Unemployment and the Labor Share

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Abstract:
How do labor market conditions such as the unemployment rate influence the labor share? To answer this question, I develop a search-theoretic model of the labor market that generates a simple relationship between unemployment, workers' reservation wage, and the labor share. I derive expressions for the dynamic evolution of factor shares and present some comparative statics results regarding factor shares in the steady state equilibrium. I simulate the model and compare its predictions for factor shares to the U.S. data from 1951-2012. The results suggest that labor market conditions – specifically, unemployment fluctuations and changes in workers' reservation wage – can account for much of the variation in the U.S. labor share at an annual frequency during this period.

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1 Introduction

What determines the relative income shares of workers and the owners of capital? Historically, this question was considered so fundamental that in 1817 Ricardo called it "the principal problem in Political Economy."¹ A common approach in modern macroeconomics is to assume either a Cobb-Douglas or constant-elasticity-of-substitution (CES) aggregate production function, and competitive factor and product markets. In such a setting, factor shares are either constant or they are driven by changes in the capital-output ratio. Labor’s share is either increasing or decreasing in the capital-output ratio depending on the elasticity of substitution.

There have been recent signs of a resurgence of interest in the labor share. King and Watson (2012) document a sharp fall in the U.S. labor share during the 2000s. Elsby, Hobijn, and Sahin (2013) discuss the recent decline in the U.S. labor share and identify globalization and offshoring as potentially important factors. Karabarbounis and Neiman (2014) document a global decline in labor’s share in the corporate sector since the early 1980s and attribute this to a fall in the relative price of investment goods by estimating an elasticity of substitution that is greater than one. Piketty (2014) and Piketty and Zucman (2014) suggest that the global decline in labor’s share arises from capital accumulation and an elasticity of substitution greater than one.

In fact, there is a large body of empirical evidence that this elasticity is below one. For example, Oberfield and Raval (2014) find that the aggregate elasticity of substitution for the U.S. manufacturing sector is significantly below one.² If this elasticity is indeed below one, the mechanism proposed by Karabarbounis and Neiman (2014) appears to work in the wrong direction. Moreover, even if we accept that this elasticity is greater than one in the U.S., almost all of the decline in the relative price of investment goods occurred steadily from the late 1970s to 2002, a period during which the U.S. labor share is either stationary or trending slightly upwards.

This paper takes a different approach. I consider the effects of labor market conditions, specifically unemployment and workers’ reservation wage, on the U.S. labor share during the entire period 1951-2012. While undoubtedly these are not the only determinants of factor shares, the results of this paper suggest that we can go a surprisingly long way towards accounting for the behavior of the U.S. labor share at an

²See also Antras (2004). Chirinko (2008) provides a summary of estimates of the elasticity of substitution. As discussed in Rognlie (2014), the vast majority of estimates are below one.
annual frequency using labor market conditions alone. The results contrast starkly with the standard neoclassical view that factor shares are either constant or are driven primarily by the capital-output ratio or the effective capital-labor ratio.

In a companion paper, Mangin (2015), I develop a search-theoretic model of the labor market with a novel feature: the frictional process of matching between workers and firms, which gives rise to unemployment, and the production technology, are unified processes. The theory provides microfoundations for an aggregate production function that incorporates labor market frictions. Unlike Diamond-Mortensen-Pissarides style models with random matching and Nash bargaining, wages are determined by local competition between firms who compete directly to hire workers. This endogenizes labor’s share, which depends on the labor market tightness, the reservation wage, and the distribution of firm productivities. This framework is useful for examining the relationship between labor market conditions and factor shares since it yields a simple, microfounded expression relating unemployment and labor’s share.

The present paper consists of two parts. The first part is theoretical. I extend the static framework developed in Mangin (2015) to a dynamic setting with an endogenous reservation wage and provide comparative statics results for factor shares in the steady state equilibrium. The second part is quantitative. After deriving expressions for the dynamic evolution of factor income shares, I simulate the model in order to examine the theory’s predictions regarding the relationship between unemployment, workers’ reservation wage, and the U.S. labor share during the period 1951-2012.

In general, the theory predicts that labor’s share is increasing in the equilibrium labor market tightness – or decreasing in the unemployment rate – and increasing in workers’ reservation wage, which depends on the flow value of unemployment as well as future employment and wage prospects. The theory provides a very simple way of capturing the basic intuition that labor’s share should vary with the degree of competition to hire workers. When competition to hire workers is more intense, it is intuitive that labor’s share should increase. When workers struggle to find jobs during periods of high unemployment, we expect labor’s share to fall. Of course, these effects may not be instantaneous; there may be a time delay before current labor market conditions are fully reflected in labor’s share.

Rios-Rull and Santaeulalia-Llopis (2010) document some business cycle properties of factor shares in the U.S. At a quarterly frequency, the labor share falls instantaneously in response to a positive productivity shock, but it rises to a higher level one
year later and the overall effect on labor’s share is positive and persistent. While my paper does not explain the counter-cyclical behavior of labor’s share at a quarterly frequency, if we focus on movements at an annual frequency the theory’s predictions are consistent with these facts. The mechanism is clear and intuitive: when unemployment is lower, labor’s share tends to rise because greater competition to hire workers leads to an increase in their endogenous ”bargaining” position.

In the quantitative exercise, I simulate the model and compare its predictions regarding factor shares to the U.S. data from 1951-2012. To predict factor shares, I use only data on unemployment rates and unemployment insurance coverage. Remarkably, these two factors alone can deliver much of the observed variation in labor’s share without using data on the capital-output ratio at all. Labor market conditions can account for much of the behavior of factor shares in the U.S. at an annual frequency during a period of more than fifty years from 1951 to 2002. However, the very steep decline in labor’s share in the 2000s, particularly in 2002-2005, cannot be fully explained by these factors alone and I discuss some alternative explanations.

This paper contributes to a literature that highlights the role of labor market conditions in affecting factor income shares. Gomme and Greenwood (1995) argue that optimal labor contracting between entrepreneurs and workers can explain the counter-cyclical behavior of labor’s share at a quarterly frequency. Blanchard (1997) examines the possibility of medium-run shifts in labor’s share due to labor market deregulation and changes in the degree of unionization. Bentolila and Saint-Paul (2003) consider how changes in workers’ bargaining power, among other factors, might lead to shifts in the relationship between labor’s share and the capital-output ratio.

In this paper, I focus entirely on labor market frictions and abstract from the role of product market imperfections. In Blanchard and Giavazzi (2003), factor shares depend on both the degree of labor market regulation, which is captured by workers’ bargaining power, and product market regulation, which is captured by the markups of imperfectly competitive firms. When more firms enter, there is greater competition in the product market and markups fall, increasing labor’s share. Similarly, if workers’ bargaining power increases – due to an increase in unionization, for example – labor’s share increases. In the present paper, higher firm entry implies greater competition to hire workers, which simultaneously generates both lower unemployment and a higher endogenous bargaining position of workers, thereby increasing labor’s share.
Outline. The remainder of this paper is structured as follows. Section 2 develops the model and provides some comparative statics results for the steady state equilibrium. Section 3 derives expressions for the dynamic evolution of factor shares and presents the results of the quantitative exercise. Section 4 concludes.

2 Steady state

The model extends the static framework in Mangin (2015) to a simple dynamic setting with an endogenous reservation wage. The dynamic environment enables us to calibrate the model in Section 3 and examine its predictions quantitatively. In this section, I first describe the dynamic environment and then restrict our attention to steady state equilibria.

There is an infinite number of discrete time periods. In each period, there is a continuum of homogeneous risk-neutral workers of measure $L$. At the start of period $t$, there is a continuum of measure $U_t$ of unemployed workers and a continuum of risk-neutral potential firms. The measure of entering firms is $V_t$ and the ratio of entering firms to unemployed workers is $\phi_t = V_t / U_t$.

Unemployed workers choose whether to accept or reject job offers, and potential firms simultaneously make an entry decision. We will see that there is a unique reservation wage $b_t \geq 0$ such that workers accept a job offer exactly when the wage is greater than or equal to $b_t$. Given $b_t$, the ratio $\phi_t$ is pinned down by a zero profit condition. Given $\phi_t$, workers’ reservation wage $b_t$ is given by an indifference condition that equates the expected payoffs from accepting and rejecting a wage offer $b_t$.

To enter, potential firms must pay an upfront cost $C_t$, which can be interpreted as the cost of purchasing a machine. Each machine represents one unit of capital. After paying the cost $C_t$, firms privately draw a productivity level $x$ from a distribution $G(x)$.

Assumption 1. The distribution of firm productivities $G(x)$ is differentiable, has support $[x_{\min}, \infty)$ where $x_{\min} \geq 0$, a finite mean, and no mass points.

After learning their productivity $x$, firms can choose to search (or compete) to hire a worker. If $x > b_t$, the firm searches for a worker. If $x \leq b_t$, the firm exits immediately since it cannot afford to pay workers their reservation wage. For the sake of tractability, workers who are already employed cannot be targeted by firms: there is no on-the-job search. The labor market tightness is the ratio of competing firms to unemployed
workers, \( \theta_t = \phi_t(1 - G(b_t)) \). Competing firms approach unemployed workers at random so the number of firms targeting each unemployed worker during period \( t \) is a Poisson random variable with parameter \( \theta_t \).

If a firm succeeds in hiring a worker, it can produce output and earn profits until the match is destroyed. Unsuccessful firms exit.\(^3\) Firms discount future profits using a discount factor \( \beta \in (0, 1) \). Matches are destroyed at the end of each period at an exogenous rate \( \delta \in (0, 1] \). When a match is destroyed, the worker becomes unemployed and the firm exits. For simplicity, capital is destroyed when a match is destroyed.

If no firms approach an unemployed worker in period \( t \), he receives the value of non-market activity \( z_t \in [0, 1] \) and remains unemployed. If exactly one firm, with productivity \( x \), approaches a worker, then in each period until match destruction the firm produces output \( y = x \) and pays the worker his reservation wage \( b_t \) at the time of hiring \( t \). If two or more firms compete for a worker, the highest productivity firm hires the worker and in each period until match destruction it produces output \( y = x_1 \), the highest productivity, and pays the worker a wage \( x_2 \), the second-highest productivity among competing firms.

### 2.1 Steady state unemployment and output

For now, we restrict our attention to steady state equilibria. In this section, I provide expressions for unemployment and aggregate output in the steady state, taking the labor market tightness \( \theta \) and the reservation wage \( b \) as given. As we will see, the labor market tightness \( \theta \) is the key variable driving unemployment, output, and factor shares. In Section 2.2, I prove the existence and uniqueness of the steady state equilibrium \((\phi, b)\) and hence \((\theta, b)\).

**Unemployment.** The unemployment dynamics are similar to those in a standard directed search model of the labor market such as Julien, Kennes, and King (2000). Equating the inflows and outflows from unemployment in each period, we obtain steady state unemployment,

\[
U = \frac{\delta L}{1 - (1 - \delta)e^{-\theta}}.
\]

\(^3\)For simplicity, I assume that a firm’s capital is destroyed when it is unsuccessful in hiring.
The steady state unemployment rate \( u(\theta) \) is the proportion of the labor force who are initially unemployed at the start of the period but who are not approached by a firm during that period, which occurs with probability \( e^{-\theta} \). That is,

\[
(2) \\
\quad u(\theta) = \frac{\delta e^{-\theta}}{1 - (1 - \delta)e^{-\theta}}.
\]

**Aggregate output.** To obtain a tractable aggregate production function, I focus on the Pareto distribution, \( G(x) = 1 - x^{-1/\lambda} \) for \( x \in [1, \infty) \) and \( G(x) = 0 \) otherwise.\(^4\) The parameter \( \lambda \in [0, 1] \) is called the shape parameter since it governs the degree of dispersion of \( G(x) \). Both the mean and the variance of \( G(x) \) are increasing in \( \lambda \).\(^5\)

Given the reservation wage \( b \), let \( G_b(x) \) be the (possibly) truncated distribution of competing firms’ productivity levels, \( G_b(x) = \Pr(X \leq x|x \geq b) \). This distribution remains Pareto, \( G_b(x) = 1 - \left(\frac{x}{x_0}\right)^{-1/\lambda} \) where \( x_0 = \max\{1, b\} \). The minimum of this distribution is \( x_0 \geq 1 \) since only firms with productivity greater than \( b \) search for workers. The mean of the distribution \( G_b(x) \) is \( x_0/(1 - \lambda) \).

We first derive the endogenous productivity distribution across workers who are initially unemployed at the start of the period. Suppose that \( n \) firms compete to hire a given worker. The cdf of a worker’s output conditional on \( n \) firms competing for that worker is \( H(x|n) = G_b(x)^n \), the distribution of the maximum of \( n \) draws from \( G_b(x) \). To obtain the unconditional cdf \( H(x; \theta, x_0) \), the conditional cdf \( H(x|n) \) is weighted by the Poisson probability that \( n \) firms compete. For the Pareto distribution,

\[
(3) \\
\quad H(x; \theta, x_0) = \sum_{n=0}^{\infty} \frac{\theta^n e^{-\theta}}{n!} G_b(x)^n = e^{-\theta(1-G_b(x))} = \begin{cases} \\
\quad e^{-\theta}\left(\frac{x}{x_0}\right)^{-1/\lambda} & \text{if } x \in [x_0, \infty) \\
\quad e^{-\theta} & \text{otherwise}
\end{cases}
\]

Observe that in the limit as \( \theta \to \infty \), the distribution \( H(x; \theta, x_0) \) converges to a Type II Extreme Value Distribution or Fréchet distribution.\(^6\) Our focus here is not on this asymptotic distribution but instead on the exact distribution for finite \( \theta \) given by (3). This distribution automatically builds in the possibility of unemployment, since the mass point at zero with probability mass \( u(\theta) = e^{-\theta} \) represents initially unemployed

\(^4\)Gabaix (2009) and Gabaix (2014) survey applications of the Pareto distribution in economics.

\(^5\)The mean of the distribution \( G(x) = 1 - x^{-1/\lambda} \) is \( \frac{1}{1-\lambda} \) and the variance is \( \frac{\lambda^2}{(1-2\lambda)(1-\lambda)^2} \), both of which are increasing in \( \lambda \). Note that the variance is defined only for \( \lambda < 1/2 \).

\(^6\)The Fréchet extreme value distribution is found in Kortum (1997), Eaton and Kortum (1999), and Eaton and Kortum (2002).
workers who remain unemployed (i.e. who are not approached).

Steady state expected output from a new match is just the expected value of the distribution \( H(x; \theta, x_0) \). Multiplying by the measure of unemployed workers \( U \), total output from new matches \( \bar{Y} \) is given by
\[
\bar{Y} = x_0 \gamma(1 - \lambda, \theta) \theta^\lambda U
\]
where \( \gamma(s, x) \) is the Lower Incomplete Gamma Function defined by
\[
\gamma(s, x) = \int_0^x t^{s-1} e^{-t} \, dt.
\]

Aggregate output \( Y_t \) at time \( t \) equals output from new matches \( \bar{Y} \) plus output from existing matches, \( Y_t = \bar{Y} + (1 - \delta)Y_{t-1} \), so steady state output \( Y \) is given by
\[
Y = \frac{x_0 \gamma(1 - \lambda, \theta) \theta^\lambda U}{\delta}.
\]

To understand this expression better, consider the special case where \( \delta = 1 \) and \( b \leq 1 \). Since each machine represents one unit of capital, \( K = V \) in this case and \( U = L \). The aggregate production function is therefore
\[
Y = \gamma(1 - \lambda, \theta) K^\lambda L^{1-\lambda}.
\]
In the limit where \( \theta \to \infty \) and unemployment disappears, this function is asymptotically Cobb-Douglas, \( Y = \Gamma(1 - \lambda) K^\lambda L^{1-\lambda} \) where \( \Gamma(s) = \lim_{x \to \infty} \int_0^x t^{s-1} e^{-t} \, dt \), the Gamma function.\(^8\) Alternatively, if we set \( \lambda = 0 \) so that firm heterogeneity disappears but unemployment remains, we recover an urn-ball matching function, \( Y/L = 1 - e^{-\theta} \).

Our focus here is on the unified aggregate production function (4) which incorporates unemployment. Substituting in the steady state measure of unemployed workers from (1), we can determine steady state output per capita, \( y = Y/L \), as
\[
y(\theta, b) = \frac{x_0 \gamma(1 - \lambda, \theta) \theta^\lambda}{1 - (1 - \delta)e^{-\theta}}.
\]
Output per capita is increasing in \( \theta \) (taking \( b \) as given). (See Appendix A1.)

2.2 Steady state equilibrium

Consider a stationary environment where \( z_t = z \) and \( C_t = C \) for all \( t \). Taking workers’ reservation wage \( b \) as given, the level of firm entry (given by the ratio \( \phi \)) is

\(^7\)See Appendix A0 for properties of the function \( \gamma(s, x) \).

\(^8\)This limiting aggregation result is similar to Jones (2005), who derives a Cobb-Douglas aggregate production function using a Pareto distribution of ideas. Lagos (2006) also derives a Cobb-Douglas aggregate production function using the Pareto distribution in a Mortensen-Pissarides style search-theoretic model with random matching and Nash bargaining.
pinned down by the following zero profit condition:

\[ C = \frac{x_0^{-1/\lambda} (x_0 \lambda \theta \lambda^{-1} \gamma (1 - \lambda, \theta) + e^{-\theta} (x_0 - b))}{1 - \beta (1 - \delta)}, \]

where \( \theta = \phi(1 - G(b)) \), the labor market tightness. Firms who pay the cost \( C \) receive a machine with productivity \( x \) drawn from \( G(x) \). With probability \( 1 - G(b) = x_0^{-1/\lambda} \), firms search for a worker. Given that they are searching, their expected flow payoff is essentially the expected payoff for bidders in a second-price auction where each firm’s valuation of a worker’s labor equals their productivity \( x \) given by the distribution \( G_b(x) \) with minimum \( x_0 = \max\{1, b\} \). This flow payoff is discounted by the effective discount factor \( \beta (1 - \delta) \), which incorporates both the discount factor \( \beta \) and the exogenous match survival rate, \( 1 - \delta \). (See Appendix A2 for details.)

**Assumption 2.** The cost of purchasing capital is not too high:

\[ C < \frac{1}{1 - \beta (1 - \delta)} \left( \frac{1}{1 - \lambda} - z \right). \]

Assumption 2 implies that there is a critical value \( \bar{b}(\lambda, \beta, \delta, C) > z \) such that for any \( b < \bar{b} \), there exists a unique level of firm entry \( \phi > 0 \) which satisfies (6). If \( b \geq \bar{b} \), there is no firm entry and hence \( \phi = \theta = 0 \). So for any given \( b \geq 0 \), there is a unique level of firm entry \( \phi \) and we have a function \( \phi_r : \mathbb{R}^+ \to \mathbb{R}^+ \). When \( b < \bar{b} \), this function is differentiable and \( \phi_r'(b) < 0 \). Since \( \phi = 0 \) when \( b \geq \bar{b} \), \( \phi_r \) is weakly decreasing in \( b \). (See Appendix A3 and A4.)

**Reservation wage.** Let \( V^U \) be the expected value of being unemployed at the start of a period and \( V^E(x) \) be the expected value of being employed at wage \( x \). We have:

\[ V^E(x) = x + \beta ((1 - \delta) V^E(x) + \delta V^U), \]

\[ V^U = e^{-\theta} (z + \beta V^U) + (1 - e^{-\theta}) V^E(\bar{w}), \]

where \( \theta = \phi(1 - G(b)) \) and \( \bar{w} \) is the expected wage for new matches. If a worker is employed at wage \( x \) in the current period, he receives a flow payoff \( x \). If the match survives, he receives the discounted expected value of being employed at wage \( x \) in the next period. If the match is destroyed, he receives the discounted expected value of being unemployed at the start of the next period. If a worker is unemployed at
the start of the current period, there are two possibilities. With probability $e^{-\theta}$, he remains unemployed and receives a flow payoff $z$ plus the discounted value of being unemployed at the start of the next period. With probability $1 - e^{-\theta}$, the worker is employed at an expected wage of $\bar{w}$. (See Appendix A5 for details.)

When offered a job paying wage $x$, a worker can either accept it and receive $V^E(x)$ or reject it and remain unemployed, in which case he receives $z + \beta V^U$. Unemployed workers will accept a job offer at wage $x$ if $V^E(x) \geq z + \beta V^U$ and reject it otherwise. For any given $\phi \geq 0$, there exists a unique reservation wage $b$ such that workers will accept a job offer if and only if the wage offered is greater than or equal to $b$. So we have a function $b_r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $b'_r(\phi) > 0$ for all $\phi$. (See Appendix A5 and A6.)

The reservation wage $b$ satisfies the following indifference condition,

$$V^E(b) = z + \beta V^U. \tag{8}$$

Substituting $V^E(b)$ and $V^U$ from (7) into (8), the steady state reservation wage is

$$b = \frac{z(1 - \beta(1 - \delta)) + \beta(1 - \delta)w(\theta, b)}{1 - \beta(1 - \delta)e^{-\theta}}, \tag{9}$$

where $w(\theta, b) = (1 - e^{-\theta})\bar{w}$, the expected wage for all workers who are initially unemployed (including those who remain unemployed). It is clear from (9) that when $\phi = \theta = 0$, we have $b_r(\phi) = z$, the value of non-market activity. Since $b'_r(\phi) > 0$, we have $b_r(\phi) \geq z$ for all $\phi$.

A steady state equilibrium is a pair $(\phi, b)$ that simultaneously satisfies the zero profit condition (6) and workers’ indifference condition (8), as well as condition (1) for steady state unemployment. We know that for any given $\phi \geq 0$, (9) has a unique solution $b_r(\phi)$. At the same time, we know that for any given $b < \bar{b}$, equation (6) has a unique solution $\phi_r(b)$, and if $b \geq \bar{b}$ we have $\phi_r(b) = 0$. The function $b_r$ is increasing in $\phi$, and $\phi_r$ is decreasing in $b$. Hence there exists a unique steady state equilibrium $(\phi^*, b^*)$ that satisfies both (9) and (6), and in the limit as $t \rightarrow \infty$ the economy also satisfies (1).

Proposition 1 contains some comparative statics results regarding the effects of the key parameters – the value of non-market activity $z$, the cost of purchasing capital $C$, and the shape parameter $\lambda$ from the underlying productivity distribution – on the equilibrium $(\phi^*, b^*)$. Using the steady state expressions (2) and (5), we can infer some comparative statics results regarding the steady state equilibrium unemployment, $u^* = u(\theta^*)$, and output per capita, $y^* = y(\theta^*)$. 

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Proposition 1. There is a unique steady state equilibrium $(\phi^*, b^*)$ where $z \leq b^* < \bar{b}$. If $b^* < x_{\min}$, then: (i) workers’ reservation wage $b^*$ is increasing in $\lambda$ and $z$, and decreasing in $C$; (ii) the labor market tightness $\theta^*$ is decreasing in $z$ and $C$, and increasing in $\lambda$; (iii) the unemployment rate $u^*$ is increasing in both $z$ and $C$, and decreasing in $\lambda$; and (iv) output per capita $y^*$ is decreasing in $z$ and $C$, and increasing in $\lambda$.

Proof. See Appendix A7. 

In Proposition 1, we focus on the case where $b^* < x_{\min} = 1$ and $\theta^* = \phi^*$. We will see in Section 2.3 that this is the interesting case regarding the behavior of factor shares, since factor shares turn out to be constant when $b^* \geq x_{\min}$. It is possible to impose a restriction on the parameters to ensure $b^* < x_{\min}$ by making the upper bound $\bar{b} < 1$. However, we do not make this restriction in order to retain the greatest generality.

2.3 Factor income shares

Steady state expected wages for all workers who are initially unemployed is

\[ w(\theta, b) = x_0(1 - \lambda)\theta^\lambda \gamma(1 - \lambda, \theta) - (x_0 - b)\theta e^{-\theta}. \]

This is just the expected output of a new match minus the expected flow payoff for a firm that is successful in hiring. In the steady state, average wages across employed workers is the expected wages for initially unemployed workers, given by (10), conditional on employment, $w(\theta, b)/(1 - e^{-\theta})$, which is just $\bar{w}$. Multiplying $\bar{w}$ by the proportion of workers who are employed in the steady state, $1 - u(\theta)$, and then dividing by output per capita $y(\theta, b)$, we obtain labor’s share:

\[ s_L = 1 - \lambda - \left(1 - \frac{b}{x_0}\right)\varepsilon(1 - \lambda, \theta), \]

where $x_0 = \max\{1, b\}$ and $\varepsilon(s, x)$ is the elasticity of $\gamma(s, x)$ with respect to $x$.

Steady state capital share is defined as $s_K \equiv 1 - s_L$, given by

\[ s_K = \lambda + \left(1 - \frac{b}{x_0}\right)\varepsilon(1 - \lambda, \theta). \]

\[ ^9\text{Specifically, } \bar{b} < 1 \text{ and therefore } b^* < 1 \text{ if } \frac{\lambda}{(1-\beta(1-\theta))(1-\lambda)} < C. \text{ See Appendix A3 for details.} \]

\[ ^{10}\text{See Appendix A0 for properties of the function } \varepsilon(s, x). \]
In general, steady state capital share is decreasing in the labor market tightness ratio $\theta$ (taking $b$ as given) since the elasticity $\varepsilon(1 - \lambda, \theta)$ is decreasing in $\theta$. Intuitively, as the ratio of competing firms to unemployed workers leads to an increase in labor’s share. Since greater competition for workers – reflected in a higher labor market tightness – also leads to a lower unemployment rate $u(\theta)$, this gives rise to a negative relationship between unemployment and labor’s share (taken $b$ as given). At the same time, capital share is decreasing in $b$ (taking $\theta$ as given) since in matches where firms face no competition, workers are paid more when the reservation wage is higher.

Steady state factor shares are constant in two distinct cases. First, there is an asymptotic result: factor shares are constant in the limit as $\theta \to \infty$ and unemployment disappears. Second, factor shares are constant when workers’ reservation wage equals or exceeds the minimum firm productivity.

When two or more firms compete for a worker, the successful firm receives on average the expected value of the productivity difference between the two highest productivity firms, which is a constant share of match output. In the first case, as $\theta \to \infty$, all firms face direct competition and hence we have constant factor shares. In the second case, $b$ equals or exceeds the minimum productivity for entering firms, $x_{\text{min}} = 1$. In matches where workers are paid their reservation wage, firms receive on average the expected value of the difference between a single productivity draw from the distribution $G_b(x)$ and its minimum value, $x_0 = b$. Since $G_b(x)$ is again Pareto, this is a constant share of match output and we have constant factor shares.

Only the second case can arise as an equilibrium outcome. However, since the reservation wage is endogenous, it is not clear a priori whether this case is empirically relevant. In Section 3, when we calibrate the model, it turns out that the endogenous value of the reservation wage is always less than one and hence factor shares are variable not constant. Importantly, this is a result, not an assumption.

**Comparative statics.** For the interesting case where $b^* < 1$, the steady state equilibrium capital share is $s_{K}^* = \lambda + (1 - b^*) \varepsilon(1 - \lambda, \theta^*)$, where $b^*$ is the reservation wage and $\theta^*$ is the equilibrium labor market tightness. How do equilibrium factor shares respond to a change in the cost of purchasing capital? Suppose there is an increase in $C$. There is an indirect effect on capital share through the equilibrium ratio $\theta^*$. A higher

\[11\]This is because as $\theta \to \infty$ and hence $\phi \to \infty$, we have $b_\varepsilon(\phi) > \bar{b}$ but $\phi_\varepsilon(b) = 0$ when $b > \bar{b}$. 

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cost $C$ means that entry is less attractive for firms, which decreases the equilibrium ratio $\theta^*$. This has a positive effect on capital share since there is less competition for workers. Overall, equilibrium capital share is increasing in the cost $C$.

**Proposition 2.** The equilibrium labor share $s_L^*$ is increasing in the value of non-market activity $z$ and decreasing in the cost of purchasing capital $C$.

*Proof.* See Appendix A8.

We can also compare two economies with different values of non-market activity $z$. When $z$ increases, the reservation wage $b^*$ increases and there is a negative effect on capital share. There is also a positive indirect effect on capital share through the labor market tightness ratio $\theta^*$, since a higher $z$ decreases the level of firm entry, which has a positive impact on capital share. Overall, the negative effect dominates and capital share is decreasing in the value of non-market activity. The effect of the shape parameter $\lambda$ from the distribution $G(x)$ is ambiguous when $b^* < 1$.

### 3 Quantitative exercise

In this section, I examine one of the model’s predictions quantitatively by asking the following question: Through the lens of the model, can variations in unemployment rates and the value of non-market activity account for the behavior of factor shares in the U.S. over the period 1951-2012? To answer this question, I calibrate the model and use annual data on unemployment rates and unemployment insurance (UI) coverage to predict the movements in factor shares during this period.

I use unemployment data to directly pin down the labor market tightness ratio $\theta_t$ for each period, in order to match the unemployment rates exactly. I therefore abstract from the well-known difficulties that search models have in generating fluctuations in unemployment that are as volatile as in the data. This enables me to examine the relationship between unemployment and factor shares predicted by the model, taking the fluctuations in unemployment as exogenous.

#### 3.1 Unemployment, output, and factor shares

For the purpose of this exercise, consider an environment similar to that described in Section 2, except for the following key difference: a sequence of labor market tightness
ratios $\{\theta_t\}$ is exogenously given.\textsuperscript{12} Given the sequence of labor market tightness ratios $\{\theta_t\}$, there is a unique reservation wage $b_t \geq 0$ for each period $t$. First, I present expressions for the dynamics of unemployment, output, and factor shares, taking a sequence $\{(\theta_t, b_t)\}$ as given. I then discuss how to determine the sequence of reservation wages $\{b_t\}$ given the sequence $\{\theta_t\}$.

**Unemployment.** Given an initial value for the measure of unemployed workers, $U_0$, the evolution of unemployment is given by the following:

\begin{equation}
U_{t+1} = U_t - (1 - e^{-\theta_t})U_t + \delta(L - U_t + (1 - e^{-\theta_t})U_t).
\end{equation}

The measure of unemployed workers at the start of period $t + 1$ equals the measure who were unemployed at the start of period $t$, minus the unemployed who found jobs in period $t$, plus the matches that were destroyed in period $t$. Simplifying, for all $t \geq 0$

\begin{equation}
U_{t+1} = (1 - \delta)e^{-\theta_t}U_t + \delta L.
\end{equation}

The unemployment rate $u_t$ for period $t$ is given by

\begin{equation}
u_t = \frac{U_t e^{-\theta_t}}{L}.
\end{equation}

**Aggregate output.** Let $G_{bt}(x)$ be the (possibly) truncated distribution of competing firms’ productivity levels, $G_{bt}(x) = \Pr(X \leq x | x \geq b_t)$. This distribution is Pareto, $G_{bt}(x) = 1 - \left(\frac{x}{x_{0t}}\right)^{-1/\lambda}$ where $x_{0t} = \max\{1, b_t\}$. The total output $\bar{Y}_t$ produced by new matches during period $t$ is the expected value of the endogenous productivity distribution, $H(x; \theta_t, x_{0t}) = e^{-\theta_t(x_{0t})^{-1/\lambda}}$, multiplied by $U_t$,

\begin{equation}
\bar{Y}_t = x_{0t} \gamma (1 - \lambda, \theta_t) \theta_t U_t.
\end{equation}

\textsuperscript{12}Assuming the equilibrium condition for firm entry holds for all periods $t$, we could of course back out the implied sequence of values for the cost of purchasing capital, $\{C_t\}$, which could be interpreted as investment-specific shocks. Alternatively, we could introduce productivity shocks and back out the implied sequence of such shocks necessary to generate the observed sequence of unemployment rates.
Aggregate output $Y_t$ at time $t$ can be decomposed into output from new matches $\tilde{Y}_t$ plus output from existing matches. Given an initial value for $Y_0$, for all $t \geq 0$ we have

$$Y_{t+1} = \tilde{Y}_{t+1} + (1 - \delta) Y_t.$$  

**Factor income shares.** Factor shares are a *weighted average* of the income shares in both new matches and previously formed matches that are still active. To start with, expected wages in period $t$ for initially unemployed workers is

$$w(\theta_t, b_t) = x_{0t}(1 - \lambda)\gamma(1 - \lambda, \theta_t) - (x_{0t} - b_t)\theta_t e^{-\theta_t}.$$  

Multiplying expected wages (18) by $U_t$, we obtain the total wages for workers who are newly employed. Dividing total wages by $\tilde{Y}_t$, the total output from new matches given by (16), labor’s share for *new* matches at time $t$ is

$$\tilde{s}_{L,t} = 1 - \lambda - \left(1 - \frac{b_t}{x_{0t}}\right) \varepsilon(1 - \lambda, \theta_t).$$  

Capital’s share in *new* matches, defined by $\tilde{s}_{K,t} \equiv 1 - \tilde{s}_{L,t}$, is therefore

$$\tilde{s}_{K,t} = \lambda + \left(1 - \frac{b_t}{x_{0t}}\right) \varepsilon(1 - \lambda, \theta_t).$$

Given an initial value, $s_{K,0}$, capital’s overall share at time $t + 1$ is

$$s_{K,t+1} = \frac{\tilde{s}_{K,t+1} \tilde{Y}_{t+1} + s_{K,t}(1 - \delta)Y_t}{Y_{t+1}},$$

for all $t \geq 0$, where $\tilde{s}_{K,t+1}$ is weighted by new matches’ proportion of output, $\tilde{Y}_{t+1}/Y_{t+1}$, and $s_{K,t}$ is weighted by the proportion of output produced by "old" matches.

**Reservation wage.** The reservation wage $b_t$ depends on both current labor market conditions, given by $\theta_t$ and $z_t$, and expectations of future conditions. For this quantitative exercise, I present two extreme cases with regard to workers’ expectations. In the myopic equilibrium, workers expect current labor market conditions to continue in the future. That is, $E_t(\theta_{t+s}) = \theta_t$ and $E_t(z_{t+s}) = z_t$ for all $t, s \geq 0$. In the perfect foresight equilibrium, there is no uncertainty about the future values of $\theta_t$ and $z_t$. That is, $E_t(\theta_{t+s}) = \theta_{t+s}$ and $E_t(z_{t+s}) = z_{t+s}$ for all $t, s \geq 0$.  

14
Let $V_t^E(x)$ be the expected value of being employed at wage $x$ in period $t$, and let $V_t^U$ be the expected value of being unemployed at the start of the period $t$. For each time $t$, the reservation wage $b_t$ satisfies

$$V_t^E(b_t) = z_t + \beta V_{t+1}^U,$$

where

$$V_t^E(x) = x + \beta((1 - \delta)V_{t+1}^E(x) + \delta V_{t+1}^U),$$

$$V_t^U = e^{-\theta_t}(z_t + \beta V_{t+1}^U) + (1 - e^{-\theta_t})V_t^E(\bar{w}_t),$$

and $\bar{w}_t$ is the expected wage for new matches at time $t$.

### 3.2 Strategy

Suppose that $z_t = z$ and $\theta_t = \theta$ from time $T$ onwards. For all $t \geq T$, there exists a unique reservation wage $b_T = b \in \mathbb{R}^+$ given by (9) and hence we can determine the value of unemployment, $V^U(\theta, b, z)$, in the end steady state.

Taking the sequence of unemployment rates $\{u_t\}_{t=0}^T$ from the data, we can determine the corresponding sequence $\{\theta_t\}_{t=0}^T$ using (14) and (15). Using this sequence $\{\theta_t\}_{t=0}^T$ we can calculate the entire sequence of reservation wages $\{b_t\}_{t=0}^T$ by working backwards from period $T$ using (22), provided we can also determine $\{z_t\}_{t=0}^T$. Given the sequence $\{(\theta_t, b_t)\}_{t=0}^T$, a sequence for aggregate output $\{Y_t\}_{t=0}^T$ is obtained using (17) and (16), and finally a sequence for capital share $\{s_{K,t}\}_{t=0}^T$ is obtained using (20) and (21).

**Value of non-market activity.** In the search literature, it is standard to think of the flow payoff from unemployment $z_t$ in terms of a "replacement rate", i.e. the value of non-market activity as a percentage of average wages. Often, a single value for this replacement rate is chosen and the issue of eligibility for benefits is ignored. The exact conditions governing eligibility for UI in the U.S. are complex and vary across states. There are many factors, such as an individual’s work history, that determine eligibility. I abstract from many of these factors and consider just one important factor – the UI coverage rate – which has a strong time trend until 1978.

Over the post-war period, the after-tax replacement rate for insured workers has been relatively constant (Anderson and Meyer (1997)). However, the UI coverage rate has increased dramatically in the U.S. In 1948, only 59% of workers were covered by UI, compared with 92% in 1978.\footnote{Economic Report of the President (2009, 1983)} As Figure I indicates, while UI coverage increased steadily from 1948 until 1978, it has been relatively stable since then.
For this quantitative exercise, I define the value of non-market activity $z_t$ in a way that incorporates UI coverage. It is important to use the UI coverage rate, not the actual rate of receipt of UI benefits, since a worker’s outside option depends on whether or not he could potentially receive UI benefits if necessary. Actual receipt of UI benefits depends on both eligibility and take-up rates, which are affected by many different factors, including cultural and political factors that vary significantly across states, as discussed in Blank and Card (1991).

I start by choosing a target level $\alpha$ for the average value of non-market activity as a percentage of average wages throughout the entire period. This enhances comparability with the search-theoretic literature, where a single value for $\alpha$ is generally chosen. The value of non-market activity is defined as $z_t \equiv \eta p_t w_{t-1}^{\alpha}$ where $p_t$ is the probability of being eligible for UI, $\eta$ is a "normalization" parameter that enables us to match the target $\alpha$, and $w_{t-1}^{\alpha}$ is average wages for workers employed in the previous period.

The value of $p_t$ is determined by both UI coverage and long-term unemployment.
In particular, define \( p_t = e_t(1 - LTU_t) \), where \( LTU_t \) is the long-term unemployment rate and \( e_t \) is the UI coverage rate. The inclusion of long-term unemployment is a simple proxy for the fact that covered workers who are long-term unemployed may lose eligibility for UI due to exhaustion of benefits. Greater incidence of long-term unemployment decreases the average value of non-market activity \( z_t \) by decreasing \( p_t \).

**Data sources.** The factor shares data source is the Bureau of Labor Statistics Multifactor Productivity (BLS MFP) historical release (updated in August 2014) which covers the period 1948-2012. This data is derived by the BLS from the NIPA. The BLS MFP release provides data for the private business sector, which excludes both general government and government enterprises.\(^{14}\) As discussed in detail in Elsby, Hobijn, and Şahin (2013), the BLS MFP measure is preferable to the alternative measure found in the BLS Labor Productivity and Costs release due to the latter’s inadequate treatment of self-employment income, which leads to a spurious decline in labor’s share.\(^ {15}\)

The unemployment data is the BLS civilian annual average unemployment rate from 1948-2013. The long-term unemployment rate is the BLS measure of the annual average number of unemployed workers who are unemployed for 27 weeks or longer as a proportion of the total unemployed from 1948-2013. The UI coverage rate is the proportion of employed workers in the civilian labor force who are eligible for UI benefits. This data is obtained from the Economic Report of the President (2009, 1983) and covers the period 1948-2007.\(^ {16}\)

**Calibration.** I set the initial conditions \( U_0, Y_0, \) and \( s_{K,0} \) to match the unemployment rate and capital’s share in 1948. From 1949 onwards, I use the unemployment data to generate predictions for factor shares, but I test the model’s predictions only for 1951 onwards. At the end of the period 1948-2013, I assume a long-term unemployment rate of 15.0% and that the economy converges to a steady state equilibrium with an unemployment rate of 5.83% (both long-run averages for 1948-2013). Due to the time

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\(^{14}\)Labor’s share is labor compensation divided by the total of labor and capital cost. Labor compensation is defined as wages and salaries of employees plus employers’ contributions for social insurance and private benefit plans, and all other fringe benefits. An estimate of the wages, salaries, and supplemental payments of the self-employed is included.

\(^ {15}\)One limitation of using the BLS MFP data is that it is only available annually, not quarterly.

\(^ {16}\)Through 1996, UI coverage includes the following programs: State, Unemployment Compensation for Federal Employees (UCFE), Railroad Retirement Board (RRB), and Unemployment Compensation for Ex-Servicemembers (UCX). From 1997, it includes only State and UCFE programs.
trend, the coverage rate $e_t$ equals the 2007 level in the end steady state. The results are not sensitive to these assumptions.

I set $\beta = 0.95$, $\delta = 0.45$, and $\alpha = 0.5$. The annual match destruction rate represents a monthly rate of 3.75%. The replacement rate of $\alpha = 0.5$ is relatively standard in the search literature; in Shimer (2005), a value of 0.4 is chosen. After setting $\delta$, $\beta$ and $\alpha$, I simultaneously choose a value for $\lambda$ so that the model’s predictions match the mean capital share in the data, a value for $\eta$ to match the target $\alpha$, and a sequence $\{\theta_t\}_{t=1}^T$ to match the unemployment rates for 1948-2013.

In Section 3.3, I concentrate on the period 1951-2002. As we will see, labor’s share experienced a very sharp fall in around 2002-2005 when it decreased by four percentage points from 0.674 to 0.634. For the period 1951-2002, however, factor shares are stationary and hence we focus on this period for now.18

### 3.3 Results

Tables I and II present the results of the baseline calibration. The endogenous value of workers’ reservation wage is always less than one, hence factor shares are variable. For completeness, I present results for both 1951-2002 and the full period 1951-2012. Notice that the mean and standard deviation of capital’s share are much higher for the period 1951-2012 due to the dramatic fall in labor’s share in the 2000s. For now, our primary focus is on the period 1951-2002 in which labor’s share is stationary.

During 1951-2002, the correlation between the data and the model’s predictions is 0.653 when workers are myopic and 0.556 for the perfect foresight equilibrium. The standard deviation of capital’s share is lower than in the data, 0.0070 when workers are myopic and 0.0064 for the perfect foresight equilibrium compared with 0.0101 in the data. The autocorrelation of the model’s predictions for capital’s share is significantly higher than in the data, 0.828 for the myopic model and 0.884 for the perfect foresight equilibrium compared with 0.567 in the data.

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17This includes both job-to-job transitions and transitions from employment into unemployment. Job-to-job transitions are not explicit in the model but correspond to cases where a worker loses their job at the end of one period and instantly gains another at the start of the next period without going through any spell of unemployment. Estimates of the job separation rate, including both quits and layoffs, are usually between 3-5% monthly or 35-60% annually. Davis, Faberman, Haltiwanger, and Rucker (2010) discuss the downward bias in the published JOLTS separations rate and provide an adjusted estimate of 4.65% monthly (quits plus layoffs).

18An augmented Dickey-Fuller unit root test rejected non-stationarity for 1951-2002 at the 1% level.
## 1951-2002 Data Model

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model $s_{K,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean capital share</td>
<td>0.341</td>
<td>0.341</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.0101</td>
<td>0.0070</td>
</tr>
<tr>
<td>Correlation b/n model and data</td>
<td>1.000</td>
<td>0.653</td>
</tr>
<tr>
<td>Autocorrelation (1 period lag)</td>
<td>0.567</td>
<td>0.828</td>
</tr>
</tbody>
</table>

Table I: Capital Share - Data, Myopic Model, Perfect Foresight, 1951-2002

## 1951-2012 Data Model

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model $s_{K,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean capital share</td>
<td>0.345</td>
<td>0.345</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.0148</td>
<td>0.0093</td>
</tr>
<tr>
<td>Correlation b/n model and data</td>
<td>1.000</td>
<td>0.643</td>
</tr>
<tr>
<td>Autocorrelation (1 period lag)</td>
<td>0.748</td>
<td>0.789</td>
</tr>
</tbody>
</table>

Table II: Capital Share - Data, Myopic Model, Perfect Foresight, 1951-2012

At least part of the discrepancy in the standard deviation and autocorrelation is likely due to measurement error. Part of it may also be due to the simplifying assumptions of the model, such as the fact that wages for existing matches cannot be re-negotiated and there is no on-the-job search: only new matches respond to current labor market conditions, leading to lower volatility in factor shares.

Figure II compares the model’s predictions for labor’s share when workers are myopic with the U.S. data for the full period 1951-2012. For the perfect foresight equilibrium, the fit is similar except that the model’s predictions are smoothed out because current conditions do not have quite as strong an impact on factor shares.

In the model, labor’s share is driven by both unemployment and workers’ reservation wage, which is in turn influenced by the value of non-market activity $z_t$. As Figure I shows, the UI coverage rate has a strong upward trend during 1951-1977 and is stable from 1978 onwards. While labor’s share is relatively stable during the period 1951-1977, as Figure II shows, this is because the upward trend in the value of non-market activity $z_t$ during 1951-1977 is at least partially offset by an upward trend in unemployment during this period. From 1978 onwards, almost all of the variation in the model’s predictions for labor’s share is driven by unemployment. From the late
1970s to the early 2000s, unemployment trended downwards in the U.S. while labor’s share trended slightly upwards, consistent with the theory’s predictions.

**Discussion.** Leaving aside the model for a moment and looking at the raw data on capital share and unemployment for 1951-2002, one might think that there is no relationship between factor shares and unemployment. The correlation between capital share and the unemployment rate is 0.049 for this period at an annual frequency. If we consider the two periods 1951-1977 and 1978-2002 separately, however, the relationship appears very different. For the earlier period, 1951-1977, the correlation between capital share and the unemployment rate is -0.129. For the later period, 1978-2002, the correlation between the unemployment rate and capital share is 0.468. This suggests that the relationship between unemployment and labor’s share has changed over time.

The model predicts that significant changes in UI coverage during the period 1951-1977, and hence in the value of non-market activity $z_t$, will affect the relationship between unemployment and labor’s share. During periods in which $z_t$ exhibits no trend, such as 1978-2002, the relationship between unemployment and labor’s share
should be more apparent in the data. This is consistent with what we observe.

Importantly, the model predicts that factor shares depend not only on the current unemployment rate and reservation wage, but also on *previous* periods’. Factor shares are a weighted average of the income shares in both new matches and earlier matches that are still active. The relationship between labor’s share and lags of unemployment may therefore be crucial to the model’s ability to match the data.

Table III presents the correlations in the data between capital share $s_{K,t}$ and lags of unemployment $u_{t-l}$ for both time periods. During the early period 1951-1977, there is a weak correlation of 0.301 between capital share $s_{K,t}$ and lagged unemployment $u_{t-1}$. During the later period 1978-2002, there is a much stronger correlation of 0.700 between $s_{K,t}$ and $u_{t-1}$. The fact that the relationship is stronger during 1978-2002 is again consistent with the model’s predictions.

<table>
<thead>
<tr>
<th>Correlation, data $s_{K,t}$ and $u_{t-l}$</th>
<th>1951-1977</th>
<th>1978-2002</th>
</tr>
</thead>
<tbody>
<tr>
<td>lag $l$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-0.129</td>
<td>0.468</td>
</tr>
<tr>
<td>1</td>
<td>0.301</td>
<td>0.700</td>
</tr>
<tr>
<td>2</td>
<td>0.235</td>
<td>0.742</td>
</tr>
<tr>
<td>3</td>
<td>0.208</td>
<td>0.609</td>
</tr>
</tbody>
</table>

Table III: Correlations between capital share and unemployment in the data

**Incorporating a time delay.** As Table III highlights, in both the earlier and later periods the relationship between lagged unemployment and capital’s share is stronger than the contemporaneous one. This suggests that the model’s fit, particularly for the period 1951-1977, may be improved by incorporating a time delay.19

In the model, all workers and all firms instantly observe the current labor market tightness and it is immediately reflected in the current period’s labor share. This is perhaps an unrealistic assumption. In reality, it is likely that informational frictions impede the ability of agents to instantaneously observe current labor market conditions. Workers and firms may take time to learn about labor market conditions.

19 If we consider the correlation between the cyclical component of capital share and the cyclical component of unemployment at annual frequencies (HP filter 6.25), the presence of the time lag is even more apparent. For the period 1951-2002, the contemporaneous correlation between the cyclical components is 0.018, but when unemployment is lagged by one year the correlation is 0.539. Labor’s share appears to lag unemployment by one year, consistent with the finding of Rios-Rull and Santaelulalia-Llopis (2010) that labor’s share lags output by one year.
While we abstract from the details of modelling this process, we can consider how the model’s fit is affected if labor market conditions take one year to be reflected in labor’s share. To be precise, suppose that capital’s share in new matches, \( s_{K,t} \), depends on the previous year’s labor market tightness and reservation wage, \( \theta_{t-1} \) and \( b_{t-1} \). This shifts the model’s predictions for capital share \( s_{K,t} \) forward by one year, i.e. the new prediction for capital share is \( \hat{s}_{K,t} = s_{K,t-1} \) for all \( t \). Everything else is the same as in the baseline calibration. Table IV and Figure III present the results of this exercise.

![Figure III: Labor Share - Data vs Model 2 (shifted), Myopic workers, 1951-2012](image)

### Table IV: Capital Share - Data, Myopic Model (shifted) Perfect Foresight (shifted), 1951-2002

<table>
<thead>
<tr>
<th>1951-2002</th>
<th>Data</th>
<th>Model 2, ( \hat{s}_{K,t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean capital share</td>
<td>0.341</td>
</tr>
<tr>
<td></td>
<td>Standard deviation</td>
<td>0.0101</td>
</tr>
<tr>
<td></td>
<td>Correlation b/n model and data</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>Autocorrelation (1 period lag)</td>
<td>0.567</td>
</tr>
</tbody>
</table>

Table IV: Capital Share - Data, Myopic Model (shifted) Perfect Foresight (shifted), 1951-2002
The results in Table IV appear significantly stronger than Table I. While the standard deviation is similar, the correlation between the data and the model’s predictions for capital share increases significantly to 0.761 when workers are myopic and 0.730 for the perfect foresight equilibrium. If we consider the two periods 1951-1977 and 1978-2002 separately, a clear picture emerges of why this is the case.

<table>
<thead>
<tr>
<th>Model, ( s_{K,t} )</th>
<th>Myopic model</th>
<th>Perfect foresight</th>
<th>Myopic model</th>
<th>Perfect foresight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1951-1977</td>
<td>0.522</td>
<td>0.345</td>
<td>0.748</td>
<td>0.668</td>
</tr>
<tr>
<td>1978-2002</td>
<td>0.725</td>
<td>0.674</td>
<td>0.751</td>
<td>0.756</td>
</tr>
<tr>
<td>1951-2002</td>
<td>0.653</td>
<td>0.556</td>
<td>0.761</td>
<td>0.730</td>
</tr>
</tbody>
</table>

Table V: Correlations between model’s predictions for capital share and data

As Table V indicates, the fit of the model improves dramatically for the earlier period 1951-1977 when we incorporate a time delay of one year, but only slightly improves for the later period 1978-2002. Looking at Table III, the stronger effect on the earlier period is what we might expect given the changing relationship between unemployment and labor’s share across these two periods.

3.4 What happened to labor’s share in the 2000s?

A striking feature of Figures II and III is the sharp fall in labor’s share in the early 2000s. In particular, the dramatic fall in around 2002-2005 is the main feature of the data that the model’s predictions fail to capture. Labor’s share decreased by four percentage points from 0.674 in 2002 to 0.634 in 2005. Despite the model’s relatively good fit from 1951 to 2002, this episode is not predicted by the theory. Through the lens of the model, it certainly appears that something very unusual happened to factor shares in the U.S. in the early 2000s which cannot be easily reconciled with their behavior through the entire post-war period prior to this point.

Oberfield and Raval (2014) suggest that the steep decline in the 2000s in the U.S. manufacturing labor share is due to an acceleration in the bias of technical change. The "bias of technical change" is defined as a residual that incorporates a multitude of different factors. Elsby, Hobijn, and Şahin (2013) identify globalization and the increased off-shoring of labor-intensive tasks as one factor that may explain part of the recent decline in labor’s share. The authors show that U.S. industries with greater
import exposure experienced greater labor share declines. While this is undoubtedly an important factor influencing the recent behavior of the U.S. labor share, it seems unlikely that globalization can explain the particularly steep fall in labor’s share during 2002-2005 since globalization is a relatively slow process, not a sudden change.

Karabarbounis and Neiman (2014) argue that the declining relative price of investment goods is responsible for the global decline in the corporate labor share since 1980. However, almost all of the decline in the relative price of investment goods in the U.S. occurred steadily over the period 1975-2002, a period during which the U.S. labor share is either stationary or trending slightly upwards.\(^{20}\) This mechanism cannot therefore explain the sharp fall in the U.S. labor share in the early 2000s, the period in which most of the decline took place.

Employee stock options may be relevant for the recent behavior of the U.S. labor share. Stock options are counted as labor income at the time of exercise, and the rate at which these options are exercised varies dramatically due to stock market fluctuations. Moylan (2008) argues that from the late 1990s onwards, this issue has had a quantitatively significant impact on official measures of labor’s share. During periods of high stock prices when options are exercised at a higher rate, labor’s share will appear higher. In particular, Elsby, Hobijn, and Şahin (2013) find that roughly half of the rise and fall of the U.S. labor share between 1998 and 2003 is due to a small number of industries in the investment banking and IT sectors. Once this is eliminated, the sharp fall in labor’s share in around 2002-2005 may appear less dramatic.

A deeper examination of the changes in factor shares at a disaggregated level reveals that a small number of industries that are heavily leveraged to commodity prices experienced by far the most dramatic increases in capital share during the 2000s, with a particularly sharp rise during the period 2002-2005. These industries are Petroleum and Coal Products (324) and Primary Metals (331) from the non-durable manufacturing sector; and Mining (21) from the non-manufacturing sector.\(^{21}\) I refer to this small but highly volatile bundle of industries as the PCMM industries.

The PCMM industries have an overall capital share of 61.8% in 2002, which rises dramatically to 78.3% in 2005. This is likely due to the steep rise in global commodity prices at this time. These industries already have a much higher capital share than

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\(^{20}\)In the BLS MFP data, labor’s share trends upwards during this period. In the updated data set from Karabarbounis and Neiman (2014), the U.S. corporate labor share is stationary from 1975-2002. The data on the relative price of investment goods in the U.S. is taken from the same data set.

\(^{21}\)The data source is the Multifactor Productivity and Related KLEMS Measures from the NIPA Industry Database, 1987 to 2012.
the aggregate and they jump from just 2.6% of total value added in the U.S. in 2002 to 4.3% in 2005. There are therefore two factors that could potentially contribute to the rise in the aggregate capital share: the increase in capital’s share within these industries; and the greater weighting of these industries in total value added.

Could this small cluster of industries be responsible for the sharp jump in the aggregate capital share in the U.S. in 2002-2005? To answer this question, I exclude the PCMM industries from the calculation of capital share in the BLS MFP data from 1987 onwards. The adjusted capital share jumps 2.9 percentage points during 2002-2005 compared with 4.0 percentage points for the unadjusted data. While labor’s share still declines during this period when we exclude these industries, the adjustment eliminates roughly 30% of the decline from 2002 to 2005.

Clearly, both commodity prices and stock options are factors that lie outside the model’s scope. Both appear to have had a quantitatively significant impact on the U.S. labor share in the 2000s. It seems likely that focussing on a "core" measure of labor share that eliminates the excess volatility due to both the PCMM industries and the exercise of stock options would reduce much – although not all – of the sharp decline in labor’s share during the 2000s. A "core" measure of the U.S. labor share for the full period 1951-2012 may well behave in a more similar way to labor’s share during the period 1951-2002, in which it appears to be both stationary and driven primarily by labor market conditions, particularly unemployment.

4 Conclusion

This paper presents a novel theoretical framework that provides microfoundations for a negative relationship between unemployment and the labor share. The quantitative results suggest that labor market conditions such as the unemployment rate and workers’ reservation wage are the key drivers of the behavior of factor shares at an annual frequency. While undoubtedly these are not the only determinants of income shares, these two factors alone can account for a high degree of the variation in factor shares in the U.S. during a period of more than fifty years from 1951 to 2002 without using any data on the capital-output ratio at all.

The success of the model’s predictions for the entire period 1951-2002 highlights the fact that something very unusual happened to the U.S. labor share during the 2000s. While the very recent decline in around 2009-2012 can be attributed to higher
unemployment, as the model predicts, the sharp fall in around 2002-2005 cannot be fully explained by changing labor market conditions. Instead, this decline appears to have been influenced by factors outside the model, such as commodity prices and employee stock options, and potentially also globalization and offshoring.
References


Notations

\( \alpha \) : value of non-market activity as \% of average wages, "replacement rate"

\( b_t \) : workers' reservation wage

\( b_r(\phi) \) : best response function for workers

\( \bar{b} \) : critical value of reservation wage

\( \beta \) : discount factor

\( C_t \) : cost of purchasing capital

\( \delta \) : match destruction rate

\( e_t \) : unemployment insurance coverage rate

\( \varepsilon(s,x) \) : elasticity of \( \gamma(s,x) \) with respect to \( x \)

\( \eta \) : normalization parameter for value of non-market activity

\( G(x) \) : distribution of productivity levels of entering firms

\( G_{bt}(x) \) : distribution of productivity levels of competing firms

\( \gamma(s,x) \) : Lower Incomplete Gamma function

\( \Gamma(s) \) : Gamma function

\( H(x;\theta_t,x_{0t}) \) : endogenous productivity distribution across potential workers

\( L \) : measure of potential workers, or labor force

\( LTU_t \) : long-term unemployment rate

\( \lambda \) : shape parameter of Pareto distribution

\( p_t \) : probability of being eligible for unemployment insurance

\( \phi_t \) : ratio of entering firms to unemployed workers

\( \phi_r(b) \) : best response function for firms

\( \tilde{s}_{L,t} \) : labor share for new matches

\( \tilde{s}_{K,t} \) : capital share for new matches

\( s_{L,t} \) : labor share

\( s_{K,t} \) : capital share

\( \theta_t \) : ratio of competing firms to unemployed workers, "labor market tightness"

\( U_t \) : measure of unemployed workers

\( u_t \) : unemployment rate

\( u(\theta) \) : steady state unemployment rate

\( V_t \) : measure of entering firms

\( V^U \) : steady state value of unemployment

\( V^E(x) \) : steady state value of employment at wage \( x \)

\( \tilde{w} \) : expected wage for new matches

\( w(\theta_t,b_t) \) : expected wages for all workers unemployed at start of period

\( w_t^e \) : average wages of employed workers

\( x_{0t} \) : minimum value of distribution \( G_{bt}(x) \)

\( \tilde{Y}_t \) : total output from new matches

\( Y_t \) : aggregate output

\( y(\theta,b) \) : steady state output per capita

\( z_t \) : value of non-market activity
Appendix A – Proofs for online publication

A0. Useful facts

Here I reproduce some useful facts presented in Lemma 2 and Fact 1 in Mangin (2015) that will be used in the proofs in this Appendix. For any $s \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$, the Lower Incomplete Gamma function is defined by

\begin{equation}
\gamma(s, x) \equiv \int_0^x t^{s-1}e^{-t} \, dt.
\end{equation}

Observe that $\lim_{x \to \infty} \gamma(s, x) = \Gamma(s)$, the Gamma function, and $\gamma(1, x) = 1 - e^{-x}$.

**Fact 1.** Recurrence relation: $\gamma(s, x) = (s-1)\gamma(s-1, x) - x^{s-1}e^{-x}$

**Fact 2.** $\frac{\partial}{\partial x} \gamma(s, x) = x^{s-1}e^{-x}$

**Fact 3.** $\frac{\partial}{\partial s} \gamma(s, x) = \int_0^x t^{s-1}e^{-t}(\ln t) \, dt$

**Fact 4.** For $x > 0$, the elasticity of $\gamma(s, x)$ with respect to $x$ is

\begin{equation}
\varepsilon(s, x) = \frac{x^s e^{-x}}{\gamma(s, x)}.
\end{equation}

**Fact 5.** $\varepsilon(s, x)$ is increasing in $s$, $\frac{\partial}{\partial x} \varepsilon(s, x) > 0$

**Fact 6.** $\varepsilon(s, x)$ is decreasing in $x$, $\frac{\partial}{\partial x} \varepsilon(s, x) < 0$

**Fact 7.** $\lim_{x \to 0} \varepsilon(s, x) = s$ and $\lim_{x \to \infty} \varepsilon(s, x) = 0$

A1. Proof that output per capita is increasing in $\theta$

Let $y(\theta, b) = \frac{x_0 \gamma(1-\lambda, \theta)^{\lambda}}{1-(1-\delta)e^{-\theta}}$. Differentiating $y$ with respect to $\theta$, we have

\[
\frac{\partial y}{\partial \theta} = x_0 \left( \frac{(e^{-\theta} + \gamma(1-\lambda, \theta)\lambda\theta^{\lambda-1})(1-(1-\delta)e^{-\theta}) - \gamma(1-\lambda, \theta)\theta^{\lambda}(1-\delta)e^{-\theta})}{(1-(1-\delta)e^{-\theta})^2} \right).
\]

Rearranging, using the fact that $(1-\delta)e^{-\theta} < 1$ and dividing both sides by $\theta^{\lambda-1}\gamma(1-\lambda, \theta)$, we obtain $\frac{\partial y}{\partial \theta} > 0$ if and only if

\begin{equation}
\lambda + \frac{\theta^{1-\lambda} e^{-\theta}}{\gamma(1-\lambda, \theta)} > \frac{(1-\delta)\theta e^{-\theta}}{1-(1-\delta)e^{-\theta}}.
\end{equation}

Since $\frac{(1-\delta)\theta e^{-\theta}}{1-(1-\delta)e^{-\theta}}$ is decreasing in $\delta$, it suffices to show that $\lambda + \frac{\theta^{1-\lambda} e^{-\theta}}{\gamma(1-\lambda, \theta)} > \frac{\theta e^{-\theta}}{1-e^{-\theta}}$, which is true provided that $\frac{\partial h}{\partial \lambda} > 0$ where $h(\lambda, \theta) = \lambda + \frac{\theta^{1-\lambda} e^{-\theta}}{\gamma(1-\lambda, \theta)}$, since $h(0, \theta) = \frac{\theta e^{-\theta}}{1-e^{-\theta}}$. Differentiating
$h(\lambda, \theta)$ with respect to $\lambda$: 
\[
\frac{\partial h}{\partial \lambda} = \frac{\partial}{\partial \lambda} \left( \lambda + \frac{\theta e^{-\theta}}{\theta + e^{\lambda}} \right) = 1 - \frac{\theta e^{-\theta}}{(\theta + e^{\lambda})^2} \frac{\partial}{\partial \lambda} (\lambda) = (\lambda + 1 - \lambda = 1).
\]
Substituting in \(\frac{\partial}{\partial \lambda} (\theta^\lambda (1 - \lambda, \theta))\) and simplifying, we have \(\frac{\partial h}{\partial \lambda} = 1 - \frac{\theta e^{-\theta} \beta}{\gamma(1 - \lambda, \theta)}\), where

\[
(26) \quad B = \int_0^\theta t^{-\lambda}e^{-t}(\ln \theta - \ln t)dt.
\]

So \(\frac{\partial h}{\partial \lambda} > 0\) if and only if \(\theta^\lambda e^{-\theta} < \gamma(1 - \lambda, \theta)^2\). Substituting in expression (58) for $B$ derived in Appendix A11 of Mangin (2015), we require that

\[
\left( \frac{\theta^\lambda}{1 - \lambda} \right)^2 e^{-\theta} F_{2,2}(1 - \lambda, 1 - \lambda; 2 - \lambda, 2 - \lambda; -\theta) < \gamma(1 - \lambda, \theta)^2.
\]

Using the identity \(\gamma(x, z) = z^x x^{-1} F_{1,1}(x; x + 1; -z)\) from Andrews, Askey, and Roy (2000) this is equivalent to

\[
(27) \quad e^{-\theta} F_{2,2}(1 - \lambda, 1 - \lambda; 2 - \lambda, 2 - \lambda; -\theta) < F_{1,1}(1 - \lambda; 2 - \lambda; -\theta)^2.
\]

Now Lemma 4 in Mangin (2014) implies that the left-hand side of (27) is less than or equal to \(F_{1,1}(1 - \lambda; 2 - \lambda; -\theta) F_{1,1}(2 - \lambda; 3 - \lambda; -\theta)\), so it suffices to show that \(F_{1,1}(2 - \lambda; 3 - \lambda; -\theta) < F_{1,1}(1 - \lambda; 2 - \lambda; -\theta)\). Applying Kummer's first transformation, \(F_{1,1}(y; z; -x) = e^{-x} F_{1,1}(z - y; z; x)\) from Andrews, Askey, and Roy (2000), we require that \(F_{1,1}(1; 3 - \lambda; \theta) < F_{1,1}(1; 2 - \lambda; \theta)\). This is true since the function \(F_{1,1}(a_1; b_1; x)\) is decreasing in its second argument. Hence \(\frac{\partial h}{\partial \theta} > 0\), so the original inequality (25) holds and therefore \(\frac{\partial y}{\partial \theta} > 0\).

A2. Derivation of zero profit condition

Let $J_1(x)$ be the discounted expected revenue net of wages for a firm with productivity $x$ that faces no competition when hiring a worker. Let $J_2(x)$ be the discounted expected revenue net of wages for a successful firm with productivity $x$ that faces competition when hiring. If $w(x)$ is the expected value of $x_2$ given that $x$ is highest, then

\[
J_1(x) = \frac{x - b}{1 - \beta(1 - \delta)}; \quad J_2(x) = \frac{x - w(x)}{1 - \beta(1 - \delta)}.
\]

Let $\eta(x)$ be the probability of a firm successfully hiring, given productivity $x$ and given that there are two or more firms competing. We have $\eta(x) = \frac{e^{-\theta (1 - G_b(x))} - e^{-\theta}}{1 - e^{-\theta}}$. The value function for firms, $\tilde{V}$, is

\[
\tilde{V} = -C + (1 - G(b)) \left( 1 - e^{-\theta} \int_{x_0}^\infty J_1(x) dG_b(x) + (1 - e^{-\theta}) \int_{x_0}^\infty \eta(x) J_2(x) dG_b(x) \right).
\]
Using analogous reasoning to that found in Appendix A4 of Mangin (2015), we obtain:

\[
\int_{x_0}^{\infty} \eta(x)(x - w(x))dG_b(x) = \int_{x_0}^{\infty} \eta(x)(1 - G_b(x))dx,
\]

Setting \( V = 0 \), we have

\[
C = \frac{1 - G(b)}{1 - \beta(1 - \delta)} \left( e^{-\theta} \int_{x_0}^{\infty} (x - b)dG_b(x) + (1 - e^{-\theta}) \int_{x_0}^{\infty} \eta(x)(1 - G_b(x))dx \right).
\]

Rearranging and again using integration by parts, the zero profit condition is

\[
C = \frac{(1 - G(b)) \left( \int_{x_0}^{\infty} e^{-\theta(1-G_b(x))} (1 - G_b(x))dx + e^{-\theta}(x_0 - b) \right)}{1 - \beta(1 - \delta)}.
\]

For the Pareto distribution, \( G_b(x) = 1 - \left( \frac{x}{x_0} \right)^{-1/\lambda} \) where \( x_0 = \max\{1, b\} \), we have

\[
C = \frac{x_0^{-1/\lambda} (x_0\lambda^{1/\lambda} - 1) (1 - \lambda, \theta) + e^{-\theta}(x_0 - b)}{1 - \beta(1 - \delta)},
\]

where \( \theta = \phi(1 - G(b)) \).

### A3. Existence and uniqueness of \( \phi \) for any given \( b \)

**Existence.** Let \( F(\theta) = (1 - G(b)) \left( \int_{x_0}^{\infty} e^{-\theta(1-G_b(x))} (1 - G_b(x))dx + e^{-\theta}(x_0 - b) \right), b \in \mathbb{R}^+ \). The zero profit condition holds if and only if \( F(\theta) = C(1 - \beta(1 - \delta)) \), where \( C(1 - \beta(1 - \delta)) > 0 \). Now \( F(\theta) \) is continuous in \( \theta \) on \([0, \infty)\) and \( F(\theta) \to 0 \) as \( \theta \to \infty \). If we can ensure that \( F(0) > C(1 - \beta(1 - \delta)) \), the intermediate value theorem implies there exists \( \theta > 0 \) such that \( F(\theta) = C(1 - \beta(1 - \delta)) \). Now, \( F(0) = (1 - G(b)) \left( \int_{x_0}^{\infty} (1 - G_b(x))dx + (1 - b) \right) = (1 - G(b))(E_{G_b}(x) - b) \). If \( G(x) \) is Pareto, we have \( F(0) > C(1 - \beta(1 - \delta)) \) provided the following condition holds:

\[
(28) \quad C < \frac{x_0^{-1/\lambda}}{1 - \beta(1 - \delta)} \left( \frac{x_0}{1 - \lambda} - b \right).
\]

If condition (28) holds, there exists \( \theta > 0 \) and hence there exists \( \phi > 0 \) such that the zero profit condition holds, where \( \phi = \theta/(1 - G(b)) \). If (28) fails, then no firms enter and \( \theta = \phi = 0 \).

If Assumption 1 holds, there is a unique critical value \( \bar{b}(\lambda, \beta, \delta, C) > z \) such that condition (28) holds whenever \( b < \bar{b} \). To see this, let \( f(b) = \frac{x_0^{-1/\lambda}}{1 - \beta(1 - \delta)} \left( \frac{x_0}{1 - \lambda} - b \right) - C \). Condition (28) holds if and only if \( f(b) > 0 \). First we prove that \( f'(b) < 0 \). If \( b \leq 1 \), we have \( f'(b) = \frac{1}{1 - \beta(1 - \delta)} \left( \frac{1}{1 - \lambda} - b \right) - C \), so \( f'(b) = \frac{-1}{1 - \beta(1 - \delta)} < 0 \). If \( b > 1 \), we have \( f(b) = \frac{\lambda^{-1/\lambda}}{(1 - \lambda)(1 - \beta(1 - \delta))} - C \), so \( f'(b) = \frac{-b^{-1/\lambda}}{1 - \beta(1 - \delta)} < 0 \). So for all \( b \in \mathbb{R}^+ \), we have \( f'(b) < 0 \). Now
if \( f(z) > 0 \), then since \( f'(b) < 0 \) and \( f(b) \to -C \) as \( b \to \infty \), there exists a unique \( \bar{b} > z \) such that \( f(\bar{b}) = 0 \). We have \( f(z) > 0 \) provided that

\[
(29) \quad C < \frac{1}{1 - \beta(1 - \delta)} \left( \frac{1}{1 - \lambda} - z \right).
\]

Condition (29) ensures that there exists a unique critical value \( \bar{b} > z \) such that \( f(\bar{b}) = 0 \). We know that for any \( b < \bar{b} \) condition (28) also holds and therefore there exists \( \theta > 0 \) and \( \phi > 0 \) that satisfy the zero profit condition, where \( \phi = \theta/(1 - G(b)) \). If \( b \geq \bar{b} \), then (28) fails and hence \( \theta = \phi = 0 \).

**Uniqueness.** To prove the uniqueness of \( \theta \) which satisfies \( F(\theta) = C(1 - \beta(1 - \delta)) \), and hence the uniqueness of \( \phi = \theta/(1 - G(b)) \), it suffices to show that \( F'(\theta) < 0 \). Applying Leibniz’ integral rule, \( F'(\theta) = -(1 - G(b)) \left( \int_{x_0}^{\infty} (1 - G_b(x))^2 e^{-\theta(1 - G_b(x))} dx + (1 - b)e^{-\theta} \right) < 0 \). So for any given \( b \in \mathbb{R}^+ \), there exists a unique \( \theta \) and hence a unique \( \phi \) that satisfies the zero profit condition. In other words, we have a best-response function \( \phi_r(b) : \mathbb{R}^+ \to \mathbb{R}^+ \).

**A4. Proof that \( \phi_r'(b) \leq 0 \)**

If \( b \geq \bar{b} \), we have \( \phi_r(b) = 0 \) and so \( \phi_r'(b) = 0 \) for \( b > \bar{b} \). Assume instead that \( b < \bar{b} \).

Let \( F_1(\theta, b) = x_0^{-\lambda} (x_0 \theta^{\lambda-1} \gamma(1 - \lambda, \theta) + e^{-\theta}(x_0 - b)) - C(1 - \beta(1 - \delta)) = 0 \) where \( x_0 = \max\{1, b\} \). When \( b < 1 \), \( F_1(\theta, b) = \lambda \theta^{\lambda-1} \gamma(1 - \lambda, \theta) + e^{-\theta}(1 - b) - C(1 - \beta(1 - \delta)) \) and \( \theta = \phi \), so \( \partial F_1/\partial b = -e^{-\theta} \) and \( \partial F_1/\partial \theta = -(\lambda \theta^{\lambda-2} \gamma(2 - \lambda, \theta) + e^{-\theta}(1 - b)) \). By the implicit function theorem, \( \theta'(b) = \phi_r'(b) = -\frac{\partial F_1/\partial b}{\partial F_1/\partial \theta} \), which gives the following expression:

\[
(30) \quad \phi_r'(b) = \theta_r'(b) = \frac{-e^{-\theta}}{\lambda \theta^{\lambda-2} \gamma(2 - \lambda, \theta) + e^{-\theta}(1 - b)} < 0, \quad b < 1.
\]

When \( b \geq 1 \), we have \( F_1(\theta(\phi, b), b) = b^{-1/\lambda} \lambda \theta^{\lambda-1} \gamma(1 - \lambda, \theta) - C(1 - \beta(1 - \delta)) \) where \( \theta(\phi, b) = \phi(1 - G(b)) = \phi b^{-1/\lambda} \) and hence \( \frac{\partial \theta}{\partial b} = -\frac{1}{\lambda} \phi b^{-1} \). Now \( \phi_r'(b) = -\frac{\partial F_1/\partial b}{\partial F_1/\partial \phi} \) where \( \partial F_1/\partial \phi = -b^{-1/\lambda} \lambda \theta^{\lambda-2} \gamma(2 - \lambda, \theta) \) is obtained by differentiating and then applying Fact 1. Again using Fact 1 to simplify the following, we have \( \frac{dF_1}{db} = \frac{\partial F_1}{\partial \phi} \frac{\partial \phi}{\partial b} + \frac{\partial F_1}{\partial \phi} = -b^{-1/\lambda} e^{-\theta} \). Also, \( \frac{dF_1}{d\phi} \) is given by \( \frac{dF_1}{d\phi} = \frac{\partial F_1}{\partial \phi} \frac{\partial \theta}{\partial \phi} \), so we have the following expression for \( \phi_r'(b) \), which is again negative:

\[
\phi_r'(b) = -\frac{dF_1/\partial b}{dF_1/\partial \phi} = \frac{-e^{-\theta}}{b^{-1/\lambda} \lambda \theta^{\lambda-2} \gamma(2 - \lambda, \theta)} < 0, \quad b \geq 1.
\]

In general, for any \( b < \bar{b} \) we have

\[
(31) \quad \phi_r'(b) = \frac{-e^{-\theta}}{x_0^{-1/\lambda} (x_0 \lambda \theta^{\lambda-2} \gamma(2 - \lambda, \theta) + e^{-\theta}(x_0 - b))} < 0, \quad x_0 = \max\{1, b\}.
\]
A5. Steady state reservation wage

Let \( V_E(w) \) be the expected value of being employed at wage \( w \) and let \( V_U \) be the expected value of being unemployed at the start of the period. We have

\[
V_E(w) = w + \beta((1 - \delta)V_E(w) + \delta V_U).
\]

Workers decide whether to accept or reject wage offers \( w \), taking \( \phi \) as given. If \( V_E(w) \geq z + \beta V_U \), workers accept the wage offer \( w \), while if \( V_E(w) < z + \beta V_U \) they reject it. We show that for any given \( \phi \) there exists a unique reservation wage \( b \) such that workers will accept a wage offer \( w \) if and only if \( w \geq b \). The reservation wage \( b \) satisfies \( V_E(b) = z + \beta V_U \).

Rearranging (32), we have

\[
V_E(w) = \frac{w + \beta \delta V_U}{1 - \beta(1 - \delta)}.
\]

Since \( dV_E(w)/dw > 0 \), \( \lim_{w \to \infty} V_E(w) = +\infty \), and it is easily verified that \( V_E(0) < z + \beta V_U \), there exists a unique reservation wage \( b \) for any given \( \phi \) such that \( V_E(b) = z + \beta V_U \).

This gives us a best-response function \( b_r(\phi) : \mathbb{R}^+ \to \mathbb{R}^+ \). Given the existence of the unique reservation wage \( b \), we can then derive the expression for \( V_U \) in equation (7). Letting \( \theta = \phi(1 - G(b)) \), the expected value of being unemployed at the start of a period is

\[
V_U = e^{-\theta}(z + \beta V_U) + (1 - e^{-\theta}) \int_{x_0}^{\infty} \max(V_E(w), z + \beta V_U) d\tilde{F}(w, \theta, x_0),
\]

where \( x_0 = \max\{1, b\} \) and \( \tilde{F}(w, \theta, x_0) \) is the distribution of wage offers given \( \theta \). Substituting in \( V_E(b) = z + \beta V_U \), we have \( \max(V_E(w), z + \beta V_U) = \max(V_E(w), V_E(b)) = V_E(w) \) since \( b \leq w \) for all \( w \). Since \( \int_{x_0}^{\infty} V_E(w) d\tilde{F}(w, \theta, x_0) = V_E(\tilde{w}) \) where \( \tilde{w} \) is the expected value of wage offers for new matches, we obtain

\[
V_U = e^{-\theta}(z + \beta V_U) + (1 - e^{-\theta})V_E(\tilde{w}).
\]

A6. Proof that \( b_r(\phi) > 0 \)

We start with expression (9) for the reservation wage:

\[
b = \frac{z(1 - \beta(1 - \delta)) + \beta(1 - \delta)w(\theta, b)}{1 - \beta(1 - \delta)e^{-\theta}}.
\]

Let \( F_2(\theta, b) = b(1 - \beta(1 - \delta)e^{-\theta}) - z(1 - \beta(1 - \delta)) - \beta(1 - \delta)w(\theta, b) = 0 \) and let \( w(\theta, b) = x_0(1 - \lambda)\theta^{\lambda}((1 - \lambda)\theta - \theta e^{-\theta}(x_0 - b)) \) where \( x_0 = \max\{1, b\} \). We have

\[
\frac{\partial F_2}{\partial \theta} = \beta(1 - \delta)\left(b e^{-\theta} - \frac{\partial w}{\partial \theta}\right),
\]

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Substituting these into (33) and (34) and simplifying, we have

$$\frac{\partial F_2}{\partial b} = 1 - \beta(1 - \delta)e^{-\theta} - \beta(1 - \delta)\frac{\partial w}{\partial b}. $$

If \( b < 1 \), then \( \theta = \phi \) and \( w(\theta, b) = (1 - \lambda)\theta^\lambda \gamma(1 - \lambda, \theta) - \theta e^{-\theta}(1 - b) \). Differentiating,

$$\frac{\partial w}{\partial \theta} = (1 - \lambda)(\lambda\theta^{\lambda-1}\gamma(1 - \lambda, \theta) + e^{-\theta}) - (1 - b)(e^{-\theta}(1 - \theta)) \text{ and } \frac{\partial w}{\partial b} = \theta e^{-\theta}. $$

Substituting these into (33) and (34) and simplifying using Fact 1,

$$\frac{\partial F_2}{\partial \theta} = -\beta(1 - \delta)(\lambda\theta^{\lambda-1}\gamma(2 - \lambda, \theta) + (1 - b)\theta e^{-\theta}),$$

$$\frac{\partial F_2}{\partial b} = 1 - \beta(1 - \delta)e^{-\theta} - \beta(1 - \delta)\theta e^{-\theta}. $$

and hence using \( b'_r(\phi) = b'_r(\theta) = -\frac{\partial F_2/\partial \theta}{\partial F_2/\partial b} \) we have

$$b'_r(\phi) = b'_r(\theta) = \frac{\beta(1 - \delta)(\lambda\theta^{\lambda-1}\gamma(2 - \lambda, \theta) + (1 - b)\theta e^{-\theta})}{1 - \beta(1 - \delta)e^{-\theta} - \beta(1 - \delta)\theta e^{-\theta}} > 0, \ b < 1. $$

The numerator is positive when \( b < 1 \) and the denominator is positive since \( 1 - e^{-\theta} - \theta e^{-\theta} > 0 \) and \( \beta(1 - \delta) < 1 \), so \( b'_r(\phi) > 0 \) when \( b < 1 \).

If \( b \geq 1 \), then we have \( F_2(\theta(\phi, b), b) = b(1 - \beta(1 - \delta)e^{-\theta}) - z(1 - \beta(1 - \delta)) - \beta(1 - \delta)w(\theta, b) \) where \( w(\theta, b) = b(1 - \lambda)\theta^\lambda \gamma(1 - \lambda, \theta) \) and \( \theta(\phi, b) = \phi(1 - G(b)) = b^{-1/\lambda} \), and hence \( \frac{\partial \theta}{\partial b} = -\frac{b}{\lambda} \). Differentiating, we have

$$\frac{\partial w}{\partial \theta} = b(1 - \lambda)(\lambda\theta^{\lambda-1}\gamma(1 - \lambda, \theta) + e^{-\theta}) \text{ and } \frac{\partial w}{\partial b} = (1 - \lambda)\theta^\lambda \gamma(1 - \lambda, \theta). $$

Substituting into (33) and (34) and simplifying using Fact 1,

$$\frac{\partial F_2}{\partial \theta} = -\beta(1 - \delta)b\lambda\theta^{\lambda-1}\gamma(2 - \lambda, \theta),$$

$$\frac{\partial F_2}{\partial b} = 1 - \beta(1 - \delta)e^{-\theta} - \beta(1 - \delta)(1 - \lambda)\theta^\lambda \gamma(1 - \lambda, \theta). $$

Now \( b'_r(\phi) = -\frac{dF_2/\partial \phi}{dF_2/\partial \theta} \), where \( \frac{dF_2}{d\phi} = \frac{\partial F_2}{\partial \theta} \frac{\partial \theta}{\partial \phi} = -b^{1 - 1/\lambda} \beta(1 - \delta)\lambda\theta^{\lambda-1}\gamma(2 - \lambda, \theta) \) and

$$\frac{dF_2}{db} = \frac{\partial F_2}{\partial \theta} \frac{\partial \theta}{\partial b} + \frac{\partial F_2}{\partial b} = 1 - \beta(1 - \delta)e^{-\theta} - \beta(1 - \delta)\theta e^{-\theta}, \text{ by applying Fact 3. So we have}$$

$$b'_r(\phi) = -\frac{dF_2/\partial \phi}{dF_2/\partial b} = \frac{b^{1 - 1/\lambda} \beta(1 - \delta)\lambda\theta^{\lambda-1}\gamma(2 - \lambda, \theta)}{1 - \beta(1 - \delta)e^{-\theta} - \beta(1 - \delta)\theta e^{-\theta}} > 0, \ b \geq 1. $$

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In general, we have
\[ b_r(\phi) = x_0^{-\frac{1}{\lambda}} \beta (1 - \delta) (x_0 \lambda \theta^{\lambda - 1} \gamma (2 - \lambda, \theta) + (x_0 - b) \theta e^{-\theta}) \]
\[
\frac{1 - \beta (1 - \delta) e^{-\theta}}{1 - \beta (1 - \delta) e^{-\theta}} > 0, \quad x_0 = \max\{1, b\}.
\]

A7. Proof of Proposition 1

Here we establish some comparative statics results for the equilibrium \((\phi^*, b^*)\) with respect to the parameters \(p_i \in \mathbb{P} = (\lambda, z, C)\). We restrict our attention to the case where \(b < 1\) and \(\phi = \theta\).

Consider the best-response function \(\theta_r(b; p)\) defined in Appendix A3 and the best-response function \(b_r(\theta; p)\) defined in Appendix A5. The function \(\theta_r\) is differentiable for any \(b < \bar{b}\) and the function \(b_r\) is differentiable for any \(\theta \geq 0\). Let \(x = (\theta, b)\) and define the following function:
\[
G(x; p) = \begin{bmatrix}
    b_r(\theta; p) - b \\
    \theta_r(\theta; p) - \theta
\end{bmatrix}.
\]

By definition, \(x^* = (\theta^*, b^*)\) is an equilibrium if and only if \(G(x, p) = 0\). By the implicit function theorem, for any \(p_i \in \mathbb{P}\) we have
\[
Dx^*(p_i) = -(D_x G(x^*(p_i); p_i))^{-1} D_{p_i} G(x^*(p_i); p_i)
\]
\[
= -\begin{bmatrix}
    \frac{\partial b_r}{\partial p_i} & -1 \\
    -1 & \frac{\partial \theta_r}{\partial p_i}
\end{bmatrix}^{-1}
\begin{bmatrix}
    \frac{\partial b_r}{\partial \theta} \\
    \frac{\partial \theta_r}{\partial p_i}
\end{bmatrix}.
\]

For notational simplicity, denote the matrix \(D_x G(x^*(p_i); p_i)\) by \(J_G\). Using the derivatives \(\partial b_r / \partial \theta\) and \(\partial \theta_r / \partial b\) given by (35) and (30) respectively, we obtain
\[
\det J_G = \frac{-(1 - \beta (1 - \delta) e^{-\theta})}{1 - \beta (1 - \delta) e^{-\theta} - \beta (1 - \delta) \theta e^{-\theta}},
\]
where \(\det J_G < -1\) and hence \(J_G\) is invertible. Multiplying out (37), for any \(p_i \in \mathbb{P}\),
\[
\frac{\partial \theta^*}{\partial p_i} = -(\det J_G)^{-1} \left( \frac{\partial \theta_r}{\partial b} \frac{\partial b_r}{\partial p_i} + \frac{\partial \theta_r}{\partial \theta} \right),
\]
\[
\frac{\partial b^*}{\partial p_i} = -(\det J_G)^{-1} \left( \frac{\partial b_r}{\partial b} \frac{\partial b_r}{\partial p_i} + \frac{\partial b_r}{\partial \theta} \frac{\partial \theta_r}{\partial p_i} \right).
\]

The best response functions \(\theta_r(b; p)\) and \(b_r(\theta; p)\) are implicitly defined by (39) and (40):
\[
F_1(\theta, b; p) = \lambda \theta^{\lambda - 1} \gamma (1 - \lambda, \theta) + e^{-\theta} (1 - b) - C (1 - \beta (1 - \delta)) = 0,
\]
\[
F_2(\theta, b; p) = b (1 - \beta (1 - \delta) e^{-\theta}) - z (1 - \beta (1 - \delta)) - \beta (1 - \delta) w(\theta, b; p) = 0,
\]
where \( w(\theta, b; p) = (1 - \lambda)\theta^\lambda \gamma(1 - \lambda, \theta) - \theta e^{-\theta}(1 - b). \) We can use the implicit function theorem in relation to \( F_1(\theta, b; p) \) and \( F_2(\theta, b; p) \) and apply some earlier results to obtain the following comparative statics.

### Comparative statics for \( z \).

\[
\frac{\partial \theta^*}{\partial z} = - (\det J_G)^{-1} \left( \frac{\partial \theta_r}{\partial b} \frac{\partial b_r}{\partial z} + \frac{\partial \theta_r}{\partial z} \right) \\
= \frac{-(1 - \beta(1 - \delta))}{1 - \beta(1 - \delta)e^{-\theta}} \left( \frac{e^{-\theta}}{\lambda \theta^{\lambda-2}\gamma(2 - \lambda, \theta) + e^{-\theta}(1 - b)} \right) < 0
\]

(41)

\[
\frac{\partial b^*}{\partial z} = - (\det J_G)^{-1} \left( \frac{\partial b_r}{\partial z} + \frac{\partial b_r}{\partial \theta} \frac{\partial \theta_r}{\partial z} \right) \\
= \frac{1 - \beta(1 - \delta)}{1 - \beta(1 - \delta)e^{-\theta}} > 0
\]

(42)

### Comparative statics for \( C \).

\[
\frac{\partial \theta^*}{\partial C} = - (\det J_G)^{-1} \left( \frac{\partial \theta_r}{\partial b} \frac{\partial b_r}{\partial C} + \frac{\partial \theta_r}{\partial C} \right) \\
= \frac{-(1 - \beta(1 - \delta))}{1 - \beta(1 - \delta)e^{-\theta}} \left( 1 - \beta(1 - \delta)e^{-\theta} - \beta(1 - \delta)\theta e^{-\theta} \right) < 0
\]

(43)

\[
\frac{\partial b^*}{\partial C} = - (\det J_G)^{-1} \left( \frac{\partial b_r}{\partial C} + \frac{\partial b_r}{\partial \theta} \frac{\partial \theta_r}{\partial C} \right) \\
= \frac{-(1 - \beta(1 - \delta))\beta(1 - \delta)\theta}{1 - \beta(1 - \delta)e^{-\theta}} < 0
\]

(44)

### Comparative statics for \( \lambda \). For \( b < 1 \), using (30) we have

\[
\frac{\partial \theta^*}{\partial \lambda} = - (\det J_G)^{-1} \left( \frac{\partial \theta_r}{\partial b} \frac{\partial b_r}{\partial \lambda} + \frac{\partial \theta_r}{\partial \lambda} \right) \\
= \frac{\theta^{\lambda-1}(\gamma(1 - \lambda, \theta) + (\lambda - \mu)B)}{\lambda \theta^{\lambda-2}\gamma(2 - \lambda, \theta) + e^{-\theta}(1 - b)}, \text{ where } \mu = \frac{\beta(1 - \delta)\theta e^{-\theta}}{1 - \beta(1 - \delta)e^{-\theta}}
\]

and \( B \) is given by (26). We have \( \frac{\partial \theta^*}{\partial \lambda} > 0 \) if and only if \( \gamma(1 - \lambda, \theta) + (\lambda - \mu)B > 0 \). If \( \lambda \geq \mu \), this is clearly true for \( \theta > 0 \). Suppose instead that \( \mu > \lambda \). Rearranging and multiplying both sides by \( 1 - \lambda \), we have \( \frac{\partial \theta^*}{\partial \lambda} > 0 \) if and only if \( \frac{B(1 - \lambda)}{\gamma(1 - \lambda, \theta)} < \frac{1 - \lambda}{\mu - \lambda} \). Now \( (1 - \lambda)/(\mu - \lambda) > 1/\mu \) provided \( \mu < 1 \), which is true since \( 1 - \beta(1 - \delta)e^{-\theta} - \beta(1 - \delta)\theta e^{-\theta} > 0 \). So it suffices to
show that

\[ \frac{B(1-\lambda)}{\gamma(1-\lambda, \theta)} \frac{\beta(1-\delta)\theta e^{-\theta}}{1 - \beta(1-\delta)e^{-\theta}} < 1. \]

Using equation (57) derived in Appendix A11 of Mangin (2015), we have

\[ \frac{B(1-\lambda)}{\gamma(1-\lambda, \theta)} \frac{(2-\lambda)\gamma(2-\lambda, \theta)}{\theta^2 e^{-\theta}}. \]

Hence to establish inequality (45), it is sufficient to show that \( m(\theta) \leq 1/(2-\lambda) \) where
\[ m(\theta) = \frac{\beta(1-\delta)^{\lambda-1} \gamma(2-\lambda, \theta)}{1 - \beta(1-\delta)e^{-\theta}}. \]
It can be shown that max \( m(\theta) = \frac{1-\zeta}{2-\lambda-\zeta} \) where \( \zeta \) is the unique solution to \( 1 - \beta(1-\delta)e^{-\zeta} = \zeta \). To ensure that \( \frac{\partial y^*}{\partial x} > 0 \), it suffices to show that \( \frac{1-\zeta}{2-\lambda-\zeta} \leq \frac{1}{2-\lambda} \), which is always true since \( \lambda < 1 \) and \( \zeta > 0 \). So we have \( \frac{\partial y^*}{\partial x} > 0 \) for \( b < 1 \).

We also have \( \frac{\partial b^*}{\partial \lambda} > 0 \) for \( b < 1 \) and \( \theta > 0 \).

\[ \frac{\partial b^*}{\partial \lambda} = - (\text{det} \ J_G) ^{-1} \left( \frac{\partial b^*}{\partial x} \frac{\partial \theta^*}{\partial \lambda} + \frac{\partial b^*}{\partial \theta} \frac{\partial \theta^*}{\partial \lambda} \right) \]
\[ = \frac{B\beta(1-\delta)\theta^\lambda}{1 - \beta(1-\delta)e^{-\theta}} > 0. \]

**Proof of (iii).** Consider \( u^* = u(\theta^*) \). Since \( \frac{\partial \theta^*}{\partial \lambda} < 0 \), \( \frac{\partial \theta^*}{\partial C} < 0 \), and \( \frac{\partial \theta^*}{\partial \theta} > 0 \) for \( b < 1 \), if \( u'(\theta) < 0 \) then \( \frac{\partial u^*}{\partial y} = \frac{du}{dy} > 0 \), \( \frac{\partial u^*}{\partial \theta} = \frac{du}{d\theta} > 0 \), and \( \frac{\partial u^*}{\partial \lambda} = \frac{du}{d\lambda} > 0 \) for \( b < 1 \).

Rearranging (2), we have \( u(\theta) = \delta/(\theta^\theta e^{\theta} - (1-\delta)) \), which is clearly decreasing in \( \theta \) so \( u'(\theta) < 0 \) for \( \theta > 0 \).

**Proof of (iv).** Consider \( y^* = y(\theta^*, \lambda) \). Since \( \frac{\partial \theta^*}{\partial \lambda} \leq 0 \), we have \( \frac{\partial y^*}{\partial C} = \frac{dy}{dC} \frac{\partial \theta^*}{\partial \lambda} < 0 \) provided that \( \frac{dy}{dC} > 0 \), which was proven in Appendix A1. Starting with (5), we have
\[ y(\theta, \lambda) = \frac{\gamma(1-\lambda, \theta)^{\lambda}}{1 - (\lambda - \theta)e^{-\lambda}} \] for \( b < 1 \). So \( \frac{\partial y^*}{\partial C} = \frac{dy}{d\theta} \frac{\partial \theta^*}{\partial \lambda} + \frac{dy}{d\lambda} \frac{\partial \theta^*}{\partial \lambda} \), where \( \frac{\partial \theta^*}{\partial \lambda} > 0 \) and \( \frac{dy}{d\theta} > 0 \). In order to prove that \( \frac{\partial y^*}{\partial \lambda} > 0 \), it suffices to show that \( \frac{\partial y}{\partial \lambda} > 0 \). Differentiating \( y(\theta, \lambda) \) with respect to \( \lambda \) yields
\[ \frac{\partial y}{\partial \lambda} = \theta^\lambda \left( \int_0^\theta t^{-\lambda} e^{-t} (\ln \theta - \ln t) dt \right) > 0. \]
for any \( \theta > 0 \). So \( \frac{\partial y^*}{\partial \lambda} > 0 \) when \( b < 1 \). Finally, \( \frac{\partial y^*}{\partial \lambda} = \frac{dy}{d\theta} \frac{\partial \theta^*}{\partial \lambda} < 0 \) if \( b < 1 \), since \( \frac{\partial \theta^*}{\partial \lambda} < 0 \).

**A8. Proof of Proposition 2**

Let \( s^*_K = \lambda + (1 - b^*)\varepsilon(1 - \lambda, \theta^*) \). Differentiating \( s^*_K \) with respect to \( C \), we have
\[ \frac{ds^*_K}{dC} = -\frac{\partial b^*}{\partial C} \varepsilon(1 - \lambda, \theta) + (1 - b^*) \frac{\partial \theta^*}{\partial \lambda} \frac{\partial \varepsilon}{\partial \lambda}(1 - \lambda, \theta). \]
Since $\frac{\partial b^*}{\partial C} < 0$ from (44), $\frac{\partial}{\partial \theta} \varepsilon(1 - \lambda, \theta) < 0$ by Fact 6, and $\frac{\partial \theta^*}{\partial C} < 0$ from (43), we have $\frac{ds_K^*}{dC} > 0$ and hence $\frac{ds_K^*}{dC} < 0$. Next, differentiating $s_K^*$ with respect to $z$, we have

$$\frac{ds_K^*}{dz} = -\frac{\partial b^*}{\partial z} \varepsilon(1 - \lambda, \theta) + (1 - b^*) \frac{\partial \theta^*}{\partial z} \frac{\partial}{\partial \theta} \varepsilon(1 - \lambda, \theta).$$

Substituting in $\frac{\partial}{\partial x} \varepsilon(s, x)$, where $s = 1 - \lambda$ and $x = \theta$, we have

$$\frac{ds_K^*}{dz} = -\frac{\partial b^*}{\partial z} \varepsilon(1 - \lambda, \theta) + (1 - b^*) \frac{\partial \theta^*}{\partial z} \frac{\theta - \lambda e^{-\theta} (1 - \lambda - \theta - \varepsilon(1 - \lambda, \theta))}{\gamma(1 - \lambda, \theta)}.$$

Substituting in $\frac{\partial b^*}{\partial z}$ from (42) and $\frac{\partial \theta^*}{\partial z}$ from (41), and using Fact 4, we have

$$\frac{ds_K^*}{dz} = -\frac{\theta - \lambda e^{-\theta} (1 - \beta(1 - \delta))}{\gamma(1 - \lambda, \theta)(1 - \beta(1 - \delta)e^{-\theta})} \left( \theta + \frac{e^{-\theta}(1 - b) (1 - \lambda - \theta - \varepsilon(1 - \lambda, \theta))}{\lambda \theta^{\lambda - 2}\gamma(2 - \lambda, \theta) + e^{-\theta}(1 - b)} \right).$$

Rearranging and simplifying, we have $\frac{ds_K^*}{dz} < 0$ if $\lambda \theta^{\lambda - 1}\gamma(1 - \lambda, \theta) + e^{-\theta}(1 - b) > 0$, which is true. Hence $\frac{ds_K^*}{dz} < 0$ or equivalently $\frac{ds_L^*}{dz} < 0$. 

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