}{2} \right)^{2i} C_{2i+2(m+1)j}^i \right) - \frac{A(k/m)}{2} \sum_{i=0}^{\infty} \left(\frac{\beta}{2} \right)^{2i} \left(C_{2i}^i + \frac{\beta}{2} C_{2i+1}^i \right).$$

Using formula

$$\sum_{i=0}^{\infty} \beta^i C_{2i+r}^i = \frac{2^r}{\sqrt{1 - 4\beta} \left(1 + \sqrt{1 - 4\beta} \right)^r}$$

from [Graham et al. \(1994\)](#) (p. 203) and introducing new discounting coefficient

$$\tilde{\beta} = \frac{\beta}{1 + \sqrt{1 - \beta^2}},$$

we finally get

$$U_s^C = \frac{1}{2(1 - \beta)} \left(-B \left(\frac{k}{m} \right) + \tilde{\beta} A \left(\frac{k}{m} \right) - \frac{2\sqrt{1 - \beta^2}}{\beta} \frac{\tilde{\beta}^{m+1}}{1 - \tilde{\beta}^{2(m+1)}} \left(-B \left(\frac{k}{m} \right) + \tilde{\beta}^{m+1} A \left(\frac{k}{m} \right) \right) \right). \quad (24)$$

□

Proof of Proposition 3. Let's prove formula 10 first. In this case of four possible states, the system of equations (6) takes the following form:

$$\left\{ \begin{array}{l} V_t(0) = \frac{1}{2} (\beta V_{t+1}(0) - B(X) + \beta V_{t+1}(X)), \\ V_t(X) = \frac{1}{2} (A(X) + \beta V_{t+1}(0) - B(k - X) + \beta V_{t+1}(k)), \\ V_t(k - X) = \frac{1}{2} (A(k - X) + \beta V_{t+1}(0) - B(X) + \beta V_{t+1}(k)), \\ V_t(k) = \frac{1}{2} (A(X) + \beta V_{t+1}(k - X) + \beta V_{t+1}(k)). \end{array} \right. \quad (25)$$

It can be rewritten in a matrix form

$$V_t = P + \beta \cdot Q \cdot V_{t+1}, \quad (26)$$

where

$$V_t = \begin{pmatrix} V_t(0) \\ V_t(X) \\ V_t(k-X) \\ V_t(k) \end{pmatrix}, \quad P = \begin{pmatrix} -B(X)/2 \\ A(X)/2 - B(k-X)/2 \\ A(k-X)/2 - B(X)/2 \\ A(X)/2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

To calculate power t of matrix Q , we find the Jordan decomposition $Q = T \cdot J \cdot T^{-1}$ of Q .

Here,

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -2 & 0 & 1 \\ 1 & -2 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

so $Q^t = T \cdot J^t \cdot T^{-1}$.

For $\beta < 1$, we can find from (26) that

$$\begin{aligned} V_0 &= \sum_{i=0}^t \beta^i Q^i \cdot P + \beta^{t+1} Q^{t+1} \cdot V_{t+1} \xrightarrow{t \rightarrow \infty} \sum_{i=0}^{\infty} \beta^i Q^i \cdot P = \\ &= P + T \cdot \begin{pmatrix} \frac{\beta}{1-\beta} & 0 & 0 & 0 \\ 0 & -\frac{\beta}{2+\beta} & 0 & 0 \\ 0 & 0 & \frac{\beta}{2-\beta} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot T^{-1} \cdot P, \end{aligned}$$

from where we finally get $U_s^X = V_0(0)$:

$$\begin{aligned} U_s^X &= -\frac{1}{2} \left(B(X) + \frac{\beta}{2-\beta} B(k-X) \right) + \\ &\quad + \frac{\beta}{4(1-\beta)} \left(A(X) - B(X) + \frac{\beta^2}{4-\beta^2} (A(k-X) - B(k-X)) \right), \end{aligned}$$

which is exactly (10).

Formulae (11) – (13) can be proven exactly the same way using expression (26). For case

$m = 2$ ($1/3 < X < 1/2$), we have

$$V = \begin{pmatrix} V_t(0) \\ V_t(X) \\ V_t(2X) \\ V_t(k-2X) \\ V_t(k-X) \\ V_t(k) \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(k-2X)}{2} \\ \frac{A(k-2X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Case $m = 3$ ($1/4 < X < 1/3$) gives us

$$V = \begin{pmatrix} V_t(0) \\ V_t(X) \\ V_t(2X) \\ V_t(3X) \\ V_t(k-3X) \\ V_t(k-2X) \\ V_t(k-X) \\ V_t(k) \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(k-3X)}{2} \\ \frac{A(k-3X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Finally, in case $m = 4$ ($1/5 < X < 1/4$), we have

$$V = \begin{pmatrix} V_t(0) \\ V_t(X) \\ V_t(2X) \\ V_t(3X) \\ V_t(4X) \\ V_t(k-4X) \\ V_t(k-3X) \\ V_t(k-2X) \\ V_t(k-X) \\ V_t(k) \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(k-4X)}{2} \\ \frac{A(k-4X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} - \frac{B(X)}{2} \\ \frac{A(X)}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

For smaller X ($m \geq 5$), we observe polynomials of higher degrees when calculating eigenvalues of matrix Q , and matrix decomposition turns out to be problematic. Nevertheless, the problem may be solved computationally for any given m . \square

Proof of Proposition ??. Let's start with the possible deviation of storage. If the storage unit decides to enter the market and make a purchase under monopolistic quantities set by producers, it has to sell energy under Cournot quantities. The expected storage payoffs during one round-trip cycle are

$$-\left(\frac{1-a}{2} + k\right)k + \frac{\beta}{2-\beta} \frac{1+a-\delta k}{n+1} \delta k.$$

Storage doesn't gain any profits if

$$\frac{\beta}{2-\beta} \leq \frac{(n+1)\left(\frac{1-a}{2} + k\right)}{\delta(1+a-\delta k)},$$

which is equivalent to the right side of (??). Hence, as long as producers can maintain their collusive equilibrium, the storage unit should not purchase any energy and so never operates.

Let's analyze the possible deviations of a producer. First, we consider the case when inequality (??) doesn't hold. It means that storage is not interested in entering the market with Cournot bids. A producer deviates from its monopolistic quantity when the shock is positive: $\varepsilon = a$. Let the deviation be γ . Then the producer's profit after deviating is

$$\left(\frac{1+a}{2n} + \gamma\right) \left(\frac{1+a}{2} - \gamma\right) + \frac{\beta}{1-\beta} \left(\frac{1}{2} \left(\frac{1+a}{n+1}\right)^2 + \frac{1}{2} \left(\frac{1-a}{n+1}\right)^2\right).$$

A producer increases its quantity by γ , which results in the price going down by γ also. Because of that, all other producers switch from monopolistic quantities to Cournot quantities, which results in the discounted expected payoff over infinite horizon expressed by the second item. This move is unprofitable if

$$\begin{aligned} \left(\frac{1+a}{2n} + \gamma\right) \left(\frac{1+a}{2} - \gamma\right) + \frac{\beta}{1-\beta} \left(\frac{1}{2} \left(\frac{1+a}{n+1}\right)^2 + \frac{1}{2} \left(\frac{1-a}{n+1}\right)^2\right) &\leq \\ &\leq \frac{1+a}{2n} \frac{1+a}{2} + \frac{\beta}{1-\beta} \left(\frac{1}{2} \frac{(1+a)^2}{4n} + \frac{1}{2} \frac{(1-a)^2}{4n}\right), \end{aligned}$$

which can be simplified to

$$\frac{n-1}{n}(1+a)\gamma - 2\gamma^2 \leq \frac{\beta}{1-\beta} \frac{(1+a^2)(n-1)^2}{2n(n+1)^2}.$$

The maximum on the left side can be achieved when $\gamma_{max} = (1+a)(n-1)/(4n)$. Then we obtain

$$\frac{(1+a)^2}{4n} \leq \frac{\beta}{1-\beta} \frac{(1+a^2)}{(n+1)^2}, \quad (27)$$

which is equivalent to the left side of (??).

Now assume that a producer deviates from its monopolistic quantity when the shock is negative: $\varepsilon = -a$. This move is unprofitable if

$$\begin{aligned} \left(\frac{1-a}{2n} + \gamma\right) \left(\frac{1-a}{2} - \gamma\right) + \frac{\beta}{1-\beta} \left(\frac{1}{2} \left(\frac{1+a}{n+1}\right)^2 + \frac{1}{2} \left(\frac{1-a}{n+1}\right)^2\right) &\leq \\ &\leq \frac{1-a}{2n} \frac{1-a}{2} + \frac{\beta}{1-\beta} \left(\frac{1}{2} \frac{(1+a)^2}{4n} + \frac{1}{2} \frac{(1-a)^2}{4n}\right), \end{aligned}$$

which can be simplified to

$$\frac{(1-a)^2}{4n} \leq \frac{\beta}{1-\beta} \frac{1+a^2}{(n+1)^2}.$$

This inequality is weaker than (27).

Now consider the case when inequality (??) holds. It means that storage finds profitable to enter the market immediately after any of the producers has deviated. Since the payoffs of a deviating producer with the participating storage unit are lower than the ones without it, the inequality (27) is sufficient to make this deviation unprofitable.

Finally, we should also prove that our pool of equilibrium strategies forms an SPNE even at information sets that are off the equilibrium path. Namely, producers must not want to deviate even if storage enters the market. Indeed, if the shock is positive, all the producers set static Cournot quantities, and it becomes unprofitable to deviate. If the shock is negative, any deviation from monopolistic quantities implies punishment that was already considered earlier. Hence, all the possible deviations of a producer are unprofitable, and the proposed strategies of all the players form Nash equilibrium.

To calculate the expected payoff of a producer, we obtain a recursive equation for producer's

payoff Z_t starting from period t :

$$Z_t = \frac{1}{2}G_{01} + \frac{1}{2}G_{10} + \beta Z_{t+1}.$$

We can easily find $Z_0 = U_g^0$ from this equation.

Since positive and negative shocks are equally likely, the cumulative consumers' expected payments per period are

$$C^0 = \frac{1}{2} \cdot \left(\frac{1+a}{2}\right)^2 + \frac{1}{2} \cdot \left(\frac{1-a}{2}\right)^2 = \frac{1+a^2}{4}.$$

□

Proof of Proposition 4. To justify inequalities (15), (16), and (17), we need to find the expected payoffs of the storage unit. Let the value function $V_t^{\{-,+\}}(i)$, $i \in \{0, k/2, k\}$ be the total expected payoff of the storage operator from t on if the current state is empty ($i = 0$), half-full ($i = k/2$), or full ($i = k$) and the current shock is either negative ($-$) or positive ($+$). We have a system of recursive equations:

$$\begin{cases} V_t^-(k) = y \cdot \beta V_{t+1}^-(k) + (1-y) \cdot \left(A\left(\frac{k}{2}\right) + \beta V_{t+1}^+\left(\frac{k}{2}\right) \right), \\ V_t^-\left(\frac{k}{2}\right) = y \cdot \left(-B\left(\frac{k}{2}\right) + \beta V_{t+1}^-\left(\frac{k}{2}\right) \right) + (1-y) \cdot \left(A\left(\frac{k}{2}\right) + \beta V_{t+1}^+(0) \right), \\ V_t^+\left(\frac{k}{2}\right) = (1-x) \cdot \left(-B\left(\frac{k}{2}\right) + \beta V_{t+1}^-\left(\frac{k}{2}\right) \right) + x \cdot \left(A\left(\frac{k}{2}\right) + \beta V_{t+1}^+(0) \right), \\ V_t^+(0) = (1-x) \cdot \left(-B + \beta V_{t+1}^-\left(\frac{k}{2}\right) \right) + x \cdot \beta V_{t+1}^+(0). \end{cases}$$

for any integer $t \geq 0$. It can be rewritten in a matrix form

$$V_t = P_2 + \beta \cdot Q_2 \cdot V_{t+1}, \quad (28)$$

where

$$V_t = \begin{pmatrix} V_t^-(k) \\ V_t^-(k/2) \\ V_t^+(k/2) \\ V_t^+(0) \end{pmatrix}, \quad P_2 = \begin{pmatrix} (1-y)A(k/2) \\ (1-y)A(k/2) - yB(k/2) \\ xA(k/2) - (1-x)B(k/2) \\ -(1-x)B(k/2) \end{pmatrix}, \quad Q_2 = \begin{pmatrix} y & 0 & 1-y & 0 \\ y & 0 & 0 & 1-y \\ 1-x & 0 & 0 & x \\ 0 & 1-x & 0 & x \end{pmatrix}.$$

To calculate power t of matrix Q_2 , we find the Jordan decomposition $Q_2 = T \cdot J \cdot T^{-1}$ of Q_2 . Here,

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{(1-x)(1-y)} & 0 & 0 \\ 0 & 0 & -\sqrt{(1-x)(1-y)} & 0 \\ 0 & 0 & 0 & x+y-1 \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & -\frac{x\sqrt{1-y}}{y\sqrt{1-x}} & \frac{x\sqrt{1-y}}{y\sqrt{1-x}} & -\frac{1-y}{1-x} \\ 1 & \frac{\sqrt{(1-x)(1-y)-x}}{1-x} & -\frac{\sqrt{(1-x)(1-y)+x}}{1-x} & -\frac{1-y}{1-x} \\ 1 & \frac{x}{\sqrt{(1-x)(1-y)}} - \frac{x}{y} & -\frac{x}{\sqrt{(1-x)(1-y)}} - \frac{x}{y} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

so $Q_2^t = T \cdot J^t \cdot T^{-1}$.

For $\beta < 1$, we can find from (28) that

$$\begin{aligned} V_0 &= \sum_{i=0}^t \beta^i Q_2^i \cdot P_2 + \beta^{t+1} Q_2^{t+1} \cdot V_{t+1} \xrightarrow{t \rightarrow \infty} \sum_{i=0}^{\infty} \beta^i Q_2^i \cdot P_2 = \\ &= T \cdot \begin{pmatrix} \frac{1}{1-\beta} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\beta\sqrt{(1-x)(1-y)}} & 0 & 0 \\ 0 & 0 & \frac{1}{1+\beta\sqrt{(1-x)(1-y)}} & 0 \\ 0 & 0 & 0 & \frac{1}{1+\beta(1-x-y)} \end{pmatrix} \cdot T^{-1} \cdot P_2 = \\ &= \frac{1}{(1-\beta)(1-\beta d)(1-\beta^2(1-x)(1-y))} \times \\ &\times \begin{pmatrix} (1-y) \left(-\beta(1-x)(1+\beta^2 d) B \left(\frac{k}{2} \right) + (1-(1-y+xd)\beta^2) A \left(\frac{k}{2} \right) \right) \\ -((1-\beta y)(y-\beta d) + \beta^3(1-x)(1-y)d) B \left(\frac{k}{2} \right) + (1-y)(1-\beta x)(1+\beta^2 d) A \left(\frac{k}{2} \right) \\ -(1-x)(1-\beta y)(1+\beta^2 d) B \left(\frac{k}{2} \right) + ((1-\beta x)(x-\beta d) + \beta^3(1-x)(1-y)d) A \left(\frac{k}{2} \right) \\ (1-x) \left(-(1-(1-x+yd)\beta^2) B \left(\frac{k}{2} \right) + \beta(1-y)(1+\beta^2 d) A \left(\frac{k}{2} \right) \right) \end{pmatrix}. \end{aligned}$$

from where we finally get $U_s^2 = V_0^+(0)$:

$$U_s^2 = \frac{1-x}{(1-\beta)(1-\beta d)} \left(-B \left(\frac{k}{2} \right) + \beta(1-y)A \left(\frac{k}{2} \right) + \beta^2 y \frac{yB \left(\frac{k}{2} \right) + \beta x(1-y)A \left(\frac{k}{2} \right)}{1-\beta^2(1-x)(1-y)} \right).$$

Storage operates in this market only if $U_s^2 > 0$, which is exactly inequality (16).

For Condition ??, equation (28) reads

$$V_t = P_3 + \beta \cdot Q_3 \cdot V_{t+1},$$

where

$$V_t = \begin{pmatrix} V_t^-(k) \\ V_t^-(2k/3) \\ V_t^+(2k/3) \\ V_t^-(k/3) \\ V_t^+(k/3) \\ V_t^+(0) \end{pmatrix}, \quad P_3 = \begin{pmatrix} (1-y)A(k/2) \\ (1-y)A(k/2) - yB(k/2) \\ (1-y)A(k/2) - yB(k/2) \\ xA(k/2) - (1-x)B(k/2) \\ xA(k/2) - (1-x)B(k/2) \\ -(1-x)B(k/2) \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} y & 0 & 0 & 1-y & 0 & 0 \\ y & 0 & 0 & 0 & 1-y & 0 \\ 0 & y & 0 & 0 & 0 & 1-y \\ 1-x & 0 & 0 & 0 & x & 0 \\ 0 & 1-x & 0 & 0 & 0 & x \\ 0 & 0 & 1-x & 0 & 0 & x \end{pmatrix}.$$

The six eigenvalues of matrix Q_3 that compose the diagonal of the corresponding Jordan matrix J are

$$\lambda_1 = 1, \quad \lambda_2 = x + y - 1, \quad \lambda_{3,4,5,6} = \pm \sqrt{(1-x)(1-y) \pm \sqrt{xy(1-x)(1-y)}}.$$

Following the same argumentation as in case of $k/2$, we get

$$V_0^+(0) = \frac{1-x}{(1-\beta)(1-\beta d)} \left[-B\left(\frac{k}{3}\right) + \beta(1-y)A\left(\frac{k}{3}\right) \right. \\ \left. + \beta^3 y \frac{y^2 B\left(\frac{k}{3}\right) + x(1-y)(1+\beta^2 d)A\left(\frac{k}{3}\right)}{(1-\beta^2(1-x)(1-y))^2 - \beta^4 xy(1-x)(1-y)} \right].$$

Inequality $V_0^+(0) = U_s^3 > 0$ is exactly (17).

Four possible deviations of the storage unit should be considered. All other deviations are

just compositions of those four.

- A storage unit that is not empty deviates by not selling under the positive shock. Then there may be only loss comparing to the default strategy. Indeed, nothing changes on the market except the future profits to be discounted by β .
- A storage unit that is not full deviates by not buying under the negative shock. Also, no gains here.
- A full storage unit deviates by selling under the negative shock. In this situation, the quantities supplied by the producers are $q = (1 - a)/(n + 1)$. The resulting price after the deviation are

$$p = 1 - a - \delta \frac{k}{m} - n \frac{1 - a}{n + 1} = \frac{1 - a}{n + 1} - \delta \frac{k}{m}.$$

To make this deviation profitable, the storage operator must gain more than if it waits for the positive shock and sells in that period, but this contradicts (16) or (17).

- The nonfull storage unit deviates by buying under the positive shock. Here we have $q = (1 + a)/(n + 1)$, and the resulting price after the deviation is $p = (1 + a)/(n + 1) + k/m$. It is easy to verify that the corresponding payoff is strictly negative.

Next we must rule out possible deviations of the producers. In each round, we have a static Cournot equilibrium for all the participants. Any change of the equilibrium quantity in round t leads to decreasing the payoffs in that round and, thus, decreasing the overall payoffs. \square

A.1 Optimal Bid Values

For each bunch of parameter values, we can find the optimal proportion r_{opt} and, hence, the optimal initial bid $r_{opt}k$ for the case of proportional bids. Also, we can find the optimal constant bid X . We stay with the parameter values used in our examples in the main text with $n = 2$, $\beta = \delta = 0.95$. Table 3 presents the results.

	Proportional bids, $a = 0.6$					Proportional bids, $a = 0.2$				
k	0.25	0.45	0.65	0.85	1.05	0.1	0.15	0.2	0.25	0.3
r_{opt}	0.91	0.68	0.543	0.451	0.385	0.795	0.635	0.526	0.447	0.385
$r_{opt}k$	0.228	0.306	0.353	0.384	0.404	0.08	0.095	0.105	0.112	0.115
	Constant bids, $a = 0.6$					Constant bids, $a = 0.2$				
k	0.25	0.45	0.65	0.85	1.05	0.1	0.15	0.2	0.25	0.3
X	0.21	0.259	0.253	0.248	0.299	0.068	0.066	0.074	0.069	–

Table 3

Some intuitive conclusions we can derive from this table:

- Bids decrease along with the shock in both cases;
- The first proportional bids are always higher than constant bids. This owes to more flexibility towards the boundaries – the asymptotic behavior of the linear heuristic.
- The initial proportional bids tend to increase with k , but slowly. It is not the case for constant bids because the continuation risk is more salient under this heuristic, and so kicks in sooner. Indeed, constant bids increase as long as a number of steps (parameter m from chapter 3.2.4) is the same and decrease with the next shift.

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