Supply function equilibrium with non-convexities.

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Abstract

We study existence and characterisation of supply function equilibrium when players have fixed costs. As often the case in Walrasian markets, equilibrium fails to exist in general because of the non-convexities introduced by fixed costs. Here they manifest themselves in a novel way through *point deviations* off supply schedules. Existence is restored when demand is large enough, and players are constrained to use monotone schedules only to rule out these point deviations. Under these conditions we characterise the symmetric equilibrium, which is unique. This work informs market design.

1 Introduction

Fixed costs induce non-convexities, and it is well known that non-convexities lead to inexistence of a Walrasian equilibrium as first observed by Wald (1951). No fixed point can be found. Shapley and Scarf (1974) show this negative result extends to market design, before the term "market design" existed, that is, to strategic situations.[1] The lack of existence of an equilibrium, whether because of non-convex preferences, non-convexities in production, indivisibilities or externalities, usually spells the death knell of further analysis. It also demands an alternative to complement the toolkit of economics, for the economy goes on regardless.

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[1]The core of their game is empty because of indivisibility.

In this paper we contribute to filling this gap for the narrower problem of finding an equilibrium in supply schedules – a supply function equilibrium (SFE). The SFE concept was first introduced by Grossman (1981) and Hart (1985) and has been studied in some details since; for example, Klemperer and Meyer (1989) (now KM), Vayanos (1999), and others. Supply functions are natural objects in that they specify a willingness to sell for each realisation of an entire set of possible prices; this makes them particularly useful in uncertain environments. They are commonly used in important markets like electricity markets Green and Newbery (1992), Anderson and Philpott (2002), Anderson and Hu (2002), Anderson and Hu (2005), Anderson and Hu (2008), Holmberg (2008) to name just a few. Moreover, Jha and Leslie (2022) document not only the existence but also the impact of fixed costs ("start-up costs") in electricity market. The downside of SFE as a solution concept is that it is technically demanding.[2] As may be expected, the SFE is also not immune to the scourge of fixed costs.

Indeed, we first show that a pure-strategy supply function equilibrium does not exist in two well-known models when suppliers have to incur fixed costs of operation. Fixed costs are not rare, so this is not just an intellectual curiosity; they exist, for example, in electricity markets where generators have to incur start-up costs to operate their plants. With fixed costs, a SFE may fail to exist for two reasons. One is insufficient demand, in a sense that we make precise, to sustain all potential suppliers. Conceptually this is not unique to the SFE; a similar problem arises in a Cournot model, for example. The second reason is specific to the SFE concept; a producer can *always* add price-quantity pairs (points) to the equilibrium candidate schedule, which constitute a profitable deviation. The reason is that, with fixed costs, a schedule cannot start from 0 but from some one of many possible price-quantity pairs $(p, S(p))$ that are strictly positive. Our main result restores equilibrium in pure strategies. To do so it must do away with these point deviations, for which it is sufficient to require the schedule to be non-decreasing. In most models, notably in KM, the schedule is non-decreasing *in equilibrium*; we impose it as a condition. Arguably, requiring the schedule to be non-decreasing is a natural condition. Furthermore, market design is the art of finding more or less natural conditions that deliver a desirable outcome. Here it restores existence of an

[2]Because firms compete in schedules the best responses give rise to a set of differential equations.

equilibrium in pure strategies.

We study a model in which N firms compete to supply a homogeneous commodity in an uncertain world; they do so using schedules, that is, functions $S_i(p)$ that map the price into a quantity offered (at that price). In addition to a standard cost function $c_i(q_i)$, each of these firms must also incur a fixed cost $F_i > 0$ to operate. Individual rationality then dictates that the surplus from selling must at least cover that cost F_i. Therefore a firm chooses to either produce zero or, discontinuously, some strictly positive quantity q_i at a strictly positive price p. To establish our existence results we must stay with a symmetric equilibrium, for which we also require firms to be symmetric. We characterise this equilibrium and show it is unique. Throughout we make reference to two well-established models: that of KM and another by Holmberg (2008), which is particularly suited to our motivating application of electricity markets and delivers a unique equilibrium (absent fixed costs). Our results apply to both.

The work closest to ours is the paper of Grossman (1981). We also actively refer to Klemperer and Meyer (1989) and Holmberg (2008), both of which we describe in more details below. KM drastically reduce the number of equilibria and provide an elegant solution. Grossman (1981) studies a problem, in which Cournot players incur fixed costs. Without any uncertainty, he introduces supply functions essentially as price-contingent offers an entrant can make to consumers; that is, for any price p offered by the competition (not necessarily a market price), a firm can offer some $q(p)$. Then, with enough entrants, Grossman (1981) shows an equilibrium exists and is such that the clearing price is equal to average costs. This result disappears with sunk costs. Our concern is quite different and closer to a more modern approach to the SFE. In fact, Grossman (1981) does not make full use of supply functions, as pointed out by KM, since there is no uncertainty.[3] In an uncertain environment, and with non-trivial supply schedules, we show a SFE is not robust to point deviations. Anderson and Hu (2005) allow for the interaction of SFE, forward contracts and price caps. Vayanos (1999) is concerned with asymmetric information and supply functions; the interaction of strategic behaviour and asymmetric information can deliver more or less competitive outcomes, depending on how

[3] In Grossman (1981) a schedule has at most two points; Theorem 1.

information is correlated across suppliers. There exists a large literature that addresses the lack of existence of an equilibrium in a Walrasian market but it is not directly related to this work.

2 The Model

We describe the model in two steps, which include a specific description two models of interest that are results can apply to.

2.1 Basics

There is a market demand that is denoted by $D(p,\epsilon)$, where p stands for the market price and ϵ is an exogenous, stochastic demand shock distributed on some support $\mathcal{E}$ according to $F(\epsilon)$ with density $f(\epsilon)$. The shock ϵ may be a strict demand shock, or also reflect the fact that some of the supply is not available.[4] We impose $\frac{\partial D}{\partial p} < 0$, $\frac{\partial^2 D}{\partial p^2} \leq 0$, $\frac{\partial D}{\partial \epsilon} > 0$ and $\frac{\partial^2 D}{\partial p \partial \epsilon} = 0$. With this, setting $D(p,\epsilon) := d(p) + \epsilon$ is without loss of generality; let also $\epsilon = e(D,p)$ be the inverse function when it is defined.

Demand is served by N suppliers – for example, electricity generators – that compete in supply schedules and are index by i. While in principle they may operate different technologies characterised by their (strictly convex) marginal cost $c_i'(q)$ and their fixed cost of operation F_i, we take them to be symmetric – for two reasons. One, with fixed costs there may be asymmetric equilibria even with symmetric technology (in departure from KM, for example); two, we present results for the symmetric case only. In electricity, for example, the fixed cost F is a start-up cost. Each supplier may produce a quantity $q_i \leq k$; when $k < \infty$, capacity constraints exist and may bind. A supply schedule is a function $S_i(p)$, $S : \mathbb{R} \longmapsto \mathbb{R}^+$ that is selected by firm i and prescribes quantities supplied for each realisation of the price p. The aggregate supply is $S(p) := \sum_{i=1}^{N} S_i(p)$. Firms bid their supply schedule simultaneously before ϵ is known. Upon realisation of the shock ϵ, a clearing price $p(\epsilon)$ is obtained by equating

[4] For example, in electricity, it may embed the stochastic supply of any renewable energy producer in the demand $D(p,\epsilon)$, which is then understood as a residual demand faced by thermal generators.

demand to the available supply. A Nash equilibrium is a profile $\mathbf{S}$ of supply schedules such that $S_i(p)$ maximises firm i's expected profit given the choices S_{-i} of the other $N-1$ firms.

2.2 The Klemperer and Meyer (1989) model

In the model of Klemperer and Meyer (1989) – now KM, the shock ϵ has full support: $\mathcal{E} = [0, \infty)$ and there are no constraints except for the important continuity of the schedule, so $\forall i,\ k_i = \infty$ and p is potentially unbounded. Market clearing is obtained by solving

$$S(p(\epsilon)) = D(p(\epsilon), \epsilon). \tag{1}$$

Firms are symmetric and importantly, there are no fixed costs, so $\forall i,\ c_i = c$ as here, but $F = 0$. The combination of unbounded support, no fixed costs and symmetry supports their result, including the uniqueness of equilibrium in some cases. In particular, if the support of ϵ remains bounded, the number of equilibria explodes.

2.3 Hölmberg's (2008) model

Capacity constraints do exist and matter a great deal; in fact they are rather the norm than the exception. In electricity markets, price caps also exist; equivalently, the shock ϵ is bounded.[5] Then market clearing satisfies either (1), or

$$\sum_{i=1}^{N} k_i = D(p(\epsilon), \epsilon), \quad \text{or} \quad S(\bar{p}) = D(\bar{p}, \epsilon) \tag{2}$$

when either the capacity constraint binds, or a price cap $p(\epsilon) = \bar{p}$ does. Hölmberg's (2008) model incorporates these facts, but remains symmetric $c_i \equiv c$ and free of fixed costs $\forall i,\ F_=0$. Instead of strict continuity of the schedule(s) $S(p)$, Hölmberg assumes left continuity, from which upper hemi-continuity is not guaranteed, nor strict monotonicity of the schedule(s). With an inelastic demand, Hölmberg shows the SFE is unique, and continuous in equilibrium. The key step is to show any firm finds it optimal to deliver all is capacity at the price cap.

[5]New Zealand is the only market without a price cap.

In a symmetric equilibrium, all firms behave in this manner and the differential equation characterizing the SFE has a unique boundary precisely at the point $(\bar{p}, S(\bar{p}) = k)$. With, in addition, a unique initial condition $(0, S(0))$ and a single differential equation, the resulting boundary-value problem has a single solution.

3 No equilibrium with fixed costs

This section establishes that, in the settings of both KM and Hölmberg (and many others), no pure strategy equilibrium can resist the introduction of fixed costs. That is, non-convexities have severe implications for the conduct of trade in many markets; this includes all electricity markets, for example. This is well-known to practitioners; we identify two of the reasons. First we begin by extending the result of Hölmberg when demand is elastic. This marginal generalisation allows us to derive our results more generally.

Ignore the fixed costs for now – $F = 0$; given the other firms $-i$ bidding $S_{-i}(p)$, $S_{-i}(p) = \sum_{j \neq i} S_j(p)$, the profits of firm i are given by

$$\begin{aligned}\pi^i(p, \epsilon; S_{-i}) =& p\big(D(p, \epsilon) - S_{-i}(p)\big) - c(D(p, \epsilon) - S_{-i}(p)) \\ =& p\big(d(p) + \epsilon - S_{-i}(p)\big) - c(d(p) + \epsilon - S_{-i}(p))\end{aligned} \tag{3}$$

Firm i maximizes profits along its residual demand curve. Market clearing is equivalent to $\epsilon = S^i(p) + S_{-i}(p) - d(p)$, the first order condition $\frac{\partial \pi^i}{\partial p} = 0$ of Equation (3) implies that

$$S'_{-i}(p) = d'(p) + \frac{S_i(p)}{p - c'(S_i(p))} \tag{4}$$

For symmetric equilibria, $S_i(p) = S_j(p) = S(p)$ and Equation (4) can be reduced to the differential equation

$$S'(p) = \frac{1}{N-1}\left(d'(p) + \frac{S(p)}{p - c'(S(p))}\right) =: g(p, S) \tag{5}$$

Let $l_1(S)$ be the function of S determined by $g(p, S) = 0$ and $l_2(S) = c'(S)$. These two functions define the set of symmetric supply function equilibria that is connected and between the graph of l_1 and l_2. That is, they graph the "cone" of feasible SFEs, if they exist. This cone is labelled G.

3.1 Extending Hölmberg

Let now impose both a capacity constraint $0 < k < \infty$ and a price cap $0 < \bar{p} < \infty$. We can extend the work of Hölmberg (2008) to allow for elastic demand. Let firms bid in supply schedules $S_i(p)$ that at least left-continuous.

Proposition 1. *There exists a unique, symmetric Supply Function Equilibrium S_0 starting at $(0, S_0(0) = 0)$ passing through the point $(k, \bar{p})$.*

The proof, as all other ones, is relegated to the Appendix in Section A. There is nothing special about inelastic demand in the work of Hölmberg's. The uniqueness claim is an application of the Picard-Lindelöf Theorem to the differential equation (5), where the point $(\bar{p}, k)$ is taken as the initial condition.

3.2 No equilibrium

Now we introduce the fixed cost $F > 0$. A firm is willing to supply only if doing so generates a non-negative surplus. That is, at the clearing price one must have

$$\pi(p(\epsilon), q_i) = p(\epsilon)q_i - c(q_i) - F \geq 0, \tag{6}$$

which is a form of individual rationality. For each firm i, it defines a region of feasible supply schedules that the firm is willing to offer. We call the set of price-quantity pairs (p, q_i) satisfying this ex-post individual rationality constraint the *active region*. The boundary of the active region determined by the condition

$$pq_i - c(q_i) - F \equiv 0$$

defines a function $A(q_i; c, F) := \frac{c(q_i)+F}{q_i} \equiv A_i$ that is the contour of that set. It is the set of lowest acceptable price to firm i for each quantity q_i. We denote the interior of the active region by $\mathring{A}(q_i; c, F)$. With fixed costs there may exist asymmetric supply function equilibria. To provide analytical answers we focus on the symmetric ones and so we drop the subscripts in equilibrium. Of course then $\forall i, j,\ A_i = A_j = A$.

A firm maximises

$$\pi^i(p, \epsilon; S_{-i}) = p\big(d(p) + \epsilon - S_{-i}(p)\big) - c(d(p) + \epsilon - S_{-i}(p)) - F, \tag{7}$$

which of course gives rise to the same first-order condition (4), the same differential equation (5) and the same functions l_1 and l_2. An SFE can only exist if it is contained in the cone G defined by l_1 and l_2, and satisfies (6). Denote the intersections of $A(q; c, F)$ with $l_1(S)$ and $l_2(S)$ by (s_1, p_1) and (s_2, p_2). Since the marginal cost function c is convex, the intersection (s_2, p_2) is unique. We claim

Lemma 2. *The slopes of $A(q; c, F)$ at its intersections with l_1 and l_2 equal $d'(p)^{-1}$ and 0 respectively. $A(q; c, F)$ is decreasing and convex on $(0, s_2]$ and increasing on (s_2, ∞).*

From this Lemma we can conclude

Claim 3. *The intersection of the active region $\mathbf{A} := A(q; c, F) \cup \mathring{A}(q; c, F)$ and the feasible region G is non-empty: $\mathbf{A} \cap G \neq \varnothing$.*

The proof follows directly from Lemma 2: functions l_1 and l_2 start at the origin $(0, 0)$. Therefore $A(q; c, F)$ must intersect l_1 and l_2, and the upper contour $\mathring{A}(q; c, F) \cap G$ is non-empty. That is, there are sensible candidate equilibria. Finally we can state the first of our non-existence results.

Proposition 4. *Let $F > 0$ for all players. For any M, $0 < M \leq N$, there exists some ϵ_M such that M of the N players operate and for one such i, $\Pi_i(F, \epsilon_0) = 0$. When $\underline{\epsilon} < \epsilon_M$, there is no pure strategy ex post optimal supply function equilibrium.*

This is not the only way to show an equilibrium does not exist, but it is sufficient. The reason for this failure is reasonably standard and rooted in the discontinuities in actions that come to exist because of fixed costs. In the papers of Klemperer and Meyer (1989) and Holmberg (2008), the initial condition of a supply schedule is $(0, S(0) = 0)$, whereas here it must respect (6) and therefore start in the active region $\mathbf{A}$ bounded by $A(q; c, F)$. As a result there may be other, asymmetric equilibria than the symmetric equilibrium of either KM or Hölmberg's. When demand is too low for all N firms to operate, a subset of them may operate – say $0 < M < N$ firms. If demand is such that the marginal firm (of these M firms) makes exactly zero profit, it is a pure-strategy equilibrium for them to offer a schedule and for the remaining $N - M$ firm to offer 0. This is akin to a coordination game.[6] But of course, since ϵ is a continuous random variable, this is a probability zero event. If these M firms make positive profits, at least another firm j can offer positive quantities. Conversely, when the lower bound $\underline{\epsilon}$ of the demand shock is too low, not all M firms break even. This kind of non-existence is not specific to the SFE concept; it would also arise if firms were to play Cournot, for example.

An equilibrium also fails to exist for another reason altogether, which is *specific* to the SFE concept in the presence of fixed costs. Suppose now that demand is large enough for all N firms to operate. That is,

Assumption 5. $\exists \epsilon_N > 0$ *and* $\epsilon_N \leq \underline{\epsilon}$.

With this, Proposition 4 is ruled out. Yet,

Proposition 6. *Let* $F > 0$ *for all players and suppose Assumption 5 holds. There is no pure strategy ex post optimal supply function equilibrium.*

The SFE is not immune to point deviations, that is, pairs (q, p) that can be offered by a firm to augment the schedule it offers – as follows. In the $(p, S(p))$ plane, the boundary $A(q; c, F)$ is convex, first decreasing then increasing, so the set $\mathring{A}(q; c, F)$ is also convex (Lemma 2). There can be no equilibrium to the right of the minimum (in the $S(p)$ dimension). For any candidate equilibrium starting from the segment to the left of the minimum, say from the point $(\tilde{p}, S(\tilde{p}))$,

[6] In a symmetric equilibrium all M firms may make zero profit, but this is not the only candidate equilibrium to consider. There may be asymmetric equilibria, in which only the marginal firm breaks even.

a firm can pick a point in $\mathring{A}(q; c, F)$. That is, it can offer a quantity $q' > S(\tilde{p})$ at a lower price $p' < \tilde{p}$. The point (p', q') is an addition to the candidate schedule; it is a profitable deviation. The problem does not disappear when the demand is always large enough.

These point deviations cannot be part of a best reply in KM, where schedules are assumed to be (strictly) continuous for all prices starting from zero. Hence there is no price contingency that is not covered, and for any price p', it cannot be optimal to offer two different quantities $S(p')$ and q'. This is precisely the problem here: the set $A(q; c, F)$ that represent the constraint (6) is strictly bounded away from zero, so that for some prices, no quantities are offered. This invites a deviation. The same holds for the model of Hölmberg (2008), where the relaxation to left-continuity is not sufficient to reach any other conclusion. These smoothness conditions are not sufficient with fixed costs, because any fixed costs induce that convex set $\mathbf{A}$ bounded by $A(q; c, F)$.

Hence for an equilibrium to exist the lower bound on the shock ϵ must be large enough; that is, demand must be large enough, and these point deviations must be deterred. There is also no reason to believe an equilibrium in mixed strategies does not exist, but it is unwieldy in practice except maybe at Wimbledon.[7] These options are not appealing, neither in theory nor (much less) in practice. Instead we try to find ways to discipline the behaviour of players so that an equilibrium in pure strategies can exist.

3.3 Discussion

We see these deviations differ from those arising in a quantity game such as Cournot, for example, where the equilibrium candidate (for a firm) is a single point q^c that induces a clearing price p^c. Then, given q^c_{-i}, a deviation for firm i can only be another point q'_i. It is easy to see that even with constraint (6), there is no deviation from the standard Cournot equilibrium that satisfies the first-order conditions of the problem. What may happen instead is mixing between 0 and the candidate q^c. Whereas (6) may not be satisfied with N players supplying, it may if only $L < N$ operate instead. That is, if q^c_{-i} is such that the best response

[7] See the work of John Wooders on mixing in professional tennis; only the very best players do it correctly.

is $q_i = 0$, then players mix between 0 and the Cournot quantity $q^c(L)$ for $L < N$ players.[8] This kind of mixing here is irrelevant.[9] Indeed, the supply schedule offers quantities $S(p)$ starting from some pair $(p', S(p'))$ in $A(q; c, F)$ – that is, where the participation condition (6) just binds. Clearly there is no object in mixing between any $(p', S(p'))$ and $(0, S(0))$ since payoffs are identical. Given the schedule $S(p)$, for any $p > p'$ there is also no object in mixing with $(0, S(0))$ since the revenue $p \cdot S(p)$ strictly dominates any mixture of itself with 0.

4 Equilibrium with fixed cost

In this Section, we establish existence of a symmetric supply function equilibrium under certain conditions.

4.1 Existence

First we turn to the main result, for which we assume the demand is large enough for an equilibrium to exist under the "right" conditions. That is, Assumption 5 holds. Now we can restore existence of a pure strategy SFE with the introduction of a simple condition that curtails the response of players. We simply require their supply schedules to be non-decreasing.

Proposition 7. *Let $F > 0$ and suppose Assumption 5 holds. A symmetric, pure strategy Supply Function Equilibrium exists if the supply schedules are non-decreasing for all players.*

This simple but strong result applies to both models presented in Section 2, and of course extends to the (simpler) case of inelastic demand.

Adding the condition that $S(p)$ must be non-decreasing prevents deviations because it prohibits the point deviations discussed in Section 3.2 – as well as their converse (higher price, lower quantity). That is, a schedule $S_i(p)$ is not allowed to first start in $\mathring{A}(q; c, F)$ and then hit $A(q; c, F)$; nor can arbitrary points in $\mathring{A}(q; c, F)$ be added to the schedule. This

[8] Grossman (1981) extends the quantity offer q^c to a pair $\{q^c(p), 0\}$, where $q^c(p)$ is a quantity that is contingent on the price offered by a competitor – not the clearing price. This is quite a different model, and the terminology "supply schedule", while strictly correct, does not carry the same meaning as here and as in much of the literature.

[9] That is, a mixed-strategy equilibrium must mix differently.

monotonicity condition effectively acts like a price floor, in the sense that once a schedule has started, no greater quantities can be offered at a lower price. While it has the same effect as a price floor, it presents the advantage of not being a direct constraint on price. In addition, it is informationally parsimonious, even free, where any price floor requires knowing both equilibrium play and the state ϵ.

Monotonicity of the supply schedule is also very natural. In the KM paper, monotonicity arises as a consequence of the smoothness requirement on the supply schedules combined with equilibrium play. In the paper of Hölmberg, monotonicity is also imposed on the supply schedules $S_i(p)$ as a convenient technical assumption. A form of monotonicity is also imposed in many auctions. For example, the English auction is obviously monotone; so is the first-price auction, where bids are restricted to be non-decreasing. More sophisticated auctions also typically restrict bids to be monotone: so do the "Clock Proxy Auction" of Ausubel, Crampton and Milgrom or the "Hierarchical Package Bidding" auction of Goeree and Holt (both in **?**). This form of restriction on bidding behaviour is common in market design, where it may be used to achieve certain desiderata of the mechanism. One such, minimalist, desiderata surely is the existence of an equilibrium.

4.2 Uniqueness

Beyond existence we can characterise the SFE and claim it is unique. First, recall Proposition 1. Now let $S_0(p) \cap A(q; c, F) = (p_0, q_0)$. This intersection is unique since $S_0(p)$ is non-decreasing from 0 and $(\bar{p}, S(\bar{p}) = k) \in \mathbf{A} \cap G$. Then,

Proposition 8. *Suppose Assumption 5 holds and let schedules be non-decreasing. The symmetric supply function equilibrium is unique on* $[p_0, \bar{p}]$. *It is characterised by the differential equation (5) together with the conditions* $S(\bar{p}) = k$ *and* $S(p_0) = q_0$.

Hence the part of the unique, unconstrained schedule that satisfies (6) – so, starting from the set $A(q; c, F)$ – is the equilibrium supply schedule.

Uniqueness is not immediately obvious. With fixed costs, a supply schedule starts from the set $A(q; c, F)$; so in principle there are many possible starting points (initial conditions for the ODE (5)). But to start at any other point that (p_0, q_0) and terminate at $(\bar{p}, k)$, the schedule must be characterised by another differential equation, which does not correspond to a maximiser of the problem of the firms.

In a sense, the fact that the equilibrium schedule with fixed costs is the same as without should not be surprising: in the objective function (7) of the suppliers the fixed cost F enters linearly and does not interact with the instrument $S(p)$. So it is irrelevant to the optimality conditions of the firms. Rather it is a challenge to existence, as we show in Section 3.2. We hasten to add that fixed costs introduce a further complexity: there may be asymmetric equilibria that exist too. This is not the case in the model of KM (Proposition 3), nor in those of Hölmberg's (2007,2008), where the symmetric equilibrium is the unique equilibrium.

5 Conclusion

In this paper we tackle the vexing question of existence of a Supply Function equilibrium when suppliers have to handle fixed costs, which induce non-convexities. This is a natural question to consider as many markets are plagued by these non-convexities; our motivating example of electricity can be extended to airlines or shipping and to manufacturing.

Existence is challenged for two reasons. The first one is the usual discontinuity in actions that owes to the lack of convexity; this is a problem that is familiar to any Walrasian environment. The second one is peculiar to the SFE as a solution concept; players can avail point deviations that extend their supply schedule. The cure to this new problem is to restrict the supply schedules to be monotone.

A Proofs

Proof of Proposition 1. We first rule out "flat" part of the supply function equilibrium S near $(\overline{p}, \overline{q})$. Suppose on some interval $p \in [\overline{p} - \delta, \overline{p}]$, the symmetric supply $S(p) = \overline{q}$. Consider player i. We show her best response is not $S(p)$. The necessary first order condition of her to maximize profits is:

$$(M-1)S'(p) = d'(p) + \frac{S(p)}{p - f'(S(p))} \tag{8}$$

When $p \in [\overline{p} - \delta, \overline{p}]$, by our hypothesis, $S'(p) = 0$ and the supply function equilibrium has to satisfy

$$d'(p) + \frac{S(p)}{p - f'(S(p))} = 0, p \in [\overline{p} - \delta, \overline{p}] \tag{9}$$

Pick one $p_0 \in (\overline{p} - \delta, \overline{p})$, and choose a small number $\delta' > 0$ such that $p_0 - \delta' \in (\overline{p} - \delta, \overline{p})$. Because $d''(p) \leq 0$, $d'(p_0 - \delta') \geq d'(p_0)$ which implies that $\frac{S(p_0-\delta')}{p_0-\delta-f'(S(p_0-\delta'))} \leq \frac{S(p_0)}{p_0-f'(S(p_0))}$. By hypothesis, $S(p_0 - \delta') = S(p_0)$, thus we have a contradiction.

Next we rule out the "jump" discontinuity of the symmetric SFE at price $\overline{p}$. Let $\Delta S_i(\overline{p}) = \overline{q} - S_i(\overline{p})$. To deal with the jump in supply schedule at the price cap, we assume there is a rationing mechanism $R(\epsilon - \overline{\epsilon}, \Delta S_i(\overline{p}), \Delta S_{-i}(\overline{p}))$ such that $0 < R_1 + R_2 < 1$ and $R(0, \cdot, \cdot) = 0$. Suppose the symmetric supply function candidate S is smooth on some interval $[\overline{p} - \delta, \overline{p}]$, and at $\overline{p}$, $S(\overline{p})$ can be any quantities up to the capacity constraint $\overline{q}$. The market clearing condition $S_i(p) + S_{-i}(p) = d(p) + \epsilon$ determines the clearing price as a function of the demand shock $p = p(\epsilon)$. By submitting S_i, the player i also chooses the slope $p'(\epsilon) = \frac{1}{S_i'(p(\epsilon))+S_{-i}'(p(\epsilon))-d'(p(\epsilon))}$. Let $\epsilon_0 = p^{-1}(\overline{p} - \delta)$, $\overline{\epsilon} = p^{-1}(\overline{p})$, and $\epsilon' = M\overline{q} - d(\overline{p})$. Given the strategies S_{-i}, the player i

solves the following optimal control problem

$$\max_u \int_{\epsilon_0}^{\overline{\epsilon}} \big\{p(\epsilon)[d(p(\epsilon)) + \epsilon - S_{-i}(p(\epsilon))] - c(d(p(\epsilon)) + \epsilon - S_{-i}(p(\epsilon)))\big\} f(\epsilon)\mathrm{d}\epsilon + \Phi(\overline{\epsilon}) - F \quad (10)$$

$$\text{s.t.} p'(\epsilon) = u$$

$$p(\epsilon_0) = \overline{p} - \delta$$

$$p(\overline{\epsilon}) = \overline{p}$$

$$0 < u < \frac{1}{S'_{-i}(p(\epsilon)) - d'(p(\epsilon))} \quad (11)$$

where u is control variable and

$$\Phi(\overline{\epsilon}) = \int_{\overline{\epsilon}}^{\epsilon'} \big\{[d(\overline{p}) + \overline{\epsilon} - S_{-i}(\overline{p}) + R(\epsilon - \overline{\epsilon}, S_{-i}(\overline{p}) - d(\overline{p}) + \overline{q} - \overline{\epsilon}, \Delta S_(-i))]\overline{p}$$
$$- c(d(\overline{p}) + \overline{\epsilon} - S_{-i}(\overline{p}) + R(\epsilon - \overline{\epsilon}, S_{-i}(\overline{p}) - d(\overline{p}) + \overline{q} - \overline{\epsilon}, \Delta S_{-i}))\big\} f(\epsilon)\mathrm{d}\epsilon$$

The Hamiltonian of this control problem is given by

$$H(u, p, \lambda, \epsilon) = \lambda(\epsilon)u(\epsilon) + \big\{p(\epsilon)[d(p(\epsilon)) + \epsilon - S_{-i}(p(\epsilon))] - c(d(p(\epsilon)) + \epsilon - S_{-i}(p(\epsilon)))\big\} f(\epsilon) \quad (12)$$

For every $\epsilon \in [\epsilon_0, \overline{\epsilon}]$, $\frac{\partial H}{\partial u} = 0 \implies \lambda(\epsilon) = 0$ and the transversality condition implies that

$$H(u, p, \lambda, \overline{\epsilon}) + \frac{\partial \Phi}{\partial \overline{\epsilon}} = 0 \quad (13)$$

But we can compute

$$\frac{\partial \Phi}{\partial \overline{\epsilon}} = -f(\overline{\epsilon})\big\{[d(\overline{p}) + \overline{\epsilon} - S_{-i}(\overline{p})]\overline{p} - c(d(\overline{p}) + \overline{\epsilon} - S_{-i}(\overline{p}))\big\}$$
$$+ \int_{\overline{\epsilon}}^{\epsilon'} [(\overline{p} - c')(1 - R_1 - R_2)]\, f(\epsilon)\mathrm{d}\epsilon \quad (14)$$

Thus

$$H(u, p, \lambda, \overline{\epsilon}) + \frac{\partial \Phi}{\partial \overline{\epsilon}} = \int_{\overline{\epsilon}}^{\epsilon'} [(\overline{p} - c')(1 - R_1 - R_2)]\, f(\epsilon)\mathrm{d}\epsilon \quad (15)$$

Because $\overline{p} > c'$ and $1 - R_1 - R_2 > 0$, $H(u, p, \lambda, \overline{\epsilon}) + \frac{\partial \Phi}{\partial \epsilon} > 0$. Thus undercut $\overline{p}$ is best response of player i. As a result, the symmetric supply function equilibrium pass $(\overline{p}, \overline{q})$ precisely. $\square$

Proof of Lemma 2.

$$\frac{\mathrm{d}A}{\mathrm{d}s} = \frac{sc'(s) - c(s) - F}{s^2} \tag{16}$$

At the intersections, we also have

$$l_1(s_1) = \frac{c(s_1) + F}{s_1} \tag{17}$$

$$l_1(s_2) = \frac{c(s_2) + F}{s_2} \tag{18}$$

From above we can see $\frac{\mathrm{d}A}{\mathrm{d}s}\Big|_{s_1} = \frac{c'(s_1) - l_1(s_1)}{s_1}$. Also Note that $p_1 = l_1(s_1)$ solves $g = 0$ in 5, thus $d'(p) = \frac{s_1}{c'(s_1) - l_1(s_1)}$. Similarly, we can obtain the result for s_2. Note that

$$\frac{\mathrm{d}^2 A}{\mathrm{d}s^2} = \frac{c''(s)}{s} - \frac{2c'(s)}{s^2} + \frac{2(c(s) + F)}{s^3} \tag{19}$$

thus $\frac{\mathrm{d}A}{\mathrm{d}s} < 0 \implies \frac{\mathrm{d}^2 A_{bd}}{\mathrm{d}s^2} > 0$. $\square$

Proof of Proposition 4. The existence of ϵ_0 is due to continuity of the profit function associated with (asymmetric) SFEs of M players. The associated clearing price is denoted by p_0. Note that for a player $j \neq i$ of other $N - M$ players, the residual demand is everywhere lower than that of player i, thus the profits $\Pi_j(\epsilon, F)$ of player j is less than that of player i: $\Pi_j(\epsilon, F) < \Pi_i(\epsilon, F)$ after j choosing her best reply to M operating players. Since Π_j is continuous w.r.t. ϵ, there is a neighbourhood $[\epsilon_0, \epsilon_0 + \delta)$, where $\Pi_j(\epsilon, F) < 0$. As a result, player j would producing nothing. However, when $e(p_0, D)$ is high enough for some large demand shock ϵ, $\Pi_j(\epsilon, F)$ becomes positive around the clearing price p_0. Thus, no pure strategy can support this distribution of demand shock. $\square$

Proof of Proposition 6. Suppose the pure strategy equilibrium candidate is denoted by $S_j, j = 1, 2, ..., N$, and their intersection with the boundary are p_j. WLOG, we assume $P_N \geq P_{N-1} \geq ... \geq P_1$. First note that S_1 is everywhere above the marginal cost curve, thus $P_1 > p_2$ where

p_2 is the intersection of l_2 with the boundary. By lemma 2, there is a player $i \neq 1$, she can revise her schedule at a price $p_2 < p' < P_1$ to produce a positive quantity q' such that (q', p') is in the interior of her active region so that such deviation is profitable and the resulting schedule is non-monotone. □

Proof of Proposition 7. By Assumption 5 the conditions of Proposition 4 are not met. We need only show that the monotonicity of the schedules rules out the point deviations of Proposition 6. Consider a candidate schedule $S_i(p)$ that starts on the boundary $A(q; c, F)$, terminates at $(\bar{p}, k)$ and solves the differential equation (5) – so it is continuous and smooth. There are two kinds of deviations to consider only: take a point $(\hat{p}, \hat{q}$ such that $S_i(\hat{p}) < \hat{q}$, then by continuity of $S_i(p)$, $S(\hat{p} + \varepsilon) < \hat{q}$, which violates the monotonicity requirement. Conversely, take $(\tilde{p}, \tilde{q}$ such that $S_i(\tilde{p}) > \tilde{q}$. While this does not violate monotonicity, it immediately implies that S_i cannot be an equilibrium candidate. That is, the FOC (4) that induces the differential equation (5) cannot identify a maximum.

□

Proof of Proposition 8. First by Proposition 7, a solution exists. Next, by Proposition 1, the SFE is smooth in the neighborhood of the point $(\bar{p}, k)$. Applying the Picard-Lindelöf Theorem to the differential equation (5), the SFE is unique locally. Noting that the RHS of the differential equation (5) is continuously differentiable in p on the interval $[p_0, \bar{p}]$, we can repeatedly apply the Picard-Lindelöf Theorem again to claim the solution is unique on $[p_0, \bar{p}]$. □

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