Abstract

We show, first, that minimization of downside risk for portfolios with pre-specified expected returns leads to the same solution as minimization of the variance. Hence such optimal portfolios are defined by the Markovitz optimal solution. If the expected returns are not pre-specified, we show that the problem of minimization of downside risk has an analytical solution and we present this solution together with several illustrative numerical examples.
Optimal portfolios with downside risk

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1. Introduction

Markowitz optimal portfolio theory (Markowitz 1987), also known as the Mean-Variance theory, has had a tremendous impact and hundreds of papers are devoted to this topic. This theory addresses the question of minimizing risk for a given expected return and the optimal solution is found under one of the two assumptions: the distribution of the portfolio is normal, or the utility function is quadratic. In this theory, investor’s decision formulates a trade-off between the return and the risk, in which the risk is measured by the variance of the returns. However, it has also been noted frequently in the past, starting with Markowitz himself, that investors are more concerned with downside risk, i.e. the possibility of returns falling short of a specified target, rather than with the variance, which takes into account the favourable upside deviations as well as the adverse downside parts. Moreover, such a classic Mean–Variance framework does not consider the investor’s individual preferences. Thus, alternatives are proposed in the literature in the form of downside risk measures, such as target shortfall and semivariance, or more generally, the so-called lower partial moments; see, for example, Harlow and Rao (1989).

The lower partial moments for stochastic returns, as downside risk measures, are defined as the expectation of the \( n \)th power of the return’s deviation below a pre-specified target which depends on investor’s preference. The first- and the second-order lower partial moments are usually called target shortfall and below target variance, respectively. Intuitively, these risk measures are asymmetric and focus on the left tail of the portfolio returns below a given target rate rather than on their entire domain. The target shortfall is the expectation of portfolio returns dropping below the given target rate or benchmark return. By contrast, the below target variance (it is often called the semivariance when the target is set as the expectation of the return) considers the dispersion of return below the target rate (Fishburn 1977). Both criteria aim to measure the extent that the portfolio fails to reach its manager’s target or benchmark return. In this regard, these downside risk measures are more appropriate for investment risk because investors are often more interested in losses relative to target returns. Moreover, unlike the variance, semivariance can remain the same together with higher ‘upside potential’. For more details on downside risk measures, we refer to Harlow (1991), Nawrocki (1999) and Chapter 2 of McNeil et al. (2005).

A manager who does not wish the return of his portfolio to fall below the target rate would tend to compose portfolios minimizing downside risk measures, which is the so-called
downside optimization (Harlow 1991). Such portfolios with optimal downside risks are attractive as they may lower the risks while maintaining or improving the expected returns in the classic mean–variance framework. As a result, by considering downside risk measures, portfolio managers are usually able to search for more profits in the trade-off between risk and return. Empirical evidence also shows that downside measures are more efficient than Mean–Variance measures in this sense. Moreover, in asset pricing models, a downside risk framework can provide less downside exposure while preserving the same, or a greater level, of expected return, see Harlow and Rao (1989). In fact, despite the complexity of the computation, it is clear from the literature that lower partial moments, especially the target shortfall and below target variance, are not simply ad hoc measures, but are grounded in capital market theory with both appealing theoretical and intuitive features. For instance, Markowitz (2010) accords support to semivariance as a comparable risk measure with variance and there is mounting empirical evidence showing the superiority of downside risk measures (Jarrow and Zhao 2006).

Due to the formulation of partial moments, downside optimization is naturally connected with utility theory and stochastic dominance. Bawa (1978) used lower partial moments with an \( \alpha \)-degree and \( \tau \)-threshold model to capture investor’s risk aversion. More recently, Cumova and Nawrocki (2014) proposed a ratio of upside partial moment and lower partial moment to study investors’ behaviour. In contrast to the classical Mean–Variance model, which is consistent with a quadratic utility function, downside optimization is a more general framework as it allows investors to consider different orders \( n \) and to choose a favourable target. Hence, there is no doubt that the downside risk framework should provide a useful set of tools for portfolio managers considering a broad set of problems.

In addition to the literature mentioned above, there are ongoing efforts devoted to various applications of downside risks and related risk measures in finance. We list a few of them. Zhu et al. (2009) discuss robust portfolio selection with respect to parameter uncertainty under a downside risk framework. Moreover, the target shortfall is directly related to the Conditional Value-at-Risk (CVaR), which has been frequently applied as the optimizing objective in many portfolio selection studies. For more details on the risk measure CVaR, we refer to Artzner et al. (1999) and Rockefeller and Uryasev (2002). Mansini et al. (2007) consider optimal portfolios with respect to the CVaR and obtain the solution by linear programming. Sawik (2008) formulates the portfolio optimization problem as multi-objective mixed integer program. Ogryczak and Sliwinski (2011a, 2011b) use duality to improve the efficiency of linear programming in portfolio selection. Sawik (2012a) compares three different bi-criteria portfolio optimization models based on Value-at-Risk, CVaR and variance. Sawik (2012b) studies multi-objective portfolio optimization with downside risk approaches and Sawik (forthcoming) proposes further work in multi-objective models. Cumova and Nawrocki (2014) use both upside and downside partial moments to account for an investor’s utility. A literature review on downside risk measures can be found in Nawrocki (1999) and more account of robust portfolio optimization with respect to various risk measures is in Gabrel et al. (2014).

However, most of the aforementioned works have to rely on numerical method to find the solutions to the optimization problem. In fact, as Jarrow and Zhao (2006) commented, analytical solutions to optimal portfolio’s weights are generally out of reach for such downside measures. One could either employ numerical techniques to search for the optimal solution or implement optimization based on empirical estimation and simulation, see, for instance, Nawrocki (1991) and Cumova and Nawrocki (2014). Nevertheless, in the absence of massive computational efforts such as in simulation and numerical optimization, analytical solutions are always preferable in both theoretical studies and practical applications, and further, they do not suffer from computational errors.

Landsman (2008) finds an analytical solution to optimal portfolio weights in the classical Mean–Variance framework with additional constraints on the returns. Based on his results, we adopt a similar approach and derive the analytical solutions to the downside optimization in the context of normally distributed returns. This work offers several novel contributions. Firstly, we show that optimal portfolios with respect to downside risk are the same as those generated by the classical Markowitz Mean–Variance analysis in a framework in which the expected portfolio return is pre-specified. Thus, in this case the downside risk framework can not improve the classical results and investors cannot expect additional profits from the downside approach assuming normal distributions of the returns. Particularly, Jarrow and Zhao (2006) empirically observed that optimal portfolios with respect to mean–variance and downside risk frameworks are much alike. Our findings provide a theoretical proof for their results.

Secondly, we consider a more general portfolio selection with downside risk measure when the expected return of the portfolio is not pre-specified. We show that such downside optimization is a convex optimization and has a unique solution. We further derive analytical formulae for the solution and obtain optimal weights. Consequently, we can easily select optimal portfolios with respect to downside risk measures for any inputs of means and covariances. As mentioned above, such results are useful both theoretically and practically.

Thirdly, we present numerical examples to illustrate our results. We first observe that the general principal that an asset with higher return should have higher risk holds true in the downside risk measure framework. Moreover, we notice that a portfolio with minimal target shortfall can have very different weights from the one with minimal below target variance and it is plausible to argue that one could always outperform the other. Hence, instead of optimizing expected shortfall or below target variance separately, we propose a new downside optimization that targets a combination of the two downside risk measures. In this regard, an investor can choose his/her preference between the downward deviation from the target and the risk to drop below the target, and the corresponding analytical weights are still available. Similar to the classical results, we also show that one can obtain an analogue to the Mean–Variance efficient frontier based on the Mean-Downside-risk framework.

The rest of the paper is organized as follows. In section 2, we present the formulae for the risk associated with downside risk measures and state some useful properties. Section 3 provides the downside optimization and finds an analytical solution.
We offer numerical examples in section 4. We discuss further potential developments and conclude our paper in section 5.

2. Downside risk

To facilitate the expressions of the equations and formulae in the sequel, we first present a short list of the notions and notations that are useful as below.

- $X \sim N(\mu, \sigma^2)$. $X$ denotes stochastic return of an asset that is normally distributed with mean $\mu$ and variance $\sigma$.
- $\mathbf{X} \sim N(\mu, \Sigma)$. $\mathbf{X}$ denotes stochastic returns of multiple assets that are multi-normally distributed with means $\mu$ and covariance matrix $\Sigma$.
- $\Phi(x)$ and $\varphi(x)$ are the cumulative distribution function and density function of the standard normal random variable, respectively.
- For a fixed target $K$, we consider the downside risks below $K$

$$R_i(\mu, \sigma) := E[(X - K)^+],$$

for $i = 1$ and $i = 2$, i.e., $R_1(\mu, \sigma)$ is the target shortfall and $R_2(\mu, \sigma)$ is the below target variance.

- $A_K = (1, \mu - K)^T$ where 1 is a $n$-dimensional vector with identical unit components. $\delta = |A_K \Sigma^{-1} A_K^T|$ is the determinant of $A_K \Sigma^{-1} A_K^T$.
- $\sigma_1 = \delta^{-1}(11^T \Sigma^{-1} 1)$, $\sigma_2 = \delta^{-1}(\mu - K)^T \Sigma^{-1} 1$, $\sigma_3 = \delta^{-1}(\mu - K)^T \Sigma^{-1} (\mu - K)$. 

In the context of normality, we compute the downside risk functions as follows.

**Proposition 1**

$$R_1(\mu, \sigma) = E[(X - K)^+] = \sigma \varphi\left(\frac{K - \mu}{\sigma}\right) + (K - \mu) \Phi\left(\frac{K - \mu}{\sigma}\right), \quad (1)$$

$$R_2(\mu, \sigma) = E[(X - K)^+] = (\sigma^2 + (K - \mu)^2) \Phi\left(\frac{K - \mu}{\sigma}\right) + (K - \mu) \varphi\left(\frac{K - \mu}{\sigma}\right) + \sigma (K - \mu) \varphi\left(\frac{K - \mu}{\sigma}\right) \quad (2)$$

**Proof** Let $X = \sigma Z + \mu$, where $Z \sim N(0, 1)$.

$$R_1(\mu, \sigma) = E[(\sigma Z + \mu - K)^+] = \int_{-\infty}^{\frac{K - \mu}{\sigma}} e^{-\frac{z^2}{2}} \frac{1}{\sqrt{2\pi}} dz + (\mu - K) \int_{\frac{K - \mu}{\sigma}}^{\infty} e^{-\frac{z^2}{2}} \frac{1}{\sqrt{2\pi}} dz$$

$$= \sigma \varphi\left(\frac{K - \mu}{\sigma}\right) + (K - \mu) \Phi\left(\frac{K - \mu}{\sigma}\right).$$

$$R_2(\mu, \sigma) = E[(X - K)^+] = \sigma^2 E\left[\left(\frac{X - K}{\sigma}\right)^2\right] + \sigma (K - \mu) \varphi\left(\frac{K - \mu}{\sigma}\right) + \sigma (K - \mu) \varphi\left(\frac{K - \mu}{\sigma}\right).$$

Using $d\varphi(z) = -z\varphi(z)dz$ and integration by parts, we have

$$R_2(\sigma) = (\sigma^2 + (K - \mu)^2) \varphi\left(\frac{K - \mu}{\sigma}\right) + (K - \mu) \varphi\left(\frac{K - \mu}{\sigma}\right).$$

**Theorem 2** Downside risk functions (1) and (2) are increasing functions of $\sigma$.

**Proof** Assume now that $\mu$ is given. We show that $R'_1(\sigma) > 0$ and $R'_2(\sigma) > 0$. Direct calculations using the property $\phi'(x) = -x\varphi(x)$ gives

$$R'_1(\sigma) = \varphi\left(\frac{K - \mu}{\sigma}\right) > 0,$$

and similarly,

$$R'_2(\sigma) = 2\sigma \Phi\left(\frac{K - \mu}{\sigma}\right) > 0.$$

Obviously, the conclusion of this theorem is that for a pre-specified mean of return, minimizing each of these risk measures is equivalent to minimizing the variance. Hence there is a clear equivalence to the mean–variance principle.

3. Downside risk for portfolios

In this section we establish optimal portfolios with minimal downside risk measures. We assume that short selling is permitted. Let $\mathbf{X}$ denote the vector of returns in the portfolio and $\alpha$ be the vector of corresponding weights that adds up to unity, then $\alpha^T \mathbf{X}$ is the return of portfolio. Note that in our setting the components of $\alpha$ are not necessarily positive. Assuming multivariate normal distribution for $\mathbf{X}$, $N(\mu, \Sigma)$, we have that $\alpha^T \mathbf{X} \sim N(\alpha^T \mu, \alpha^T \Sigma \alpha)$.

Theorem 2 indicates that finding a portfolio that minimizes $E[(\alpha^T \mathbf{X} - K)^+]$ or $E[(\alpha^T \mathbf{X} - K)^+]^2$ is equivalent to finding $\alpha$ that minimize $\alpha^T \Sigma \alpha$. Formally, given any expected return $c$ on a portfolio, the solutions to

$$\min_{\alpha} E[(\alpha^T \mathbf{X} - K)^+] \text{ subject to } B \alpha = c, \quad (3)$$

and

$$\min_{\alpha} E\left[\left((\alpha^T \mathbf{X} - K)^-ight)^2\right] \text{ subject to } B \alpha = c, \quad (4)$$

are the same as

$$\min_{\alpha} \alpha^T \Sigma \alpha \text{ subject to } B \alpha = c, \quad (5)$$

where

$$B = \begin{pmatrix} 1, 1, \ldots, 1 \\ \mu_1, \mu_2, \ldots, \mu_n \end{pmatrix}_{2 \times n}, \quad c = \begin{pmatrix} 1 \\ c \end{pmatrix}. $$

The constant $B$ allows us, in addition to requiring all weights to sum to 1, to put a constraint $(c)$ on the expected return of
the portfolio. We now present the following proposition and corollary.

**Proposition 3 (Landsman (2008))** A portfolio with return $\alpha^T X$, $X \sim N(\mu, \Sigma)$, that minimizes $\alpha^T \Sigma \alpha$ with given expected return $\alpha^T \mu = c$ is defined by weights

$$\alpha = \Sigma^{-1} B^T (B \Sigma^{-1} B^T)^{-1} c.$$

**Corollary 4** A portfolio with return $\alpha^T X$, $X \sim N(\mu, \Sigma)$, that minimizes either $E(\alpha^T X - K)^-$ or $E(\alpha^T X - K)^+$ with given expected return $\alpha^T \mu = c$ is defined by weights

$$\alpha = \Sigma^{-1} B^T (B \Sigma^{-1} B^T)^{-1} c.$$

Note that in proposition 3 and corollary 4, we tacitly assume that the means of $X$ are inhomogeneous. In case that all $X_i$, $i = 1, \ldots, n$ have a common mean $\mu$, i.e., $\mu = \mu 1$, where $1$ is a $n$-dimensional vector with identical unit components, $B \Sigma^{-1} B^T$ becomes a singular matrix which is not invertible. Also note that $\alpha^T \mu \equiv \mu$ due to the constraint $\alpha^T 1 = 1$ for homogeneous means. Thus $c$ can only be $\mu$ and (5) reduces to the classic quadratic optimization:

$$\min_{\alpha} \alpha^T \Sigma \alpha \quad \text{subject to} \quad \alpha^T 1 = 1,$$

whose solution is well known as

$$\alpha = \Sigma^{-1} 1 / \Sigma^{1/2} 1,$$

see Landsman (2008) and Luenberger and Ye (1984). Without loss of generality, we shall further assume that the means of $X$ are inhomogeneous.

Let us further consider a more general optimization problem for downside risk portfolios where the expected return on the portfolio is not pre-specified. That is, we drop off the second row in $B$ and only reserve the unity constraint,

$$\min_{\alpha} E[(\alpha^T X - K)^-], \quad \text{subject to} \quad \alpha^T 1 = 1. \quad (6)$$

We obtain the following results.

**Theorem 5** The downside risk function (6) of a portfolio with return $\alpha^T X$, $X \sim N(\mu, \Sigma)$, subject to $\alpha^T 1 = 1$, is minimized at

$$\alpha^* = \Sigma^{-1} A_K^{-1} (A_K \Sigma^{-1} A_K)^{-1} c^*,$$

where $c^* = (1, c^T)$ and $c^*$ is the solution to the equation

$$\varphi \left( \frac{-c}{\sqrt{q_1 c^2 - 2q_2 c + q_3}} \right) (q_1 c - q_2)$$

$$= \sqrt{q_1 c^2 - 2q_2 c + q_3} \Phi \left( \frac{-c}{\sqrt{q_1 c^2 - 2q_2 c + q_3}} \right).$$

c* exists and is unique.

**Proof** Note that (6) can be recast as

$$\min_{\alpha} E[(\alpha^T Y)^-], \quad \text{subject to} \quad \alpha^T 1 = 1$$

$$Y = X - K 1, \quad Y \sim N(\mu_Y, \Sigma_Y), \quad \mu_Y = \mu - K 1. \quad (7)$$

We know that for any given $\alpha^T \mu_Y = c$, $E(\alpha^T Y)^-$ is an increasing function at $\alpha^T \Sigma \alpha$ and is minimized at

$$\alpha_c = \arg \min_{\alpha} \alpha^T \Sigma \alpha = \Sigma^{-1} A_K^{-1} (A_K \Sigma^{-1} A_K)^{-1} c,$$

where $A_K^T = (I, \mu - K 1)^T$ and $c = (1, c)^T$. It is straightforward to show that

$$A_K \Sigma^{-1} A_K^T = \begin{pmatrix} I^T \Sigma^{-1} 1 & I^T \Sigma^{-1} \mu_Y \\ \mu_Y^T \Sigma^{-1} 1 & \mu_Y^T \Sigma^{-1} \mu_Y \end{pmatrix},$$

$$\left( A_K \Sigma^{-1} A_K^T \right)^{-1} = \begin{pmatrix} \mu_Y^T \Sigma^{-1} \mu_Y & -I^T \Sigma^{-1} \mu_Y \\ -\mu_Y^T \Sigma^{-1} 1 & I^T \Sigma^{-1} 1 \end{pmatrix},$$

$$\alpha_c^T \Sigma \alpha_c = \left( I^T \Sigma^{-1} 1 c^2 - 2\mu_Y^T \Sigma^{-1} 1 c^2 + \mu_Y^T \Sigma^{-1} \mu_Y \right),$$

and

$$\delta = \delta^T \Sigma^{-1} 1 \mu_Y - \mu_Y^T \Sigma^{-1} \mu_Y \times \mu_Y^T \Sigma^{-1} 1 = \delta^T \Sigma^{-1} 1 \mu_Y \times \mu_Y^T \Sigma^{-1} 1.$$
Thus, for $t > 0$, $h(t)$ are non-decreasing and attain the minimum at $0$:

$$\lim_{t \to 0} h(t) = 0,$$

which concludes the proof. □

The following theorem presents the optimal portfolio that minimizes (2).

**Theorem 7** The downside risk function (12) of a portfolio $\mathbf{a}^T \mathbf{X}$ with $\mathbf{X} \sim N(\mu, \Sigma)$, subject to $\mathbf{a}^T \mathbf{1} = 1$, is minimized at

$$\mathbf{a}^* = \Sigma^{-1} A_k' (A_k \Sigma^{-1} A_k')^{-1} \mathbf{c}^*,$$

where $\mathbf{c}^* = (1, \mathbf{c}^*)^T$, $\mathbf{c}^*$ is the solution to the equation

$$\varphi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right) \sqrt{\varphi q_1 c^2 - 2q_2 c + q_3} = ((q_1 + 1)c - q_2) \phi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right).$$

c* exists and is unique.

**Proof** Similar to theorem 5, we first recast (12) as

$$\min_{\mathbf{a}} E[(\mathbf{a}^T \mathbf{Y})^-]^2,$$

subject to $\mathbf{a}^T \mathbf{1} = 1$ and

$$\mathbf{Y} = \mathbf{X} - K \mathbf{1}, \quad \mathbf{Y} \sim N(\mu_Y, \Sigma), \quad \mu_Y = \mu - K \mathbf{1},$$

and transform it to an equivalent univariate minimization according to (2), (9) and corollary 4, i.e.

$$\min_{\mathbf{c}} E[(\mathbf{c}^T \mathbf{Y})^-]^2 := \min_{\mathbf{c}} f(c),$$

where

$$f(c) = ((q_1 + 1)c - q_2) \phi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right) - \sqrt{\varphi q_1 c^2 - 2q_2 c + q_3} - c \sqrt{\varphi q_1 c^2 - 2q_2 c + q_3} \times \varphi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right).$$

We compute the first three order derivatives of $f(c)$:

$$\frac{f'(c)}{2} = ((q_1 + 1)c - q_2) \phi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right) - \sqrt{\varphi q_1 c^2 - 2q_2 c + q_3} - c \sqrt{\varphi q_1 c^2 - 2q_2 c + q_3} \times \varphi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right).$$

$$\frac{f''(c)}{2} = (q_1 + 1) \phi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right) - \sqrt{\varphi q_1 c^2 - 2q_2 c + q_3} - c \sqrt{\varphi q_1 c^2 - 2q_2 c + q_3} \times \varphi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right) - c \times \varphi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right) \times \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}}.$$

$$\frac{f'''(c)}{2} = \phi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right) - \sqrt{\varphi q_1 c^2 - 2q_2 c + q_3} - c \sqrt{\varphi q_1 c^2 - 2q_2 c + q_3} \times \varphi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right) - c \times \varphi \left( \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \right) \times \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}} \times \frac{-c}{\sqrt{\varphi q_1 c^2 - 2q_2 c + q_3}}.$$

Due to the fact that $q_1 > 0, q_3 > 0, q_1 c^2 - 2q_2 c + q_3 > 0$ and $q_1 q_3 > q_2^2$, we have

$$g(c) = c^2 (q_2^2 - 3q_1 q_2 + 3q_2^2) + 2c (3q_2^2 - q_3 q_2 - 3q_1 q_2 q_3) + q_2^2 - 2q_2 q_3 + 3q_2 q_3^2.$$
According to the proofs of theorem 5 and 7, we can see that 0.0198453317
Sandisk Microsoft Citrix Intuit Symantec
The mixed downside risk function
Corollary 8 The mixed downside risk function (16) of a portfolio with return $\alpha^TX, X \sim N(\mu, \Sigma)$, subject to $\alpha^T1 = 1$, is minimized at $\alpha^* = \Sigma^{-1}A_K^{-1}(A_K\Sigma^{-1}A_K^{-1})^{-1}e^*$, where $e^* = (1, e^*)^T$, $e^*$ is the solution to the equation
$$\varphi\left( \frac{-c}{\sqrt{q_1c^2 - 2q_2c + q_3}} \right) = \Phi\left( \frac{-c}{\sqrt{q_1c^2 - 2q_2c + q_3}} \right) \times (\lambda(2q_1c - 2c - 2q_2 + 1) - 1).$$
$c^*$ exists and is unique.

Proof The proof is similar to that of theorems 5 and 7. □

Clearly, $\lambda$ reflects the manager’s preference in the choice of downside risk measures, i.e. the target shortfall and below target variance. As mentioned previously, by choosing $\lambda = 1$, the manager’s only concern is the variance of the portfolio below the threshold. The portfolio is, therefore, expected to be more ‘stable’ in the target zone. On the other hand, for $\lambda = 0$, the manager’s portfolio has the minimal expected shortfall in mind. We, therefore, anticipate its return to be closer to the target. Intuitively, portfolios with minimal expected shortfall should outperform the ones with minimal below target variances in the sense of the expected returns for the same target. In contrast, portfolios with minimal below target variances should generally be less risky in the sense of lower variability. In next section, we demonstrate these two alternatives using numerical examples.

4. Numerical illustration

In this section, we present a numerical example to illustrate our results. We consider a portfolio of 10 stocks from the NASDAQ stock exchanges (ADOBE Sys. Inc., Compuware Corp., NVIDIA Corp., Starles Inc., VeriSign Inc., Sandisk Corp, Microsoft Corp., Symantec Corp., Citrix Sys Inc., Intuit Inc.) for the year 2005, and denote by $X = (X_1, \ldots, X_n)^T$, $n = 10$, stocks’ weekly returns. We implement the Kolmogorov–Smirnov test to test the normality assumption and since the statistics is 0.15486 with $p$-value = 0.1484, we do not reject the hypothesis of the normality.

Moreover, our approach could also provide an analytical approximation for non-normal data. For example, in the case that the stochastic return $r$ is modelled by log-normal distribution, we could take the well-known approximations that $\log(1+r) \approx r, r \ll 1$ into account. Then, we have approximately a normal distribution again. In fact, such an approximation is pretty precise when $r$ is small, which is particularly true for short-term returns such as in daily and weekly trades.

Table 1. Expected returns (means) of the 10 stocks.

<table>
<thead>
<tr>
<th>Stock</th>
<th>Expected Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adobe</td>
<td>-0.00611022899</td>
</tr>
<tr>
<td>Compuware</td>
<td>0.0081117935</td>
</tr>
<tr>
<td>NVIDIA</td>
<td>0.0095500842</td>
</tr>
<tr>
<td>Staples</td>
<td>-0.0057958520</td>
</tr>
<tr>
<td>VeriSign</td>
<td>-0.0064370369</td>
</tr>
<tr>
<td>Sandisk</td>
<td>0.0198453317</td>
</tr>
<tr>
<td>Microsoft</td>
<td>-0.0001754431</td>
</tr>
<tr>
<td>Citrix</td>
<td>0.0037884516</td>
</tr>
<tr>
<td>Intuit</td>
<td>0.0041225491</td>
</tr>
<tr>
<td>Symantec</td>
<td>-0.0061032938</td>
</tr>
</tbody>
</table>

Note: The expected returns (means) of the 10 stocks are listed in Table 1.
Table 2. Covariance Matrix of the 10 stocks.

<table>
<thead>
<tr>
<th></th>
<th>Adobe</th>
<th>Compuware</th>
<th>NVDIA</th>
<th>Staples</th>
<th>VeriSign</th>
<th>Sandisk</th>
<th>Microsoft</th>
<th>Citrix</th>
<th>Intuit</th>
<th>Symantec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adobe</td>
<td>0.00610</td>
<td>0.001173</td>
<td>0.000118</td>
<td>0.000512</td>
<td>0.000120</td>
<td>0.00100</td>
<td>−0.000119</td>
<td>0.000395</td>
<td>0.000134</td>
<td>0.000536</td>
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<td>Compuware</td>
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<tr>
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<td>0.001053</td>
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<td>0.000905</td>
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<td>0.000772</td>
<td>−0.000546</td>
<td>0.000084</td>
<td>0.000309</td>
<td>0.000255</td>
<td>0.000435</td>
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<tr>
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<td>0.000214</td>
<td>0.000122</td>
<td>0.000077</td>
<td>0.000468</td>
<td>0.000054</td>
<td>0.000255</td>
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</tr>
<tr>
<td>Citrix</td>
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<td>0.000559</td>
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<td>0.000793</td>
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<td>0.000309</td>
<td>0.000255</td>
<td>0.000435</td>
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<td>0.000066</td>
<td>0.000131</td>
<td>0.000084</td>
<td>0.000309</td>
<td>0.000255</td>
<td>0.000435</td>
</tr>
<tr>
<td>Symantec</td>
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<td>0.000435</td>
<td>8.3298E−05</td>
<td>2.71609E−05</td>
<td>0.000876</td>
<td>2.71609E−05</td>
<td>0.000254</td>
<td></td>
</tr>
</tbody>
</table>
Table 3. Optimal portfolios with respect to Optimization (6) and Optimization (12) for different targets. Sum of the weights equals to 10. Expected returns and variances of the portfolio are reported correspondingly in the last two rows.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>( K = -0.05 )</th>
<th>( K = -0.01 )</th>
<th>( K = 0 )</th>
<th>( K = 0.01 )</th>
<th>( K = 0.05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \min R_1 )</td>
<td>( \min R_2 )</td>
<td>( \min R_1 )</td>
<td>( \min R_2 )</td>
<td>( \min R_1 )</td>
</tr>
<tr>
<td>Adobe</td>
<td>0.0646365</td>
<td>0.1819154</td>
<td>0.001813</td>
<td>0.1524191</td>
<td>0.020668</td>
</tr>
<tr>
<td>Compuserve</td>
<td>-0.078936</td>
<td>-0.237032</td>
<td>0.0105504</td>
<td>-0.197284</td>
<td>0.0359588</td>
</tr>
<tr>
<td>NVIDIA</td>
<td>0.7040149</td>
<td>0.5598366</td>
<td>0.7857061</td>
<td>0.5969082</td>
<td>0.8088861</td>
</tr>
<tr>
<td>Staples</td>
<td>-0.521878</td>
<td>-0.216597</td>
<td>-0.694849</td>
<td>-0.293377</td>
<td>-0.743930</td>
</tr>
<tr>
<td>VeriSign</td>
<td>-0.766799</td>
<td>-0.485576</td>
<td>-0.926140</td>
<td>-0.556305</td>
<td>-0.971353</td>
</tr>
<tr>
<td>Sandisk</td>
<td>1.5692155</td>
<td>1.2622548</td>
<td>1.7431389</td>
<td>1.3394571</td>
<td>1.7924901</td>
</tr>
<tr>
<td>Citrix</td>
<td>-0.042999</td>
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<td>-0.020724</td>
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<td>-0.014404</td>
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<tr>
<td>Symantec</td>
<td>0.8616230</td>
<td>0.889630</td>
<td>0.8472839</td>
<td>0.8805655</td>
<td>0.8432151</td>
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<tr>
<td>Portfolio mean</td>
<td>0.052597</td>
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<td>0.061017</td>
<td>0.041475</td>
<td>0.063406</td>
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<tr>
<td>Portfolio variance</td>
<td>0.038591</td>
<td>0.034286</td>
<td>0.041687</td>
<td>0.03523</td>
<td>0.042651</td>
</tr>
</tbody>
</table>
Table 4. Optimal portfolios with respect to Optimization (16) for different $\lambda$. Sum of the weights equals to 10. Expected returns and variances of the portfolio are reported correspondingly in the last two rows.

<table>
<thead>
<tr>
<th></th>
<th>$\lambda = 0.05$</th>
<th>$\lambda = 0.5$</th>
<th>$\lambda = 0.95$</th>
<th>$\lambda = 0.05$</th>
<th>$\lambda = 0.5$</th>
<th>$\lambda = 0.95$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adobe</td>
<td>0.00147659</td>
<td>0.0428046</td>
<td>0.1342750</td>
<td>0.0364065</td>
<td>0.0134727</td>
<td>0.1167599</td>
</tr>
<tr>
<td>Compuware</td>
<td>0.00611682</td>
<td>0.049574</td>
<td>0.172834</td>
<td>0.0571659</td>
<td>0.010048</td>
<td>0.149232</td>
</tr>
<tr>
<td>NVDIA</td>
<td>0.78166133</td>
<td>0.7308542</td>
<td>0.6184039</td>
<td>0.8282334</td>
<td>0.7669136</td>
<td>0.6399363</td>
</tr>
<tr>
<td>Staples</td>
<td>-0.6882854</td>
<td>-0.578707</td>
<td>-0.340607</td>
<td>-0.7848963</td>
<td>-0.655059</td>
<td>-0.386199</td>
</tr>
<tr>
<td>VeriSign</td>
<td>-0.9182508</td>
<td>-0.819150</td>
<td>-0.599813</td>
<td>-1.0090907</td>
<td>-0.8894852</td>
<td>-0.641813</td>
</tr>
<tr>
<td>Sandisk</td>
<td>1.73452743</td>
<td>1.6263573</td>
<td>1.3869466</td>
<td>1.83368104</td>
<td>1.7031292</td>
<td>1.432789</td>
</tr>
<tr>
<td>Microsoft</td>
<td>4.65040405</td>
<td>4.7470263</td>
<td>4.9608786</td>
<td>4.56183564</td>
<td>4.6784503</td>
<td>4.9199293</td>
</tr>
<tr>
<td>Citrix</td>
<td>-0.0218276</td>
<td>-0.035681</td>
<td>-0.066342</td>
<td>-0.0091291</td>
<td>-0.025848</td>
<td>-0.060471</td>
</tr>
<tr>
<td>Symantec</td>
<td>0.84799390</td>
<td>0.8569120</td>
<td>0.8766502</td>
<td>0.83981915</td>
<td>0.850825</td>
<td>0.8728707</td>
</tr>
<tr>
<td>Portfolio mean</td>
<td>0.060606</td>
<td>0.05536</td>
<td>0.04377</td>
<td>0.06540</td>
<td>0.05908</td>
<td>0.04599</td>
</tr>
<tr>
<td>Portfolio variance</td>
<td>0.04152</td>
<td>0.03955</td>
<td>0.03585</td>
<td>0.04348</td>
<td>0.04093</td>
<td>0.03649</td>
</tr>
</tbody>
</table>
We present the corresponding means and covariance in tables 1 and 2. Optimal weights that minimize $R_1$ and $R_2$ with respect to different targets $K$ are presented in table 3. Table 4 presents the results regarding the portfolios that minimize the combined objectives.\footnote{For the computational convenience, we consider the initial constraint on the weight as $\alpha^T 1 = 10$. In fact, for any initial wealth $W_0 \neq 0$, we are to solve}

According to tables 3 and 4, we can see that for the same given target $K$, portfolios that have minimal expected shortfall have higher expected returns as well as higher variances, compared with the ones that minimize the below target variances. Moreover, we observe that both the expected return and variance of the optimal portfolio rise with the target $K$. On the one hand, minimal expected shortfall keep the expected return of the portfolio close to the target; on the other hand, higher target results in an increase in volatility. After all, such a scenario agrees with the common sense that higher returns go hand in hand with higher risks.

Furthermore, the consistency between the expected return and the variance also implies a Mean–Variance efficient frontier for the optimal portfolio selections. The Mean–Variance efficient frontier is as important as it is useful for establishing the Capital Asset Pricing Model and for measuring the performance of the portfolio, see for example Sharpe (1971) and

\[ \min_{\alpha} E[\alpha^T X - K] \] subject to $1^T \alpha = 1$, $\mu^T \alpha = c$

then it is easy to see that the results obtained in section 3 are applicable as $\hat{X}$ is again multivariate normal distributed.

In contrast to the classic Mean-Variance efficient frontier, we can work out the Mean-Downside risk efficient frontiers, see figure 1. Thus, we may also use such Mean-Downside risks efficient frontiers to measure the performance of the portfolios.

5. Conclusions and further discussion

In this paper, we discuss optimal portfolios with respect to downside risk, i.e. $E((X - K)^-)^\beta$, where $\beta = 1, 2$, in the context of the multivariate normal distribution. We show that the two downside risks are monotone increasing in $\sigma$, which immediately suggests that the solutions to the optimal minimization problems

\[ \min_{\alpha} E(\alpha^T X - K)^- \] subject to $1^T \alpha = 1$, $\mu^T \alpha = c$

and

\[ \min_{\alpha} E((\alpha^T X - K)^-)^2 \] subject to $1^T \alpha = 1$, $\mu^T \alpha = c$

coinides with those of the classical mean-variance optimization. Hence, considering downside risk, though it appears to be more attractive for many investors compared to the classical variance, cannot improve the optimal gain when the expected return of the portfolio is fixed. This proffers theoretical support to the empirical findings in literature such as Jarrow and Zhao (2006).

Moreover, if we drop off the assumption of a pre-specified return, i.e. we consider the following optimization problems

\[ \min_{\alpha} E(\alpha^T X - K)^- \] subject to $1^T \alpha = 1$, $\mu^T \alpha = c$

and

\[ \min_{\alpha} E((\alpha^T X - K)^-)^2 \] subject to $1^T \alpha = 1$, $\mu^T \alpha = c$.\footnote{Joro and Na (2006). In contrast to the classic Mean-Variance efficient frontier, we can work out the Mean-Downside risk efficient frontiers, see figure 1. Thus, we may also use such Mean-Downside risks efficient frontiers to measure the performance of the portfolios.}
then the optimal solutions differ. In the context of normality, we obtain the analytical solutions to the two general problems and show that the solution to each optimization exists and is unique. We also provide a numerical illustration of the results. According to the numerical results, we find that the optimal portfolios with respect to the two minimizations can be very different. Hence, we further propose a new downside optimization that targets the combination of the two downside risk measures as follows,

$$\min_{\alpha} E[\lambda((\alpha^T \mathbf{X} - K)^-) + (1 - \lambda)((\alpha^T \mathbf{X} - K)^-)]$$

subject to $\alpha^T \mathbf{1} = 1$,

where $\lambda$ reflects the investors’ preference between the two downside risk measures. Again, we obtain an analytical solution to this combined optimization and show that it exists and is unique.

As a matter of fact, there are a vast number of references concerning optimal portfolios. Due to the complexity of the objective function, most of them rely on numerical techniques to find the optimal solutions. This is particularly the case for downside risks, which has been often discussed in the literature. In contrast to a numerical solution, the analytical solution guarantees the existence and uniqueness of the solution. On the one hand, it is easy to calculate and to implement; and on the other hand, it does not suffer from unavoidable computational errors nor the great computational effort involved. Clearly, analytical solution is simply superior. In this regard, our results provide useful methodology and new insights into the literature on optimal portfolios with respect to downside risks, and it gives both theoretical and practical contributions to such problems.

We rely on the assumptions of normality to obtain our results. Although the normal distribution is probably one of the most frequently applied probability laws in modeling the stochastic returns of assets, empirical evidence sometimes suggest the non-normality of such data, especially for short-term returns. While our methodology offers a sound approximation to non-normal returns, it is our aim to further extend our results to the case where the stochastic returns are modeled by non-normal distributions in general, and the elliptical distribution (Fang et al. 1990) in particular, which is much more flexible and richer than the normal distribution.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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