Locally Lipschitz BSDE driven by a continuous martingale path-derivative approach

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Abstract

Using a new notion of path-derivative, we study existence and uniqueness of solution for backward stochastic differential equation (BSDE) driven by a continuous martingale $M$ with $[M,M]_t = \int_0^t m_s m_s^* d\text{tr}[M,M]_s$:

$$Y_t = \xi([0,T]) + \int_t^T f(s, [0,t], Y_s, Z_s) m_s^* d\text{tr}[M,M]_s - \int_t^T Z_s dM_s - N_T + N_t$$

Here, for $t \in [0,T]$, $M_{[0,t]}$ is the path of $M$ from 0 to $t$, and $\xi([0,T])$ and $f(t, [0,t], y, z)$ are deterministic functions of $(t, \gamma, y, z) \in [0,T] \times D \times \mathbb{R}^d \times \mathbb{R}^{d \times n}$. In particular, we are interested in the case when $f(s, \gamma, y, z)$ is locally Lipschitz in $(y, z)$. The path-derivative is defined as a directional derivative with respect to the path-perturbation of $M$ in a similar way to the vertical functional derivative introduced by Dupire (2009), and Cont and Fournie (2013). We first prove the existence, uniqueness, and path-differentiability of solution in the case where $f(s, \gamma, y, z)$ is Lipschitz in $y$ and $z$. After proving $Z$ is a path-derivative of $Y$ when $f$ is Lipschitz and $M$ has martingale representation property, we extend the existence and uniqueness results to locally Lipschitz $f$. When the BSDE is one-dimensional, we could show the existence and uniqueness of solution. On the contrary, when the BSDE is multidimensional, we show existence and uniqueness only when $[M,M]_T$ is small enough: otherwise, we provide a counterexample that has blowing-up solution. Lastly, we investigate the applications to utility maximization problems under power and exponential utility function.
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Key words: Backward stochastic differential equation, path differentiability, functional derivative, coefficients of superlinear growth, utility maximization

1 Introduction

Let $M$ be a square integrable continuous $n$-dimensional local martingale with quadratic covariation matrix $[M, M]_t = \int_0^t m_s m_s^* d[M, M]_s$ for a $\mathbb{R}^{n \times n}$-valued process $m$. We let $D$ be the set of càdlàg $\mathbb{R}^n$-valued functions on $[0, T]$. Consider the following backward stochastic differential equation (BSDE) driven by $M$ where the terminal condition is $\xi : D \to \mathbb{R}^d$ and the driver is $f : [0, T] \times D \times \mathbb{R}^d \times \mathbb{R}^{d \times n} \to \mathbb{R}^d$:

$$Y_t = \xi(M_{[0,T]}) + \int_t^T f(s, M_{[0,s]}, Y_{s-}, Z_s m_s) d[M, M]_s - \int_t^T Z_s dM_s - N_T + N_t$$ (1.1)
Here, we denote $M_{[0,s]}$ to be the path of $M$ stopped at $s$. The solution of above BSDE is a triplet $(Y,Z,N)$ of adapted processes satisfying $[N,M] = 0$. We study the existence and uniqueness of solution when $f(s,\gamma,y,z)$ are Lipschitz in $\gamma$ and locally Lipschitz in $(y,z)$. In order to do so, we study the differentiability of solutions under the perturbation of the path of $M$. Then, we apply our result to various utility maximization problems.

BSDE was first introduced by Bismut (1973, [2]) as a dual problem of stochastic optimization under the assumptions $d = 1$, Brownian motion $M$, and a linear function $f$. Then, Pardoux and Peng (1990, [39]) extended the well-posedness result to $d \geq 1$ and Lipschitz function $f$. One can find classical results and applications in the survey paper written by El Karoui et al. (1997, [20]). Since Pardoux and Peng’s seminal paper, researchers extended the well-posedness result in various directions.

One direction of extension is to incorporate the case where $f$ grows superlinearly in $z$. The well-posedness results for such BSDEs have numerous applications including utility maximization in incomplete market (Hu et al. 2005, [23]), dynamic coherent risk measure (Gianin, 2006, [22]), equilibrium pricing in incomplete market (Cheridito et al., 2016, [7]), and more recently, stochastic Radner equilibrium in incomplete market (Kardaras et al., 2015, [30]). When $d = 1$ and $\xi$ is bounded, Kobylanski (2000, [33]) proved the existence and uniqueness of solution when $f(s,\gamma,y,z)$ grows quadratically in $z$. Briand and Hu (2006, [3], 2008, [3]), and Delbaen et al. (2011, [17]) further extended the results to unbounded terminal condition $\xi$. Superquadratic BSDE driven by a Brownian motion also attracted the interest among mathematician. Delbaen et al. (2010, [16]) showed that such BSDE is ill-posed if there is no regularity assumption on the terminal condition and the driver. Richou (2012, [32]) studied the existence and uniqueness of solution for superquadratic Markovian BSDE. Cheridito and Nam (2014, [8]) showed the existence and uniqueness of solution for the non-Markovian case using Malliavin calculus and its connection to semilinear parabolic PDEs under the Markovian assumption.

On the contrary, when $d > 1$, Frei and dos Reis (2012, [21]) showed that a multidimensional BSDE with a quadratic driver might not be well-posed. By choosing a terminal condition which is irregular with respect to the underlying Brownian motion, they were able to construct an example such that the solution $Y$ blows up. When one does not assume regularity conditions on $\xi$ and $f$, only a few positive results are known when the terminal condition is small, or the driver satisfies certain restrictive structural conditions: see Tevzadze (2008, [13]), Cheridito and Nam (2014, [9]), Hu and Tang (2014, [25]), Jamneshan et al. (2014, [27]), Kupper et al. (2015, [31]), and Xing and Zitkovic (2016, [44]). When $\xi$ and $f$ are assumed to be regular, Nam(2014,[36]), Kupper et al. (2015, [34]), and Cheridito and Nam (2017, [19])

Researchers also tried to generalize Brownian motion to a general martingale $M$. When $M$ is a continuous martingale, El Karoui and Huang (1997, [19]) provided the existence and uniqueness of solution in the case where $f(t,\gamma_{[0,t]},y,z)$ is Lipschitz with respect to $(y,z)$ when $d \geq 1$. When $d = 1$, Morlais (2009, [35]) investigated the existence and uniqueness of solution when $f(t,\gamma_{[0,t]},y,z)$ has quadratic growth in $z$. Researchers generalized even to the case where $M$ is a general martingale with jumps. To name a few, Possamaï et al. (2015, [32]) studied the case where $d = 1$, $f$ has quadratic growth in $z$, and $M$ has jumps. On the other hand, Papapantoleon et al. (2016, [38]) treat the case where $d \geq 1$ and $f$ is (stochastically) Lipschitz in $(y,z)$. However, the following questions have not been answered when $M$ is a general martingale:

- If $d = 1$, does one have well-posedness when $f(s,\gamma,y,z)$ grows superquadratically in $z$?
- If $d > 1$, does one have well-posedness when $f(s,\gamma,y,z)$ grows superlinearly in $z$?

In this article, we answer these questions when $M$ is a continuous martingale; $\xi(\gamma)$ is Lipschitz with respect to $\gamma$; and $f(s,\gamma,y,z)$ is Lipschitz in $\gamma$ and locally Lipschitz in $(y,z)$. To be more specific, we were able to establish existence and uniqueness of solution when $d = 1$ and find a uniform almost sure bound
of the solution \( Z \). In the case where \( d > 1 \), we have the existence and uniqueness of solution as well as the bound of \( Z \) only if \([M, M]_T\) is small enough: otherwise, we provide a counterexample such that \( Z \) blows up. We apply the 1D result to various kinds of control problems for SDE driven by \( M \) using the martingale method introduced by Hu et al. (2005, [23]). In the case where \( M \) has jumps, our method does not work anymore\(^1\) and we leave this question for future papers.

The argument is based on the analysis of the stability under perturbation of \( M \). We call this stability path-differentiability of the solution. In other words, when we model stochastic optimization problem as a BSDE, the path-derivative of \( Y \) implies the stability of value process with respect to the perturbation of underlying noise. An important property, which is called delta-hedging formula, is that \( Z \) is the path-derivative of \( Y \) under appropriate notion if \( M \) possesses martingale representation property. If one can find a uniform bound of \( Z \) by estimating the derivative of \( Y \) and using delta-hedging formula, we can use the localization argument to prove the well-posedness of BSDEs with locally Lipschitz drivers.

Using Malliavin calculus on BSDE as in El Karoui et al. (1997, [20]) and Hu et al. (2012, [24]), this strategy was used in Briand and Elie (2013, [3]), Cheridito and Nam (2014, [8]), and Kupper et al. (2015, [34]) when \( M \) is a Brownian motion. However, this method cannot be trivially extended to a continuous martingale \( M \) because \( M \) is not Malliavin differentiable in general. For example, consider the case where \( M \) is a Brownian motion stopped at a hitting time. Even when \( \xi \) is smooth, \( \xi(W) \) is not Malliavin differentiable in general\(^2\). Therefore, classical Malliavin calculus method used in the papers mentioned above cannot be used to study the path-differentiability of solution for this type of BSDE\(^3\).

One may define another path-derivative notion for BSDE by assuming Markovian structure, that is, one assumes \( \xi(\gamma) = \xi(X_T) \) and \( f(t, \gamma_{[0, T]}, y, z) = f(t, X_t, y, z) \) where \( dX_t = b(t, X_t)dA_t + \sigma(t, X_t)dM_t \) for some deterministic function \( b \) and \( \sigma \). In many cases, there is a deterministic measurable function \( u \) such that \( Y_t = u(t, X_t, M_t) \). Then one can define path-differentiability as a classical differentiability of the function \( u \). This approach was used in Imkeller et al. (2012, [26]) to study existence, uniqueness, and path-differentiability of (1.1) with \( d = 1 \) and \( f \) grows quadratically in \( z \). However, this method cannot be extended BSDE with fully path-dependent coefficients.

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One of the recent definitions of “path-derivative” is the functional Itô derivative developed by Dupire (2009, [18]), and Cont and Fournie (2013, [14]). The perturbation in functional Itô derivative is given by either horizontal or vertical displacement of the path at the last time. The functional Itô calculus is general in a sense that it assumes neither Markovian structure nor Gaussian property of \( M \). Using vertical functional Itô derivative, Cont (2016, [13]) was able to get a delta-hedging formula for (1.1) when \( M \) is a continuous semimartingale determined by forward SDE driven by Brownian motion. Even though functional Itô calculus has its own strength, it is not suitable for obtaining a uniform bound of \( Z \). The reason is that we do not know the equation the functional Itô derivative of \( Y \) satisfies.

In order to find a uniform estimate of \( Z \), we modify the vertical functional Itô derivative to time-parametrized version similar to Malliavin derivative and use such notion to obtain BSDE for path-derivatives of \( Y \) and \( Z \). Then, by the classical method in BSDE, we get a uniform bound of \( Z \). However, we should note that the path-functional representation of random variables and stochastic processes are not unique and our path-derivative definition crucially depends on the representation. Therefore, it is important to select logically consistent representations of the coefficients \( \xi, f \) and our solution \( Y, Z, N \). This is done by Theorem 3.7 and it is the main reason why we cannot extend our result to the case where \( M \) has jumps.

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\(^1\)see Remark 3.8

\(^2\)Cheridito and Nam (2014, [8]) If \( \tau \) is a stopping time such that \( W_\tau \) is Malliavin differentiable, then \( \tau \) must be a constant. Indeed, for \( W_\tau = \int_0^{\infty} 1_{(s < \tau)} dW_s \in D^{1,2} \), one obtains from Proposition 5.3 of El Karoui et al. (1997, [20]) that \( 1_{(s < \tau)} \in D^{1,2} \) for almost all \( s \), and therefore, by Proposition 1.2.6 of Nualart (2006, [37]), \( P[s < \tau] = 1 \).

\(^3\)This type of BSDE is also known as BSDE with random terminal time and studied by numerous researchers including Darling and Pardoux (1997, [15]) and Jeanblanc et al. (2015, [25]).
The article is organized as follows. In Section 2, we give the definitions, notations, and assumptions we use throughout this article. In Section 3, we review the basic properties of BSDE with Lipschitz driver and driven by a continuous martingale. Then we study the differentiability of BSDE in Section 4. Using results from Section 3 and 4, we study the existence and uniqueness of solution for BSDEs with locally Lipschitz drivers in Section 5. In particular, we show the existence and uniqueness of solution when $[M, M]_T$ is small enough or $d = 1$. Otherwise, the solution may blow-up, and it is shown by an example in subsection 5.3. We study utility maximization of controlled SDE in Section 6. In Section 7, for power and exponential utility function, the scheme is applied to optimal portfolio selection under three different types of restriction: 1) when the investment strategy is restricted to a closed set; 2) when the diversification of portfolio gives the investor extra benefit; and 3) when there is information processing cost for investment.

2 Preliminaries

Real space

We denote $\mathbb{R}$ the set of real number and $\mathbb{R}_+$ the set of nonnegative real numbers. For any natural numbers $l$ and $m$, $\mathbb{R}^{l \times m}$ is the set of real $l$-by-$m$ matrices. $\mathbb{R}^m$ is the set of $m$-dimensional real vectors and we identify with $\mathbb{R}^{m \times 1}$ unless otherwise stated. For any matrix $X$, we let $X^*$ to be its transpose and we define $|X|$ to be the Euclidean norm, that is $|X|^2 := \text{tr}(XX^*)$. We always endow Borel $\sigma$-algebra on $\mathbb{R}^{l \times m}$ with respect to the norm $| \cdot |$ and denote it by $\mathcal{B}(\mathbb{R}^{l \times m})$. For $X \in \mathbb{R}^{l \times n}$, we denote $(i, j)$-entry of $X$ as $X^{ij}$. We denote $I$ to be the identity matrix of appropriate size.

Probability space and the driving martingale

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})$ be a filtered probability space. We assume the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$ is complete, quasi-left continuous, and right continuous. Let $M$ be a square integrable continuous $n$-dimensional martingale with a continuous predictable quadratic covariation matrix $[M, M]$ and $M_0 = 0$. We assume that there exists a $\mathbb{R}^{n \times n}$-valued predictable process $m$ such that

$$[M, M]_t = \int_0^t m_s m^*_s dA_s$$

where

$$A_t := \text{tr}[M, M]_t = \sum_{i=1}^n [M^i, M^i]_t.$$

Moreover, we always assume that $A_T$ is bounded by $K$. Then, we have two consequences:

- $|m_s| = 1 ds \otimes d\mathbb{P}$-a.e. ($\because A_t = \sum_{i=1}^n \int_0^t \sum_{k=1}^n m^i_k^2 dA_s = \int_0^t |m_s|^2 dA_s$)
- $|[M, M]_T| \sim A_T$ ($\because A_T/\sqrt{n} \leq |[M, M]_T| = \int_0^T m_s m^*_s dA_s \leq \int_0^T |m_s|^2 dA_s = A_T$)

In addition, we assume there exists a Poisson random measure $\nu$ on $[0, T] \times \mathbb{R}^n$ with mean $\text{Leb} \otimes \mu$ where $\mu$ is the uniform probability measure on a unit ball centered at 0. We let

$$\tilde{M}_t := \int_{[0,t] \times \mathbb{R}^n} x\nu(ds, dx)$$

and $\mathbb{R}^n$-valued càdlàg martingale $M' := M + \tilde{M}$. We assume that $M$ and $\nu$ are independent and moreover, for any given $\gamma \in \{\tilde{M}(\omega) : \omega \in \Omega\}$ and $\omega' \in \Omega$, there is $\omega \in \Omega$ such that $\tilde{M}(\omega) = \gamma$ and $M(\omega) = M(\omega')$. 

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We also assume that $\mathbb{F}^M$, the augmentation of $\sigma(M_s : s \leq t)$, is quasi-left continuous and right continuous. This condition is true when $M$ is a Hunt process: see Proposition 2.7.7 of Karatzas and Shreve (1991, [29]) and Section 3.1 of Chung and Walsh (2005, [11]). Therefore, any Feller process $M$ satisfies this property. This implies, $\mathbb{F}^M$, the augmentation of $\sigma(M_s' : s \leq t)$, is also quasi-left continuous and right continuous. It is noteworthy to observe $\mathbb{F}^M \subset \mathbb{F}'^M$ because $M$ is a continuous process while $\hat{M}$ is a pure jump process. 

Note that since $\mathbb{F}$ is continuous, for any $\mathbb{R}^d$-valued $(\mathbb{F}, \mathbb{P})$-martingale is of the form $\int ZdM + N$ where $Z$ is a $\mathbb{F}$-predictable $\mathbb{R}^{d \times n}$-valued process and $N$ is a $\mathbb{R}^d$-valued $(\mathbb{F}, \mathbb{P})$-martingale with $[N,M] = 0$. This statement also holds with $\mathbb{F}'^M$ or $\mathbb{F}^M$ instead of $\mathbb{F}$.

As always, we understand equalities and inequalities in $\mathbb{P}$-almost sure sense.

The space of càdlàg paths

We let $D$ be the set of all càdlàg $\mathbb{R}^n$-valued functions on $[0,T]$. For $\gamma \in D$, we denote $\gamma_t$ to be the value at time $t$ and $\gamma_{[0,t]}(s) := \gamma_{s\wedge t}$. For $\gamma, \gamma' \in D$, we define $(\gamma + \gamma')_{t} := \gamma_t + \gamma'_t$. On $D$, we endow a sup norm, $\|\gamma\|_\infty := \sup_{t\in[0,T]} |\gamma_t|$ and let $D$ be its Borel $\sigma$-algebra. Then, we have the following lemma whose proof is given at the appendix.

**Lemma 2.1.** A $\mathbb{R}^k$-valued stochastic process $X$ is adapted to $\mathbb{F}^M$ if and only if there exists a path functional $\mathcal{X} : [0,T] \times D \to \mathbb{R}^k$ such that

$$X_t = \mathcal{X}(t, M_{[0,t]}')$$

holds almost surely for each $t \in [0,T]$ and $\mathcal{X}(t, \cdot)$ is $\mathcal{D}$-measurable.

Let $x_t(\gamma) = \gamma_t$ and define a filtration $\mathcal{H}_t := \sigma(\{x_s : s \leq t\})$. We let $\mathcal{P}$ be the predictable $\sigma$-algebra on $[0,T] \times D$ associated with filtration $\{\mathcal{H}_t\}$. Then, it is easy to check that if a function $f : [0,T] \times D \to \mathbb{R}^d$ is $\mathcal{P}$-measurable, then $f(t, M_{[0,t]})$ is a predictable process since $M_{[0,\cdot]} : [0,T] \times \Omega \to D$ is a predictable process.

Banach space

We set the following Banach spaces:

- $\mathbb{L}^2$: all $d$-dimensional random vectors $X$ satisfying $\|X\|_2 := \sqrt{\mathbb{E}|X|^2} < \infty$
- $\mathbb{S}^2$: all $\mathbb{R}^d$-valued càdlàg adapted processes $(Y_t)_{0 \leq t \leq T}$ satisfying $\|Y\|_{\mathbb{S}^2} := \|\sup_{0 \leq t \leq T} |Y_t|\|_2 < \infty$
- $\mathbb{H}^2$: all $\mathbb{R}^d$-valued càdlàg adapted processes $(Y_t)_{0 \leq t \leq T}$ satisfying $\|Y\|_{\mathbb{H}^2} := \mathbb{E}\int_0^T |Y_{t-}|^2 dA_t < \infty$
- $\mathbb{H}^2_m$: all $\mathbb{R}^d$-valued predictable processes $(Z_t)_{0 \leq t \leq T}$ satisfying $\|Z\|_{\mathbb{H}^2_m} := \mathbb{E}\int_0^T |Z_{t-}|^2 dA_t < \infty$
- $\mathbb{M}^2$: all càdlàg martingale $(N_t)_{0 \leq t \leq T}$ satisfying $\|N\|_{\mathbb{M}^2} := \mathbb{E}\text{tr}[N,N]_T < \infty$, $[N,M] = 0$, and $N_0 = 0$.

BSDE and its solution

Assume that $\xi : D \to \mathbb{R}^d$ is $\mathcal{D}$-measurable and $f : [0,T] \times D \times \mathbb{R}^d \times \mathbb{R}^{d \times n} \to \mathbb{R}^d$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^{d \times n})$-measurable. The solution of BSDE($\xi, f$) is a triplet of adapted processes $(Y, Z, N) \in \mathbb{H}^2 \times \mathbb{H}^2_m \times \mathbb{M}^2$ which satisfies

$$Y_t = \xi(M_{[0,T]}) + \int_t^T f(s, M_{[0,s]}, Y_{s-}, Z_sM_s)dA_s - \int_t^T Z_sdM_s - N_T + N_t.$$ (BSDE($\xi, f$))

With a slight abuse of notation, sometimes we denote the above BSDE as BSDE($\xi(M_{[0,T]}), f$).
3 Properties of BSDE with Lipschitz Driver

In this section we present existence, uniqueness, stability, comparison, and path-representation results regarding BSDE($\xi, f$) when $f(s, \gamma, y, z)$ is Lipschitz with respect to $(y, z)$. Except for path-representation of solutions provided in Theorem 3.6 and Theorem 3.7, most of results are well-known: see e.g. El Karoui regarding BSDE

$$(\mathcal{P}')$$

Properties of BSDE with Lipschitz Driver

Proposition 3.1. Assume (STD). Then there exists a unique solution $(Y, Z, N) \in \mathbb{H}^2 \times \mathbb{H}^2_m \times \mathbb{M}^2$ of

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s m_s) dA_s - \int_t^T Z_s dM_s - N_T + N_t$$

and moreover, $Y \in \mathbb{S}^2$.

Remark 3.2. Note that $f(s, Y_s, Z_s)$ may not be predictable. Therefore, the integral with respect to $A$ should be interpreted as Lebesgue-Stieltjes integral.

Proof. Let us define the following Banach spaces:

- $\mathbb{H}_d^2$: all $\mathbb{R}^d$-valued càdlàg adapted processes $(Y_t)_{0 \leq t \leq T}$ satisfying $\|Y\|_{\mathbb{H}_d^2} := \mathbb{E} \int_0^T e^{aA_t} |Y_t|^2 dA_t < \infty$.

- $\mathbb{H}_{m,a}^2$: all $\mathbb{R}^d$-valued predictable processes $(Z_t)_{0 \leq t \leq T}$ satisfying $\|Z\|_{\mathbb{H}_{m,a}^2} := \mathbb{E} \int_0^T e^{aA_t} |Z_t m_t|^2 dA_t < \infty$.

- $\mathbb{M}_d^2$: all càdlàg martingale $(N_t)_{0 \leq t \leq T}$ satisfying $\|N\|_{\mathbb{M}_d^2} := \mathbb{E} \int_0^T e^{aA_t} dtr[N, N]_t < \infty$ and $N_0 = 0$.

For $a = 2 |C_y \vee C_z|^2 + 2$, we will use contraction mapping theorem for

$$\phi : (y, z, n) \in \mathbb{H}_d^2 \times \mathbb{H}_{m,a}^2 \times \mathbb{M}_d^2 \mapsto (Y, Z, N) \in \mathbb{H}_d^2 \times \mathbb{H}_{m,a}^2 \times \mathbb{M}_d^2.$$

where $(Y, Z, N)$ is given by the solution of BSDE

$$Y_t = \xi + \int_t^T f(s, y_s, z_s m_s) dA_s - \int_t^T Z_s dM_s - N_T + N_t,$$

or equivalently,

$$Y_0 + \int_0^t Z_s dM_s + N_t = \mathbb{E}_t \left[ \xi + \int_0^T f(s, y_s, z_s m_s) dA_s \right].$$

Then, since $e^{aA_t}$ is between 1 and $e^{aK}$, the space $\mathbb{H}_d^2 \times \mathbb{H}_{m,a}^2 \times \mathbb{M}_d^2$ is equivalent to $\mathbb{H}^2 \times \mathbb{H}_{m}^2 \times \mathbb{M}^2$ and the fixed point we get by contraction mapping theorem is the unique solution in $\mathbb{H}^2 \times \mathbb{H}_{m}^2 \times \mathbb{M}^2$.

First, let us show that $\phi(y, z, n) = (Y, Z, N)$ is in $\mathbb{H}^2 \times \mathbb{H}_{m}^2 \times \mathbb{M}^2$ and therefore in $\mathbb{H}_d^2 \times \mathbb{H}_{m,a}^2 \times \mathbb{M}_d^2$.

From Theorem 27 Corollary 3 of II.6 of Protter (2004, [11]),

$$\mathbb{E} \left| \mathbb{E}_t \left[ \xi + \int_0^T f(s, y_s, z_s m_s) dA_s \right] \right|^2 \leq \mathbb{E} \left| \xi + \int_0^T f(s, y_s, z_s m_s) dA_s \right|^2 < \infty.$$
for all \( t \in [0, T] \) implies
\[
\|Z\|_{H^{2,a}}^2 + \|N\|_{M^2}^2 = \text{Etr} \left[ \int_0^T Z_s dM_s + N, \int_0^T Z_s dM_s + N \right]_T < \infty.
\]
On the other hand,
\[
|Y_t| \leq |\xi| + \int_0^T |f(s, y_s, z_s, m_s)| dA_s + \sup_{t \in [0, T]} \int_t^T Z_s dM_s + N_T - N_t.
\]
From Burkholder-Davis-Gundy inequality, for some constant \( C' \),
\[
\mathbb{E} \sup_{t \in [0, T]} \left| \int_0^T Z_s dM_s + N_T - N_t \right|^2 \leq 2\mathbb{E} \left| \int_0^T Z_s dM_s + N_T \right|^2 + 2\mathbb{E} \sup_{t \in [0, T]} \left| \int_t^T Z_s dM_s + N_t \right|^2 \leq C' \left( \|Z\|_{H^{2,a}}^2 + \|N\|_{M^2}^2 \right).
\]
Therefore, \( Y \in \mathbb{S}^2 \) and this implies \( Y \in H^2 \).

Next, let us show the contraction. Let \((Y, Z, N) := \phi(y, z, n)\) and \((Y', Z', N') := \phi(y', z', n')\). Let us denote \( \delta Y_s := Y_s - Y'_s, \delta Z_s := Z_s - Z'_s, \delta N := N_s - N'_s, \) and \( \delta f := f(s, y_s, z_s, m_s) - f(s, y'_s, z'_s, m_s) \). Then,
\[
\delta Y_t = \int_t^T \delta f_s dA_s - \int_t^T \delta Z_s dM_s - \delta N_T + \delta N_t.
\]
By Itô formula,
\[
0 \leq |\delta Y_0|^2 = \int_0^T e^a A_s (2\delta Y_s^* \delta f_s - |\delta Z_s m_s|^2 - a|\delta Y_s|^2) dA_s - \int_0^T e^a A_s d\text{tr} [\delta N, \delta N]_s
\]
\[
- 2\int_0^T e^{2A_s} \delta Y_s^* \delta Z_s dM_s - 2\int_0^T e^{2A_s} \delta Y_s^* \delta N_s
\]
\[
\leq \int_0^T e^{a A_s} ((a - 1)|\delta Y_s|^2 + \frac{1}{a - 1} |\delta f_s|^2 - |\delta Z_s m_s|^2 - a|\delta Y_s|^2) dA_s - \int_0^T e^{a A_s} d\text{tr} [\delta N, \delta N]_s
\]
\[
- 2\int_0^T e^{2A_s} \delta Y_s^* \delta Z_s dM_s - 2\int_0^T e^{2A_s} \delta Y_s^* \delta N_s.
\]
If we take expectation on both side and rearrange it, by Lemma \( \text{A.1} \) we get
\[
\mathbb{E} \int_0^T e^{a A_s} \left( |\delta Y_s|^2 + |Z_s m_s|^2 \right) dA_s + \mathbb{E} \int_0^T e^{a A_s} d\text{tr} [\delta N, \delta N]_s
\]
\[
\leq \frac{1}{a - 1} \mathbb{E} \int_0^T e^{a A_s} |\delta f_s|^2 dA_s \leq \frac{a - 2}{a - 1} \mathbb{E} \int_0^T e^{a A_s} \left( |y_s - y'_s|^2 + |z_s m_s - z'_s m_s|^2 \right) dA_s
\]
and \( \phi \) is a contraction on \( \mathbb{H}_a^2 \times \mathbb{H}_{m,a}^2 \times \mathbb{M}_a^2 \). Therefore, there exists a unique fixed point, which is our solution, in \( \mathbb{H}_a^2 \times \mathbb{H}_{m,a}^2 \times \mathbb{M}_a^2 \). Therefore, there is a unique solution in \((Y, Z, N) \in \mathbb{H}^2 \times \mathbb{H}_{a}^2 \times \mathbb{M}^2 \) and \( Y \in \mathbb{S}^2 \) from the argument at the beginning of the proof. \( \square \)

**Proposition 3.3.** Assume (STD). Moreover, assume that there exist \( C_\xi, C_f \in \mathbb{R}_+ \) such that \( |\xi| \leq C_\xi \) and \( \int_0^T |f(s, 0, 0)|^2 dA_s \leq C_f^2 \). Then, for solution \((Y, Z, N)\) of (3.1), we have \( |Y_t| \leq \sqrt{C_\xi^2 + C_f^2 e^{\frac{1}{2}K(2C_\phi + C_\xi^2 + 1)}}. \)
Proof. Since (STD) are satisfied, there exists a unique solution \((Y, Z, N) \in \mathbb{H}^2 \times \mathbb{H}^2_{m} \times \mathbb{M}^2\). By Itô formula, when \(a = 2C_y + C_z^2 + 1\), we have
\[
e^{aA_t} |Y_t|^2 = e^{aA_t} |\xi|^2 + \int_t^T e^{aA_s} \left(2Y_s^* f(s, Y_s, Z_s m_s) - |Z_s m_s|^2 - a|Y_s|^2\right) dA_s - \int_t^T e^{aA_s} dtr [N, N]_s
- 2 \int_t^T e^{aA_s} Y_s^* Z_s dM_s - 2 \int_t^T e^{aA_s} Y_s^* dN_s
\leq e^{aA_t} |\xi|^2 + \int_0^T e^{aA_s} |f(s, 0, 0)|^2 dA_s - 2 \int_t^T e^{aA_s} Y_s^* Z_s dM_s - 2 \int_t^T e^{aA_s} Y_s^* dN_s.
\]
because
\[
2Y_s^* f(s, Y_s, Z_s m_s) \leq |f(s, 0, 0)|^2 + (2C_y + C_z^2 + 1) |Y_s|^2 + |Z_s m_s|^2.
\]
If we take \(\mathbb{E}(\cdot |\mathcal{F}_t)\) on both side, by Lemma \[A.1\], we get
\[
|Y_t|^2 \leq \mathbb{E}_t \left[ e^{aA_T} |\xi|^2 \right] + \mathbb{E}_t \left[ e^{aA_T} \int_0^T |f(s, 0, 0)|^2 dA_s \right] \leq (C_y^2 + C_z^2 e^{K(2C_y + C_z^2 + 1)})
\]
\[\square\]

**Proposition 3.4. (Stability) Assume that \((\xi, f)\) satisfies (STD) with Lipschitz coefficients \(C_y\) and \(C_z\). Also assume that \((\bar{\xi}, \bar{f})\) satisfies (STD) possibly with different Lipschitz coefficients. Let \((Y, Z, N)\) and \((\bar{Y}, \bar{Z}, \bar{N})\) are solutions of
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s m_s) dA_s - \int_t^T Z_s dM_s - N_T + N_t
\]
\[
\bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s m_s) dA_s - \int_t^T \bar{Z}_s dM_s - \bar{N}_T + \bar{N}_t,
\]
Then, we have the following estimate:
\[
\|Y_t - \bar{Y}_t\|^2_2 + \|Y - \bar{Y}\|^2_{\mathbb{H}^2} + \|Z - \bar{Z}\|^2_{\mathbb{H}^2_{m}} + \|N - \bar{N}\|^2_{\mathbb{M}^2}
\leq 2 e^{K(2C_y^2 + 2C_z^2 + 2)} \left( \|\xi - \bar{\xi}\|^2_2 + \|f(\cdot, \bar{Y}, \bar{Z}, \bar{m}) - \bar{f}(\cdot, \bar{Y}, \bar{Z}, \bar{m})\|^2_{\mathbb{H}^2} \right)
\]
**Proof.** Denote \(\delta Y := Y - \bar{Y}, \delta Z := Z - \bar{Z}, \delta N := N - \bar{N}, \delta \xi := \xi - \bar{\xi}\), and
\[
g(s, y, z) := f(s, \bar{Y}_s + y, \bar{Z}_s m_s + z) - \bar{f}(s, \bar{Y}_s + \bar{Y}_s, \bar{Z}_s m_s).
\]
Then, we have
\[
\delta Y_t = \delta \xi + \int_t^T g(s, \delta Y_s, \delta Z_s m_s) dA_s - \int_t^T \delta Z_s dM_s - \delta N_T + \delta N_t,
\]
where \((\delta \xi, g)\) satisfies (STD). By applying Itô formula on \(e^{aA_t} |\delta Y_t|^2\) where \(a = 2C_y + 2C_z^2 + 2\), we have
\[
e^{aA_t} |\delta Y_t|^2 = e^{aA_t} |\delta \xi|^2 + \int_t^T e^{aA_s} (2\delta Y_s^* g(s, \delta Y_s, \delta Z_s m_s) - a|\delta Y_s|^2 - |\delta Z_s m_s|^2) dA_s
- \int_t^T e^{aA_s} dtr[\delta N, \delta N]_s - \int_t^T 2e^{aA_s} \delta Y_s^* Z_s dM_s - \int_t^T 2e^{aA_s} \delta Y_s^* dN_s.
\]
and this implies, by Lemma A.1
\[
\mathbb{E}e^{aA_t}|\delta Y_t|^2 + \mathbb{E} \int_t^T e^{aA_s}|\delta Y_{s-}|^2 dA_s + \frac{1}{2} \mathbb{E} \int_t^T e^{aA_s}|\delta Z_s m_s|^2 dA_s + \mathbb{E} \int_t^T e^{aA_s}d\text{tr}[\delta N, \delta N]_s \\
\leq \mathbb{E}e^{aA_t}|\delta \xi|^2 + \mathbb{E} \int_t^T e^{aA_s}|g(s, 0, 0)|^2 dA_s \leq e^{K(2C_y+2C_z^2+2)} \left( \mathbb{E}|\delta \xi|^2 + \mathbb{E} \int_0^T |g(s, 0, 0)|^2 dA_s \right)
\]
since
\[
2\delta Y^a_s(g(s, \delta Y_{s-}, \delta Z_s m_s)) \leq |g(s, 0, 0)|^2 + (2C_y + 2C_z)|\delta Y_{s-}|^2 + \frac{1}{2}|\delta Z_s m_s|^2.
\]
Therefore,
\[
\sup_{t \in [0, T]} \mathbb{E}|\delta Y_t|^2 + \mathbb{E} \int_0^T |\delta Y_{s-}|^2 dA_s + \mathbb{E} \int_0^T |\delta Z_s m_s|^2 dA_s + \mathbb{E}\text{tr}[\delta N, \delta N]_T \\
\leq 2 \left( \sup_{t \in [0, T]} \mathbb{E}e^{aA_t}|\delta Y_t|^2 + \mathbb{E} \int_0^T e^{aA_s}|\delta Y_{s-}|^2 dA_s + \frac{1}{2} \mathbb{E} \int_0^T e^{aA_s}|\delta Z_s m_s|^2 dA_s + \mathbb{E} \int_0^T e^{aA_s}d\text{tr}[\delta N, \delta N]_s \right) \\
\leq 2e^{K(2C_y+2C_z^2+2)} \left( \mathbb{E}|\delta \xi|^2 + \mathbb{E} \int_0^T |g(s, 0, 0)|^2 dA_s \right).
\]

Now let us prove the comparison theorem when \(d = 1\). This result will be used in Section 4.2. We will denote \(\mathcal{E}(X) = \exp \left( X - \frac{1}{2}[X, X] \right) \).

**Theorem 3.5.** (Comparison Theorem) Let \(d = 1\) and assume that \(m_t\) is invertible for all \(t \in [0, T]\). Assume (STD) for \((\xi, f)\) and \((\bar{\xi}, f)\). Let \((Y, Z, N)\) and \((\bar{Y}, \bar{Z}, \bar{N})\) be the solutions of
\[
Y_t = \xi + \int_t^T f(s, Y_{s-}, Z_s m_s) dA_s - \int_t^T Z_s dM_s - N_T + N_t \\
\bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}(s, \bar{Y}_{s-}, \bar{Z}_s m_s) dA_s - \int_t^T \bar{Z}_s dM_s - \bar{N}_T + \bar{N}_t.
\]
If \(\xi \leq \bar{\xi}\) almost surely and \(f(t, y, z) \leq \bar{f}(t, y, z)\) dt \(\otimes\) d\(\mathbb{P}\) \(\otimes\) dy \(\otimes\) dz-almost everywhere, then \(Y_t \leq \bar{Y}_t\) a.s. for all \(t \in [0, T]\).

**Proof.** Let us denote
\[
\delta \xi := \xi - \bar{\xi} \quad \delta f_s := f(s, Y_{s-}, Z_s m_s) - \bar{f}(s, \bar{Y}_{s-}, \bar{Z}_s m_s) \\
\delta Y_s := Y_s - \bar{Y}_s \quad \delta Z_s := Z_s - \bar{Z}_s \quad \delta N_s := N_s - \bar{N}_s
\]
and
\[
(Zm)^{(i)} := ((\bar{Z}m)^1, (\bar{Z}m)^2, \ldots, (\bar{Z}m)^i, (\bar{Z}m)^{i+1}, (\bar{Z}m)^{i+2}, \ldots, (\bar{Z}m)^n) \\
F_s := \frac{f(s, Y_{s-}, Z_s m_s) - f(s, \bar{Y}_{s-}, \bar{Z}_s m_s)}{\delta Y_{s-}} \\
G_s := \frac{f(s, \bar{Y}_{s-}, (\bar{Z}m)^{(i-1)}) - f(s, \bar{Y}_{s-}, (\bar{Z}m)^{(i)})}{(\delta \bar{Z}m)_s} \\
d\Gamma_s := \Gamma_s \left( F_s dA_s + G_s^2 (m_s)^{-1} dM_s \right); \quad \Gamma_0 = 1.
\]
Note that $F$ and $G^i$ are uniformly bounded by $C_y$ and $C_z$, respectively, and $\Gamma_t \geq 0$ for all $t$. Moreover,

$$\Gamma_t = \mathcal{E} \left( \int_0^t F_s dA_s + \int_0^t G_s^*(m_s)^{-1} dM_s \right)_t \leq e^{C_y K} \mathcal{E} \left( \int_0^t G_s^*(m_s)^{-1} dM_s \right)_t$$

where $\mathcal{E} \left( \int_0^t G_s^*(m_s)^{-1} dM_s \right)$ is a martingale because of Novikov condition. Note that, by Doob’s maximal inequality,

$$\mathbb{E} \sup_{0 \leq s \leq t} \left| \mathcal{E} \left( \int_0^t G_s^*(m_s)^{-1} dM_s \right)_s \right|^2 \leq 4 \mathbb{E} \mathcal{E} \left( \int_0^t G_s^*(m_s)^{-1} dM_s \right)_t^2$$

$$\leq 4 \mathbb{E} e^{K} |G_s|^2 dA_s \mathcal{E} \left( 2 \int_0^t G_s^*(m_s)^{-1} dM_s \right)_t \leq 4 e^{C_y^2 K} < \infty$$

and therefore, $\Gamma \in \mathcal{S}^2$.

On the other hand, if we subtract both equations, we get

$$\delta Y_t = \delta \xi + \int_t^T (\delta f_s + F_s \delta Y_{s-} + \delta Z_s m_s G_s) dA_s - \int_t^T \delta Z_s dM_s - \delta N_T + \delta N_s.$$ 

If we apply Itô formula to $\Gamma_s \delta Y_s$, we get

$$\Gamma_t \delta Y_t = \delta Y_0 - \int_0^t \Gamma_s \delta f_s dA_s + \int_0^t (\delta Y_{s-} \Gamma_s G_s^*(m_s)^{-1} + \Gamma_s \delta Z_s) dM_s + \int_0^t \Gamma_s \delta N_s$$

This implies $\Gamma \delta Y + \int \Gamma_s \delta f_s dA_s$ is a local martingale. Note that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\Gamma_s \delta Y_s| \leq \frac{1}{2} ||\Gamma||_{\mathcal{S}^2}^2 + \frac{1}{2} \|\delta Y\|_{\mathcal{S}^2}^2$$

$$\leq \mathbb{E} \sup_{0 \leq s \leq t} \left| \Gamma_s \delta f_s dA_s \right| \leq \mathbb{E} \sup_{0 \leq s \leq t} |\Gamma_s| \int_0^t |\delta f_s| dA_s \leq \frac{1}{2} \|\Gamma\|_{\mathcal{S}^2}^2 + \frac{1}{2} \mathbb{E} \left( \int_0^t |\delta f_s| dA_s \right)^2 < \infty$$

Therefore, $\Gamma \delta Y + \int \Gamma_s \delta f_s dA_s$ and $\int (\delta Y_{s-} \Gamma_s G_s^*(m_s)^{-1} + \Gamma_s \delta Z_s) dM_s + \int \Gamma_s \delta N_s$ are martingales. If we take $\mathbb{E}_t$ on both side of the backward version of (3.2), we get

$$\delta Y_t = \frac{1}{\Gamma_t} \mathbb{E}_t \left[ \Gamma_T \delta \xi + \int_t^T \Gamma_s \delta f_s dA_s \right] \geq 0.$$

Now let us give the existence and uniqueness result when the terminal condition and the driver depends on the path of $M'$ or $M$. Consider the following assumptions:

(S) For any $\gamma \in D$ with $\|\gamma\|_{\infty} \leq 1$,

$$\xi(M_{[0,T]} + \gamma), \xi(M'_{[0,T]}) \in \mathbb{L}^2 \quad \text{and} \quad \mathbb{E} \int_0^T |f(s, M_{[0,s]} + \gamma, 0, 0)|^2 dA_s, \mathbb{E} \int_0^T |f(s, M'_{[0,s]}, 0, 0)|^2 dA_s < \infty.$$  

(Lip) There are nonnegative constants $C_y$ and $C_z$ such that

$$|f(t, \gamma_{[0,t]}, y, z) - f(t, \gamma_{[0,t]}, y', z')| \leq C_y|y - y'| + C_z|z - z'|$$

for all $\gamma \in D$, $t \in [0, T]$, $y, y', z, z' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d \times n}$. 

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Theorem 3.6. Assume (S) and (Lip). The following BSDE
\[ Y_t = \xi(M'_{[0,T]}(\omega)) + \int_t^T f(s, M'_{[0,s]}(\omega), Y_s, Z_s, N_s) \, ds - \int_t^T Z_s \, dM_s - N_T + N_t, \]
has a unique solution \((Y, Z, N) \in \mathbb{H}^2 \times \mathbb{H}_m^2 \times \mathbb{M}^2\). Moreover, \(Y \in \mathbb{S}^2\) and there are path functionals 
\(\mathcal{Y} : [0,T] \times D \to \mathbb{R}^d, Z : [0,T] \times D \to \mathbb{R}^{d \times n}\), and 
\(\mathcal{N} : [0,T] \times D \to \mathbb{R}^d\) such that 
\(\mathcal{Y}(t, \cdot), Z(t, \cdot), \mathcal{N}(t, \cdot)\) are 
\(\mathcal{D}\)-measurable and
\[ Y_t = \mathcal{Y}(t, M'_{[0,t]}), \quad Z_t = \mathcal{Z}(t, M'_{[0,t]}), \quad \text{and} \quad N_t = \mathcal{N}(t, M'_{[0,t]}) \]
for each \(t \in [0,T]\).

Proof. Since \(\xi(M'_{[0,T]}(\omega))\) and \(f(s, M'_{[0,s]}(\omega), y, z)\) satisfies (STD), the existence and uniqueness of solution follows 
from Proposition 3.1. Consider a BSDE with the same terminal condition and driver under the filtration 
\(\mathbb{F}\) and that \(Y, Z, N\) has a unique solution 
\((Y, Z, N) \in \mathbb{H}^2 \times \mathbb{H}_m^2 \times \mathbb{M}^2\). Moreover, \(Y \in \mathbb{S}^2\) and there are path functionals 
\(\mathcal{Y} : [0,T] \times D \to \mathbb{R}^d, Z : [0,T] \times D \to \mathbb{R}^{d \times n}\), and 
\(\mathcal{N} : [0,T] \times D \to \mathbb{R}^d\) such that 
\(\mathcal{Y}(t, \cdot), Z(t, \cdot), \mathcal{N}(t, \cdot)\) are 
\(\mathcal{D}\)-measurable and
\[ Y_t = \mathcal{Y}(t, M'_{[0,t]}), \quad Z_t = \mathcal{Z}(t, M'_{[0,t]}), \quad \text{and} \quad N_t = \mathcal{N}(t, M'_{[0,t]}) \]
for each \(t \in [0,T]\). In particular, this holds when \(h = 0\).

Proof. The existence of unique solution \((Y^h, Z^h, N^h)\) and \(Y^h\) being in \(\mathbb{S}^2\) comes from Proposition 3.1. Let us 
denote \(M^h := M_t + h e_i 1_{[u,T]}(\omega)\). Let \((Y', Z', N')\) be the unique solution of 
(3.3) and \((\mathcal{Y}, \mathcal{Z}, \mathcal{N})\) be the corresponding path functionals. Let us define \(\Omega' \subset \Omega\) so that 
\(\mathbb{P}(\Omega') = 1\) and \(Y'_t(\omega') = \mathcal{Y}(t, M'_{[0,T]}(\omega')), Z'_t(\omega') = \mathcal{Z}(t, M'_{[0,T]}(\omega')), \text{ and } N'_t(\omega') = \mathcal{N}(t, M'_{[0,T]}(\omega'))\) 
for all \(\omega' \in \Omega'\). Note that, by our assumption on \(\tilde{M}\) and \(M\), 
for all \(\omega' \in \Omega'\), there exists \(\omega \in \Omega\) such that \(M(\omega) = M(\omega')\) and \(\hat{M}(\omega) = h e_i 1_{[u,T]}(\omega')\). For such \(\omega'\),
\[ Y'_t(\omega) = \mathcal{Y}(t, M^h_{[0,t]}(\omega)), \quad Z'_t(\omega) = \mathcal{Z}(t, M^h_{[0,t]}(\omega)), \quad \text{and} \quad N'_t(\omega) = \mathcal{N}(t, M^h_{[0,t]}(\omega)). \]
and
\[ Y'_t(\omega) = \xi(M^h_{[0,T]}(\omega)) + \int_t^T f(s, M^h_{[0,s]}(\omega), Y'_s(\omega), Z'_s(\omega)m_s(\omega)) \, ds - \int_t^T Z'_s(\omega)m_s(\omega) \, dM_s(\omega) - \left(\int_t^T Z'_s(\omega)m_s(\omega) \, dM_s(\omega)\right)(\omega) - N'_t(\omega) + N'_t(\omega). \]
Since this holds for all \(\omega\) realizing all possible paths of \(M^h\) and \(M\), the triplet
\((Y^h, Z^h, N^h) := (\mathcal{Y}(t, M^h_{[0,t]}), \mathcal{Z}(t, M^h_{[0,t]}), \mathcal{N}(t, M^h_{[0,t]}))\)
is the unique solution of BSDE(\(\xi^h, f^h\)).

Remark 3.8. One may ask whether we can consider \(M\) with jumps. If \(M\) is a martingale with jumps, we 
know that the solution \((Y, Z, N)\) is adapted to the filtration generated by both \(M\) and \(M'\). However, \(M\) 
is not adapted to \(\mathbb{F}^M\) and it is not obvious that whether the solution \((Y, Z, N)\) is actually adapted to the 
filtration generated only by \(M'\).
4 Path-differentiability of BSDE with Lipschitz Driver

Assuming that every martingale can be represented by stochastic integral with respect to \( M, Z \) is often a "path-derivative" of \( Y \) in some sense; see, for example, El Karoui (1997, [20]) for Malliavin calculus sense or Cont (2016, [1]) for functional Itô calculus sense. This property is also called delta-hedging formula due to its relationship with finance. We will prove this property and use it to study locally Lipschitz BSDEs. More precisely, we will find the almost sure uniform bound of the path-derivative of \( Y \) to conclude \( Z \) is uniformly bounded. Then, our locally Lipschitz BSDE becomes essentially Lipschitz BSDE and existence, uniqueness, and stability automatically follows from the results of Section 3.

However, both Malliavin calculus and functional Itô calculus are not suitable for our problem as we described in the introduction. Therefore, we will define a new sense of path-derivative. Then, we will prove the delta-hedging formula for BSDE and show that the path-derivative of \((Y, Z, N)\) is the solution of the differentiated BSDE. This result, combined with Proposition 3.3, will be used to find the bound of \( Z \) in Section 5.

**Definition 4.1.** For a random variable \( V = \mathcal{V}(M_{[0,T]}) \) and a vector \( e \in \mathbb{R}^{1 \times n} \), we say \( V \) is \( \nabla^e \)-differentiable at \( u \) if

\[
\lim_{h \to 0, 1^2} \frac{\mathcal{V}(M_{[0,T]} + h e^1_{[u,T]}) - \mathcal{V}(M_{[0,T]})}{h}
\]

exists and we denote it by \( \nabla^e_u V \). For a stochastic process \( X = \mathcal{X}([., M_{[0,T]}]) \) and a vector \( e \in \mathbb{R}^{1 \times n} \), we say \( X \) is \( \nabla^e_\cdot, \nabla^{e,m}_\cdot, \) and \( \nabla^{e,N}_\cdot \)-differentiable at \( u \) and define the \( \nabla^e_\cdot, \nabla^{e,m}_\cdot, \) and \( \nabla^{e,N}_\cdot \)-derivative at \( u \) by

\[
\nabla^e_u X := \lim_{h \to 0, H^2} \frac{\mathcal{X}([., (M_{[0,T]} + h e^1_{[u,T]}), [., [.]) - \mathcal{X}([., M_{[0,T]}])}{h}
\]

if \( X \in H^2 \)

\[
\nabla^{e,m}_u X := \lim_{h \to 0, H^2 m} \frac{\mathcal{X}([., (M_{[0,T]} + h e^1_{[u,T]}), [., [.]) - \mathcal{X}([., M_{[0,T]}])}{h}
\]

if \( X \in H^2 m \)

\[
\nabla^{e,N}_u X := \lim_{h \to 0, M^2} \frac{\mathcal{X}([., (M_{[0,T]} + h e^1_{[u,T]}), [., [.]) - \mathcal{X}([., M_{[0,T]}])}{h}
\]

if \( X \in M^2 \)

if the corresponding limit exists. In general, we denote

\[
\nabla_u V = \sum_{i=1}^n (\nabla^e_i V) e_i, \quad \nabla^{e,m}_u X = \sum_{i=1}^n (\nabla^{e,m}_i X) e_i, \quad \text{and} \quad \nabla^{e,N}_u X = \sum_{i=1}^n (\nabla^{e,N}_i X) e_i.
\]

where \( \{e_i\}_{i=1,2,\ldots,n} \) is the standard basis of \( \mathbb{R}^{1 \times n} \). If a random variable or a stochastic process \( \nabla^e_\cdot, \nabla^{e,m}_\cdot, \) and \( \nabla^{e,N}_\cdot \)-differentiable for every \( e \in \mathbb{R}^{1 \times n} \) and at almost every \( u \in [0,T] \), then we say the random variable or stochastic process is differentiable with respect to \( M, or \nabla_\cdot, \nabla^{m}_\cdot, \nabla^{N}_\cdot \)-differentiable.

**Remark 4.2.** This is a modified version of vertical functional Itô derivative: see Dupire (2009, [18]), Cont and Fournie (2013, [13]). The key differences are that it is time-parametrized and the convergence is in \( L^2 \)-sense with respect to an appropriate measure.

Note that the above definition crucially depends on the representation path-functional \( \mathcal{V} \) or \( \mathcal{X} \). For example, let \( c : \gamma \in D \mapsto c(\gamma) \in C([0,T] : \mathbb{R}^n) \) be a function that removes any jump part of \( \gamma \). Then, the random variable \( V = \mathcal{V}(M_{[0,T]}) \) can also be written as \((V \circ c)(M_{[0,T]})\). Note that \( V \circ c \) is always \( \nabla \)-differentiable with derivative 0.
Therefore, in order to incorporate above definitions to establish meaningful results in BSDE, we need to choose path representation carefully. Since (stochastic) integral are defined as a limit of time-partitioned sum, we have already restricted our representation of $\xi$ and $f$ by writing a BSDE. Therefore, one should choose the path-functional representation of $\xi$ and $f$ as limits of path-dependent functionals which depends on finite number of time sections of the path of $M$. Otherwise, the path-derivative of left hand side of BSDE may be different from path-derivative of right hand side which is written by stochastic integrals.

Let $C_k$ be the set of continuously differentiable functions from $(\mathbb{R}^n)^k$ to $\mathbb{R}^d$. Let

$$S := \left\{ H : D \to \mathbb{R}^d : \exists k \in \mathbb{N}, g \in C_k, 0 = t_0^k \leq t_1^k \leq \cdots \leq t_k^k = T \text{ s.t. } H(\gamma) = g(\gamma_{t_1^k} - \gamma_{t_0^k}, \cdots, \gamma_{t_k^k} - \gamma_{t_{k-1}^k}) \right\}$$

For $\xi^{(k)} \in S$, we select $g \in C_k$ such that $\xi^{(k)}(\gamma) = g(\gamma_{t_1^k} - \gamma_{t_0^k}, \cdots, \gamma_{t_k^k} - \gamma_{t_{k-1}^k})$ and then, we have

$$\nabla_u \xi^{(k)}(M_{[0,T]}(\gamma)) = (\partial_i g)(M_{t_1^k} - M_{t_0^k}, \cdots, M_{t_k^k} - M_{t_{k-1}^k})e^i$$

where $i$ satisfies $t_{i-1}^k < u \leq t_i^k$. For the driver $f$ which depends on finitely many values $(\gamma_{t_i^k})_{i=1,\ldots,k}$, we choose its path functional and define the derivative similarly: for each $(s, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times n}$, we treat $f(s, \cdot, y, z)$ as in $S$. For solution $(Y, Z, N)$ of BSDE$(\xi, f)$, we will always refer to the path functional $Y, Z, N$ defined in Theorem 3.7. Note that the choice of such $(Y, Z, N)$ may not be unique but their derivatives are unique stochastic processes which together forms a solution of differentiated BSDE as we will see soon. This is consistent with the result of Cont and Fournie (2013, [14]).

We would like to emphasize that these definitions are only needed in order to estimate the bound of $Z$ process by delta-hedging formula $Z_t = \nabla_t Y_t$ which is the next section’s main result Theorem 5.1. The key idea is the following:

(i) Proposition 4.3. Consider BSDE$(\xi^{(k)}, f^{(k)})$ where $(\xi^{(k)}, f^{(k)})$ converges to $(\xi, f)$. Choose a representation of $(Y^{(k)}, Z^{(k)}, N^{(k)})$ of BSDE$(\xi^{(k)}, f^{(k)})$ by Theorem 3.7 and establish BSDE satisfied by $(\nabla_t Y^{(k)}, \nabla_t Z^{(k)}, \nabla_t N^{(k)})$.

(ii) Theorem 4.5 and Corollary 4.6. Prove $Z_t^{(k)} = \nabla_t Y_t^{(k)}$ when $M$ has martingale representation property.

(iii) Theorem 5.1. Using Proposition 3.3, Proposition 4.3 and the fact that $Z^{(k)}$ converges to $Z$, find the bound of $Z$.

**Proposition 4.3.** Assume $\xi$ and $f$ satisfy (S), (Lip)

(Diff) For each $(s, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times n}$, $\xi \in S$ and $f(s, \cdot, y, z) \in S$. In addition, for all $e \in \mathbb{R}^{1 \times n}$ and almost every $u \in [0, T]$, $\nabla^n_u \xi(M_{[0,T]}) \in L^2$ and $\nabla^n_u f(\cdot, M_{[0,T]}, y', z') \in \mathbb{R}^2$,

(D) For all $t \in [0, T], \gamma \in D$, $f(t, \gamma_{[0,t]}, y, z)$ is continuously differentiable with respect to $y$ and $z$. and let $(Y, Z, N)$ be the solution of BSDE$(\xi, f)$. Let $Y, Z, N$ be the corresponding path functional as in Theorem 3.7. Then, the solution $Y = Y(\cdot, M_{[0,T]}), Z = Z(\cdot, M_{[0,T]}), N = N(\cdot, M_{[0,T]})$ are $\nabla_t, \nabla^{m \cdot},$ and $\nabla^N$-differentiable, respectively. Moreover, for each $i = 1, 2, \cdots, n$ and almost every $u \in [0, T]$,

$$((\nabla_u^i Y)_t, (\nabla_u^{m \cdot} Z)_t, (\nabla_u^{m \cdot} N)_t) = \left\{ \begin{array}{ll} (0, 0, 0) & \text{if } u > t \\ (U_t, V_t, W_t) & \text{if } u \leq t \end{array} \right. dt \otimes d\mathbb{P}\text{-almost everywhere}$$

where $(U, V, N) \in \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2$ is the unique solution of the (3.1) with the terminal condition $\nabla_u^i \xi(M_{[0,T]})$ and the driver

$$g(t, y, z) = \zeta_t + \eta_t y + \theta_t \cdot z$$
Here, we defined
\[
\zeta_t := (\nabla_u^t f)(t, M_{[0,t]}, y', z')|_{(y', z')=(Y_t, Z_t, \eta_t)}
\]
\[
\eta_t := (\partial_y f)(t, M_{[0,t]}, Y_t, Z_t, \eta_t)
\]
\[
\theta_t := (\partial_z f)(t, M_{[0,t]}, Y_t, Z_t, \eta_t)
\]
and
\[
\theta_t \cdot z := \sum_{i,j} (\partial_{z_{ij}} f(t, M_{[0,t]}, Y_t, Z_t, \eta_t)) z_{ij}
\]

Proof. Note that \( Y_t = \mathcal{Y}(t, M_{[0,t]}), Z_t = \mathcal{Z}(t, M_{[0,t]}), \) and \( N_t = \mathcal{N}(t, M_{[0,t]}), \) and therefore, if \( u > t, \) then \( ((\nabla_u^t Y)_t, (\nabla_u^t Z)_t, \nabla_u^t N) = (0, 0, 0) \) because \((Y_t, Z_t, N_t)\) is unaffected by the perturbation of \( M \) at \( u. \)

Let us denote \( M^h_t := M_t + \h e_1^* 1_{[u,T]}(t) \) and
\[
\xi^h(\gamma_{[0,T]}):= \xi(\gamma_{[0,T]} + h e_1^* 1_{[u,T]})
\]
\[
f^h(s, \gamma_{[0,s]}, y, z) := f(s, (\gamma + h e_1^* 1_{[u,T]})(s), y, z).
\]

Note that \((\mathcal{Y}(t, M^h_{[0,t]}), \mathcal{Z}(t, M^h_{[0,t]}), \mathcal{N}(t, M^h_{[0,t]}))\) is the unique solution of \(\text{BSDE}(\xi^h, f^h)\) by Theorem 3.7. Let us define, for \( u \leq t, \)
\[
\Xi^{h,u,i} := \frac{\xi(M^h_{[0,T]}) - \xi(M_{[0,T]})}{h}
\]
\[
U^{h,u,i}_t := \frac{\mathcal{Y}(t, M^h_{[0,t]}) - \mathcal{Y}(t, M_{[0,t]})}{h}
\]
\[
V^{h,u,i}_t := \frac{\mathcal{Z}(t, M^h_{[0,t]}) - \mathcal{Z}(t, M_{[0,t]})}{h}
\]
\[
W^{h,u,i}_t := \frac{\mathcal{N}(t, M^h_{[0,t]}) - \mathcal{N}(t, M_{[0,t]})}{h}
\]

Then, for \( t \geq u, \) we have
\[
U^{h,u,i}_t = \Xi^{h,u,i} + \int_t^T (\delta^{h,u,i} f)(s, M_{[0,s]}, U^{h,u,i}_{s-}, V^{h,u,i}_{s-}, m_s) dA_s - \int_t^T V^{h,u,i}_{s-} dM_s - W^{h,u,i}_t + W^{h,u,i}_t
\]

Here, we defined
\[
(\delta^{h,u,i} f)(t, M_{[0,t]}, y, z)
\]
\[
= \frac{1}{h} \left[ f^{h}(t, M_{[0,t]}, \mathcal{Y}(t-, M_{[0,t-]})) + hy, \mathcal{Z}(t, M_{[0,t]}) m_t + hz) - f(t, M_{[0,t]}, \mathcal{Y}(t-, M_{[0,t-]})) \right]
\]
\[
\Xi^{h,u,i} + \eta^{h,u,i}_t y + \theta^{h,u,i}_t z
\]

where
\[
\zeta^{h,u,i}_t := \frac{1}{h} \left[ f^{h}(t, M_{[0,t]}, \mathcal{Y}(t-, M_{[0,t-]})) \mathcal{Z}(t, M_{[0,t]}) m_t) - f(t, M_{[0,t]}, \mathcal{Y}(t-, M_{[0,t-]})) \mathcal{Z}(t, M_{[0,t]}) m_t) \right]
\]
\[
\eta^{h,u,i}_t y + \theta^{h,u,i}_t z := \frac{1}{h} \left[ f^{h}(t, M_{[0,t]}, \mathcal{Y}(t-, M_{[0,t-]})) + hy, \mathcal{Z}(t, M_{[0,t]}) m_t + hz)
\]
\[
- f^{h}(t, M_{[0,t]}, \mathcal{Y}(t-, M_{[0,t-]})) \mathcal{Z}(t, M_{[0,t]}) m_t).]
This BSDE has a unique solution \((U^{h,u,i}, V^{h,u,i}, W^{h,u,i}) \in \mathbb{H}^2 \times \mathbb{H}_m^2 \times \mathbb{M}^2\) because \(\Xi^{h,u,i} \) and \(\delta^{h,u,i} f\) satisfies (STD). In particular, \(|\eta_t^{h,u,i}| \leq C_y\) and \(|\theta_t^{h,u,i}| \leq C_z\ dt \otimes d\mathbb{P}\)-a.s. uniformly for all \(h\) and \(u\). Also note that

\[
\lim_{h \to 0, L^2} \|\Xi^{h,u,i}\|_{\mathbb{H}^2} = 0, \quad \lim_{h \to 0} \eta_t^{h,u,i} = \eta_t, \quad \text{and} \quad \lim_{h \to 0} \theta_t^{h,u,i} = \theta_t
\]

with \(|\eta_t| \leq C_y\) and \(|\theta_t| \leq C_z\). Then, by Proposition 3.4

\[
\|U_t^{h,u,i} - U_t\|^2_{\mathbb{H}^2} + \|V_t^{h,u,i} - V_t\|^2_{\mathbb{H}^2} + \|W_t^{h,u,i} - W_t\|^2_{\mathbb{M}^2} \leq 2e^{K(2C_y + 2C^2 + 2)} \left( \|\Xi^{h,u,i} - \Xi\|^2_{\mathbb{H}^2} + \left\| \zeta^{h,u,i} - \zeta + (\eta^{h,u,i} - \eta)U + (\theta^{h,u,i} - \theta) \cdot V \right\|^2_{\mathbb{H}^2} \right). 
\]

By dominated convergence theorem, as \(h \to 0\), we have

\[
\| (\eta^{h,u,i} - \eta)U \|_{\mathbb{H}^2} \to 0 \quad \text{and} \quad \| (\theta^{h,u,i} - \theta) \cdot V \|_{\mathbb{H}^2} \to 0.
\]

Therefore,

\[
U_t^{h,u,i} \overset{L^2}{\to} U_t = \nabla_u^0 Y_t \quad \text{for all } t \in [0, T] \\
U_t^{h,u,i} \overset{\mathbb{H}^2}{\to} U = \nabla_u^0 Y \\
V_t^{h,u,i} \overset{\mathbb{H}_m^2}{\to} V = \nabla_u^{e,m} Z \\
W_t^{h,u,i} \overset{\mathbb{M}^2}{\to} W = \nabla_u^{e,N} N
\]

This implies \(Y, Z,\) and \(N\) are \(\nabla^\ast,\ \nabla^{m\ast},\) and \(\nabla^N\)-differentiable, respectively, and

\[
(\nabla_u^{e} Y, \nabla_u^{e,m} Z, \nabla_u^{e,N} N) = (U, V, W)
\]

\(\Box\)

It is widely known that the density process \(Z\) can be thought as a “derivative” of \(Y\) with respect to the driving martingale. Under the assumption that \(M\) possesses martingale representation property, we can prove this is indeed the case with our definition of path-derivative. To prove it, we need the following lemma.

**Lemma 4.4.** Consider \(Z := Z(\cdot, M_t) \in \mathbb{H}_m^2\) for \(Z : [0, T] \times D \to \mathbb{R}^{d \times n}\) such that \(Z\) is \(\nabla_u^{e,m}\)-differentiable. Then, \(\int_t^T Z_s dM_s\) is \(\nabla_u^{e}\)-differentiable at \(u \in [0, T]\), and

\[
\nabla_u^e \int_t^T Z_s dM_s = \begin{cases} 
Z_u e^s + \int_t^T (\nabla_u^{e,m} Z)_s dM_s & \text{for } u \in (t, T] \\
\int_t^T (\nabla_u^{e,m} Z)_s dM_s & \text{for } u \in [0, t]
\end{cases}
\]

**Proof.** Let us denote \(M_t^{h,u} := M_t + h e^s 1_{[u,T]}(t)\). Then

\[
\int_t^T Z(s, M_t^{h,u}) dM_s^{h,u} = \lim_{|H| \to 0} \sum_{i=0}^N Z(t_i, M_{[0,t_i]}^{h,u}) (M_{t_{i+1}}^{h,u} - M_{t_i}^{h,u})
\]

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where $\Pi$ is a partition $\{0 = t_0 \leq t_1 \leq \cdots \leq t_N = T\}$ including a point $u \in [0, T]$ and $|\Pi|$ is the largest interval of $\Pi$. Since the limit is convergence in probability, we can take an appropriate subsequence of $\Pi$ so that the convergence is almost sure sense. Likewise

$$\int_t^T Z(s, M_{[0,s]})dM_s = \lim_{|\Pi| \to 0} \sum_{i=0}^N Z(t_i, M_{[0,t_i]})(M_{t_{i+1}} - M_{t_i})$$

using the same sequence of partition as above, by taking another subsequence if necessary. Then we have

$$\int_t^T Z(s, M_{[0,s]}^{h,u})dM_s^{h,u} - \int_t^T Z(s, M_{[0,s]})dM_s$$

$$= \lim_{|\Pi| \to 0} \sum_{i=0}^N \left[ Z(t_i, M_{[0,t_i]}^{h,u})(M_{t_{i+1}}^{h,u} - M_{t_i}^{h,u}) - Z(t_i, M_{[0,t_i]})(M_{t_{i+1}} - M_{t_i}) \right]$$

$$= \lim_{|\Pi| \to 0} \sum_{i=0}^N \left[ \left( Z(t_i, M_{[0,t_i]}^{h,u}) - Z(t_i, M_{[0,t_i]}) \right)(M_{t_{i+1}}^{h,u} - M_{t_i}^{h,u}) + Z(t_i, M_{[0,t_i]}) \left( M_{t_{i+1}}^{h,u} - M_{t_i}^{h,u} - M_{t_{i+1}} + M_{t_i} \right) \right]$$

$$= \int_{t \vee u}^T \left( Z(s, M_{[0,s]}^{h,u}) - Z(s, M_{[0,s]}) \right) dM_s + hZ_u e^1_{u \in (t, T]}$$

Therefore,

$$\nabla^e u \int_t^T Z_s dM_s = Z_u e^1_{u \in (t, T]} + \lim_{h \to 0, L^2} \int_{t \vee u}^T \left[ \frac{Z(s, M_{[0,s]}^{h,u}) - Z(s, M_{[0,s]})}{h} \right] dM_s$$

$$= Z_u e^1_{u \in (t, T]} + \int_{t \vee u}^T \left[ \lim_{h \to \infty, \mathbb{R}^2} \frac{Z(s, M_{[0,s]}^{h,u}) - Z(s, M_{[0,s]})}{h} \right] dM_s$$

$$= Z_u e^1_{u \in (t, T]} + \int_{t \vee u}^T (\nabla^e u^m Z) dM_s$$

**Theorem 4.5.** Assume that $\xi$ and $f$ satisfy (S), (Lip), (Diff), and (D) and let $(Y, Z, N)$ to be the solution of BSDE$(\xi, f)$. Let $\mathcal{Y}, \mathcal{Z}, \mathcal{N}$ be the corresponding path functional as in Theorem 3.7. Then,

$$\nabla_u Y_u = Z_u + \nabla_u N_u, \quad du \otimes d\mathbb{P}-\text{almost everywhere.}$$

**Proof.** Since $M$ has martingale representation property and $(Y, Z, N)$ is $\mathbb{P}^M$-adapted by Theorem 3.7 we know $N \equiv 0$ and $Y$ has continuous path. Therefore, we have the following forward SDE.

$$Y_t = Y_0 - \int_0^t f(s, M_{[0,s]}, Y_{s-}, Z_s m_s)dA_s + \int_0^t Z_s dM_s + N_t$$

Let $u \in (0, T]$ and

$$M^h_u := M_s + h e^1_{[u,T]}(s)$$

$$f^h_u := f(s, M^h_u, \mathcal{Y}(s-), M^h_{[0,s-]}), Z(s, M^h_{[0,s]}) m_s$$
where \( \mathcal{Y} \) and \( Z \) are defined as in the proof of Proposition 4.3. Let us define \( \zeta, \eta, \theta, \zeta^{h,u,i}, \eta^{h,u,i}, \theta^{h,u,i}, \) and \( (U^{h,u,i}, V^{h,u,i}, W^{h,u,i}) \) as in the proof of Proposition 4.3. Note that \( \zeta^{h,u,i} \) converges to \( \zeta \) in \( \mathbb{H}^2; \) \( (U^{h,u,i}, V^{h,u,i}, W^{h,u,i}) \) converge to \( (\nabla_u^1 Y, \nabla_{u,m}^1 Z, \nabla_{u,-}^1 N) \) in \( \mathbb{H}^2 \times \mathbb{H}^2_m \times \mathbb{M}^2; \) \( \eta^{h,u,i}, \theta^{h,u,i} \) converge to \( \eta, \theta \) in \( dt \otimes d\mathbb{P} \)-a.e. sense; and \( \eta^{h,u,i}, \theta^{h,u,i}, \eta, \theta \) are bounded \( dt \otimes d\mathbb{P} \)-a.e. sense. Therefore,

\[
\mathbb{E} \left( \int_0^t \left( f_s^0 - f_s^0 \frac{\zeta_s + \eta_s \nabla_u^1 Y_s - \theta_s \cdot (\nabla_{u,m}^1 Z)_s m_s}{h} \right)^2 \right) \leq K \mathbb{E} \left( \int_0^t \left( f_s^0 - f_s^0 \frac{\zeta_s + \eta_s \nabla_u^1 Y_s - \theta_s \cdot (\nabla_{u,m}^1 Z)_s m_s}{h} \right)^2 \right) \leq K \left( \left\| \zeta^{h,u,i} - \zeta \right\|_{\mathbb{H}^2} + \left\| \eta^{h,u,i} \cdot (U^{h,u,i} - \eta \nabla_u^1 Y) \right\|_{\mathbb{H}^2} + \left\| \theta^{h,u,i} \cdot (V^{h,u,i} - \theta) \nabla_{u,m}^1 Z \right\|_{\mathbb{H}^2} \
+ \left\| \theta^{h,u,i} \cdot (W^{h,u,i} - \theta) \right\|_{\mathbb{H}^2} \right) \xrightarrow{h \to 0} 0
\]

by dominated convergence theorem. As a result,

\[
\lim_{h \to 0, t^2} \int_0^t \frac{f_s^0 - f_s^0}{h} dA_s = I_{u \cap t} (\zeta_s + \eta_s \nabla_u^1 Y_s - \theta_s \cdot (\nabla_{u,m}^1 Z)_s m_s) dA_s.
\]

Then, by our previous lemma, for \( u \in (0, t] \)

\[
\nabla_u^1 Y_t = Z_u e_1^* - \int_u^t (\zeta_s + \eta_s \nabla_u^1 Y_s - \theta_s \cdot (\nabla_{u,m}^1 Z)_s m_s) dA_s + \int_u^t (\nabla_{u,m}^1 Z)_s dM_s + \nabla_u^1 N_t
\]

Since \( \nabla_u Y_t \) and \( \nabla_u N_t \) are right continuous, we prove the claim by letting \( t \searrow u \).

When \( M \) has martingale representation property, then \( N = 0 \) and above theorem implies the following corollary which we will use in section 5 and 6.

**Corollary 4.6.** Assume the conditions in Theorem 4.5. In addition, assume that

(M) \( M \) has martingale representation property; that is, any \((\mathbb{F}, \mathbb{P})\) martingale \( X \) such that \( \mathbb{E} \text{tr}[X, X]_T < \infty \) can be expressed as \( X_t = X_0 + \int_0^t Z_s dM_s \) for some \( Z \in \mathbb{H}^2_m \).

Let \( (Y, Z, N) \) be the solution of BSDE(\( \xi, f_i \)). Let \( \mathcal{Y}, Z, \mathcal{N} \) be the corresponding path functional as in Theorem 3.7. Then,

\[
\nabla_u Y_u = Z_u, \quad du \otimes d\mathbb{P} \text{-almost everywhere.}
\]

5 BSDE with Locally Lipschitz Driver

In this section, we always assume (M). This implies \( Y \) has continuous path, therefore, \( Y_{s-} = Y_s \) for all \( s \) and \( N = 0 \). Using Corollary 4.6, this martingale representation property enables us to bound \( Z \) process by bounding \( \nabla \)-derivative of \( Y \).
5.1 A Priori Estimate of $Z$

Let $k$ be an integer and $0 = t^k_0 \leq t^k_1 \leq \cdots \leq t^k_K = T$ be a partition of $[0, T]$. For $x \in D$, we define

$$P^{(k)}(x) := (\gamma_{t^k_0}, \gamma_{t^k_1} - \gamma_{t^k_0}, \cdots, \gamma_{t^k_K} - \gamma_{t^k_{K-1}})$$

$$L^{(k)}(a_1, \ldots, a_k) := \sum_{i=1}^k a_i 1_{[i/k, (i+1)/k]}.$$ 

Let us denote $x \in \mathbb{R}^{kn}$ and let $\varphi \in C^\infty_c(\mathbb{R}^{kn}; \mathbb{R})$ be the mollifier

$$\varphi(x) := \left\{ \begin{array}{ll} \lambda \exp \left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{array} \right.$$ 

where the constant $\lambda \in \mathbb{R}_+$ is chosen so that $\int_{\mathbb{R}^{kn}} \varphi(x) dx = 1$. Set $\varphi^{(k)}(x) := k^{kn} \varphi(kx)$, and define

$$\xi^{(k)} := \left[ (\xi \circ L^{(k)}) \circ P^{(k)} \right]$$

$$f^{(k)}(s, \gamma, y, z) := \int_{\mathbb{R}^{kn}} f(s, L^{(k)}(P^{(k)}(\gamma) - x'), y, z) \varphi^{(k)}(x') dx'.$$

**Theorem 5.1.** Assume that $\xi$ and $f$ satisfy (M), (Lip). In addition, assume the following condition:

$$(\text{Diff}^{'}) \quad - \xi(0) < \infty \text{ and } \int_0^T f(s, 0, 0, 0)^2 dA_s < \infty.$$ 

$$- \left\| M_{[0,T]} - (L^{(k)} \circ P^{(k)})(M_{[0,T]}) \right\|_{\infty} \xrightarrow{k \to \infty} 0.$$ 

$$- \text{ There are } D_{\xi}, D_f \in \mathbb{R}_+ \text{ such that, for all } \gamma, \gamma' \in D, (s, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times n},$$

$$|\xi(\gamma) - \xi(\gamma')| \leq D_{\xi} \|\gamma - \gamma'\|_\infty \quad \text{and} \quad |f(s, \gamma, y, z) - f(s, \gamma', y, z)| \leq D_f \|\gamma - \gamma'\|_\infty.$$ 

Then, BSDE$(\xi, f)$ has a unique solution $(Y, Z, N) \in \mathbb{H}^2 \times \mathbb{H}^2_{m \times n} \times \mathbb{M}^2$. Moreover, $Y$ has continuous path, $N \equiv 0$ and

$$|Z_t| \leq \sqrt{D_{\xi}^2 + D_f^2 K e^{\frac{1}{2}K(2C_\eta + C_\eta^2 + 1)}} \quad dt \otimes d\mathbb{P} - a.e.$$ 

**Remark 5.2.** It is easy to see that the second condition of (Diff') holds when $M$ is a Brownian motion and $t^k_i = iT/k$. More generally, let $W$ be Brownian motion and assume that $M$ has a martingale representation

$$M_t = \int_0^t \eta_s dW_s$$

where there exists a constant $C$ such that $|\eta_s| \leq C$ almost surely. Then

$$\mathbb{E} \left[ \left\| M_{[0,T]} - (L^{(k)} \circ P^{(k)})(M_{[0,T]}) \right\|_{\infty}^2 \right] = \mathbb{E} \left[ \left\| \sum_{i=0}^{k-1} \sup_{t_i \leq t < t_{i+1}} \int_{t_i}^t \eta_s dW_s \right\|_{\infty}^2 \right] \leq \mathbb{E} \left[ \left\| \sum_{i=0}^{k-1} \sup_{t_i \leq t < t_{i+1}} \int_{t_i}^t \eta_s dW_s \right\|_{\infty}^4 \right]^{1/2}$$

$$\leq \left[ \sum_{i=0}^{k-1} \left\| \int_{t_i}^t \eta_s dW_s \right\|_{\infty}^4 \right]^{1/2} \leq \left[ \sum_{i=0}^{k-1} \mathbb{E} \left( \int_{t_i}^{t_{i+1}} |\eta_s|^2 ds \right) \right]^{1/2}$$

$$\leq \sqrt{CC^2} \left[ \sum_{i=0}^{k-1} \left( T/k \right)^2 \right]^{1/2} \leq \sqrt{CC^2}T \frac{1}{\sqrt{k}} \xrightarrow{k \to \infty} 0.$$
The inequalities are based on
\[ \sup_{i=0,\ldots,k-1} |a_i| \leq \sqrt{\sum_{i=0}^{k-1} |a_i|^2}, \]
Jensen inequality, and Burkholder-Davis-Gundy inequality. Here, we used \( \tilde{C} \) for the constant of Burkholder-Davis-Gundy inequality. Therefore, such \( M \) satisfies the second condition of (Diff').

Before we proceed to the proof, let us observe the following facts.

**Lemma 5.3.** Under the assumption of Theorem 5.1, we have the following results.

(i) \((\xi, f)\) satisfies (S) and \((\xi^{(k)}, f^{(k)})\) satisfies (S), (Diff), (Lip)

(ii) \(|\nabla_u \xi^{(k)}(M_{[0,T]}^t)| \leq D_\xi \) and \(|\nabla_u f^{(k)}(t, M_{[0,T]}^t, y, z)| \leq D_f \) for all \( u \in [0, T], (y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times n} \) in \( dt \otimes d\mathbb{P}\)-a.e.

(iii) For solution \((Y, Z, N) \in \mathbb{H}^2 \times \mathbb{H}^2_{\mathcal{M}} \times \mathcal{M}\) of BSDE(\(\xi, f\)), \(Y\) has a continuous path, \(N \equiv 0\), and
\[
\xi^{(k)}(M_{[0,T]}) \xrightarrow{k \to \infty} \xi(M_{[0,T]}) \quad \text{and} \quad f^{(k)}(\cdot, M_{[0,T]}, Y_{-\cdot}, (Zm)_{-\cdot}) \xrightarrow{k \to \infty} f(\cdot, M_{[0,T]}, Y_{-\cdot}, (Zm)_{-\cdot}).
\]

**Proof.** It is easy to see \((\xi^{(k)}, f^{(k)})\) satisfies (Lip). Note that
\[
\|\xi(M_{[0,T]} + \gamma)\|_2 \leq \|\xi(0) + D_\xi M_{[0,T]}\|_\infty + \|\gamma\|_\infty < \infty
\]
\[
\|\xi(M'_{[0,T]} + \gamma)\|_2 \leq \|\xi(0) + D_\xi M_{[0,T]}\|_\infty + D_\xi \|\tilde{M}_{[0,T]}\|_\infty < \infty
\]
since \(\|M_{[0,T]}\|_\infty \in \mathbb{L}^2\) by Burkholder-Davis-Gundy inequality and \(\|\tilde{M}_{[0,T]}\|_\infty\) is bounded by the number of jumps of \(M'\) which is a Poisson random variable. Therefore, \(\xi\) satisfies (S). We can use the same argument to show that \(f\) satisfies (S) as well. Note that
\[
|\xi^{(k)}(\gamma) - \xi^{(k)}(\gamma')| \leq \int_{\mathbb{R}^n_k} \left| \xi(L^{(k)}(P^{(k)}(\gamma) - x')) - \xi(L^{(k)}(P^{(k)}(\gamma') - x')) \right| \varphi^{(k)}(x') dx'
\]
\[
\leq D_\xi \int_{\mathbb{R}^n_k} \left| L^{(k)}(P^{(k)}(\gamma) - x') - L^{(k)}(P^{(k)}(\gamma') - x') \right|_\infty \varphi^{(k)}(x') dx'
\]
\[
\leq D_\xi \|\gamma - \gamma'\|_\infty.
\]
Likewise,
\[
|f^{(k)}(s, \gamma, y, z) - f^{(k)}(s, \gamma', y, z)| \leq D_f \|\gamma - \gamma'\|_\infty.
\]
Using the same argument for \((\xi, f)\), this implies \((\xi^{(k)}, f^{(k)})\) satisfies (S). Moreover, it also implies (Diff) and (ii) because of Lipschitzness and the convolution with the mollifier \(\varphi^{(k)}\).

Lastly, since \(M\) has martingale representation property and \((Y, Z, N)\) is \(\mathbb{F}^M\)-adapted by Theorem 3.7 we know \(N \equiv 0\) and \(Y\) has continuous path. Also, note that
\[
\left\| \xi(M_{[0,T]}) - \xi(M'_{[0,T]}) \right\|_2
\]
\[
\leq \left\| \xi(M_{[0,T]} - (\xi \circ L^{(k)} \circ P^{(k)})(M_{[0,T]}) \right\|_2 + \left\| \int_{\mathbb{R}^n_k} (\xi \circ L^{(k)})(P^{(k)}(\gamma)) - (\xi \circ L^{(k)})(P^{(k)}(\gamma) - x') \right\|_\infty \varphi^{(k)}(x') dx'
\]
\[
\leq D_\xi \left\| M_{[0,T]} - (L^{(k)} \circ P^{(k)})(M_{[0,T]}) \right\|_\infty + D_\xi \left\| \int_{\mathbb{R}^n_k} \max_{i} |x'_i| \varphi^{(k)}(x') dx' \right\|_2 \xrightarrow{k \to \infty} 0
\]
We can argue similarly for \(f^{(k)}\) to conclude (iii) holds. \(\square\)
Proof of Theorem 5.1. Let us denote \( x = (y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times n} \) and let \( \beta \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}^{d \times n}; \mathbb{R}) \) be the mollifier

\[
\beta(x) := \begin{cases} 
\lambda \exp \left( - \frac{1}{1-|x|^2} \right) & \text{if } |x| < 1 \\
0 & \text{otherwise}
\end{cases},
\]

where the constant \( \lambda \in \mathbb{R}_+ \) is chosen so that \( \int_{\mathbb{R}^{d+n+d}} \beta(x) dx = 1 \). Set \( \beta^{(m)}(x) := m^{d+n} \beta(mx) \), \( m \in \mathbb{N} \setminus \{0\} \), and define

\[
f^{(k,m)}(t, \gamma_{[0,t]}, x) := \int_{\mathbb{R}^{d+n+d}} f^{(k)}(t, \gamma_{[0,t]}, x - x') \beta^{(m)}(x') dx'.
\]

Then, it is easy to check that (S), (Lip), (Diff) for \( f^{(k)} \) implies that \( (\xi^{(k)}, f^{(k,m)}) \) satisfies (S), (Lip), (Diff), and (D). Therefore, there exists solution \( (Y^{(k,m)}, Z^{(k,m)}) \) of the BSDE

\[
Y_t^{(k,m)} = \xi^{(k)}(M_{[0,T]}) + \int_t^T f^{(k,m)}(s, M_{[0,s]}, Y_s^{(k,m)}, Z_s^{(k,m)} m_s) dA_s - \int_t^T Z_s^{(k,m)} dM_s
\]

From our Proposition 3.3 and Proposition 4.3 we know

\[
|\nabla u Y_t^{(k,m)}| \leq \sqrt{D_\xi^2 + D_f^2 K e^{K(2C_y + C^2 + 1)}}
\]

for all \( u \in [0, T] \). Then, from Corollary 4.6 for all \( k, m \in \mathbb{N} \),

\[
|Z_t^{(k,m)}| \leq \sqrt{D_\xi^2 + D_f^2 K e^{K(2C_y + C^2 + 1)}}
\]

dt \otimes d\mathbb{P} -almost everywhere. By Proposition 3.4 we have

\[
\left\| Y - Y^{(k,k)} \right\|_{h^2}^2 + \left\| Z - Z^{(k,k)} \right\|_{h^2_m}^2 \
\leq 2e^{2K(C_y + C^2 + 1)} \left( \left\| \xi(M_{[0,T]}) - \xi^{(k)}(M_{[0,T]}) \right\|_2^2 + \left\| f(\cdot, M_{[0,\cdot]}, Y_{-}, Z_{-}, M_{\cdot}) - f^{(k)}(\cdot, M_{[0,\cdot]}, Y_{-}, Z_{-}, M_{\cdot}) \right\|_{h^2}^2 \right).
\]

Since \( f^{(k)} \) is Lipschitz,

\[
\left| f^{(k)}(t, M_{[0,t]}, Y_t, Z_t m_t) - f^{(k,k)}(t, M_{[0,t]}, Y_t, Z_t m_t) \right|
\leq \int_{\mathbb{R}^{d+n+d}} \left| f^{(k)}(t, M_{[0,t]}, Y_t, Z_t m_t) - f^{(k)}(t, M_{[0,t]}, (Y_t, Z_t m_t) - x') \right| \beta^{(k)}(x') dx'
\leq (C_y + C_z) \int_{\mathbb{R}^{d+n+d}} |x'| \beta^{(k)}(x') dx' \xrightarrow{m \to \infty} 0.
\]

Combined with the previous lemma, this implies that

\[
\left\| f(\cdot, M_{[0,\cdot]}, Y_{-}, Z_{-}, M_{\cdot}) - f^{(k,k)}(\cdot, M_{[0,\cdot]}, Y_{-}, Z_{-}, M_{\cdot}) \right\|_{h^2} \xrightarrow{k \to \infty} 0.
\]

Therefore, \( Y^{(k,k)} \to Y \) in \( h^2 \) and \( Z^{(k,k)} \to Z \) in \( h^2_m \) with \( |Z_t^{(k,k)}| \leq \sqrt{D_\xi^2 + D_f^2 K e^{K(2C_y + C^2 + 1)}} \). Therefore,

\[
|Z_t| \leq \sqrt{D_\xi^2 + D_f^2 K e^{K(2C_y + C^2 + 1)}}.
\]

\( \square \)
5.2 Existence and uniqueness when $d \geq 1$ and $[M, M]_T$ is small.

**Theorem 5.4.** Assume the following conditions: (M), (Diff'), and (Loc)

(LOC) There exists a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, \gamma_{[0,t]}, y, z) - f(t, \gamma_{[0,t]}, y', z')| \leq \rho(|z| \vee |z'|) (|y - y'| + |z - z'|)$$

for all $t \in [0, T]$, $y, y' \in \mathbb{R}^d$ and $z, z' \in \mathbb{R}^d \times \mathbb{R}^{d \times n}$.

Assume that $K$ is small enough so that there is $R \in \mathbb{R}_+$ satisfying

$$\sqrt{D_\xi^2 + D_f^2Ke^{\frac{1}{2}K(\rho(R)+1)^2}} \leq R$$

Then, the BSDE

$$Y_t = \xi(M_{[0,T]}) + \int_t^T f(s, M_{[0,s]}, Y_s, Z_s m_s) dA_s - \int_t^T Z_s dM_s$$

has a unique solution $(Y, Z) \in \mathbb{H}^2 \times \mathbb{H}_m^2$ such that $Z$ is bounded. Moreover, $|Z_t| \leq R$ in $dt \otimes d\mathbb{P}$-almost everywhere sense.

**Proof.** For $R$ in the assumption, consider the BSDE with the terminal condition $\xi(M_{[0,T]})$ and the driver

$$g(t, \gamma_{[0,t]}, y, z) := f(t, \gamma_{[0,t]}, y, R z |z| \vee R).$$

Then, $g$ satisfies (Diff') and (Lip) with Lipschitz coefficient of the driver $\rho(R) = C_y = C_z$. Therefore, there exists a unique solution $(U, V) \in \mathbb{H}^2 \times \mathbb{H}_m^2$ for the following BSDE

$$U_t = \xi(M_{[0,T]}) + \int_t^T g(s, M_{[0,s]}, U_s, V_s m_s) dA_s - \int_t^T V_s dM_s$$

and $V$ is bounded by

$$|V_t| \leq \sqrt{D_\xi^2 + D_f^2Ke^{\frac{1}{2}K(\rho(R)+1)^2}}.$$

Therefore, since $|m_t| = 1$ for all $t$ (see Section 2),

$$|V_t m_t| \leq |V_t||m_t| \leq \sqrt{D_\xi^2 + D_f^2Ke^{\frac{1}{2}K(\rho(R)+1)^2}} \leq R.$$

This implies that $(U, V)$ is also a solution of

$$Y_t = \xi(M_{[0,T]}) + \int_t^T f(s, M_{[0,s]}, Y_s, Z_s m_s) dA_s - \int_t^T Z_s dM_s.$$

Now let us show the uniqueness. Assume that $(Y', Z')$ is another solution such that $Z'$ is bounded by $Q$. Without loss of generality, we can assume $Q \geq R$. Then, if we consider

$$h(t, \gamma_{[0,t]}, y, z) := f(t, \gamma_{[0,t]}, y, Q z |z| \vee Q),$$

then BSDE$(\xi, h)$ has a unique solution in $\mathbb{H}^2 \times \mathbb{H}_m^2$. Since $(Y, Z)$ and $(Y', Z')$ are both solution to such BSDE, we have $(Y, Z) = (Y', Z')$. 

\hfill $\square$
5.3 Explosion of solution when \( d > 1 \) and \([M, M]_T\) is large.

If \( d > 1 \), the result on the previous subsection cannot extend to arbitrary large \( K \) in general. This can be shown by the following counterexample which is inspired by Chang et al. (1992, [6]).

For \( \delta > 0 \), let \( M_t := \sqrt{2(W_t^r - W_{T-\delta})^2 [1_{T-\delta, \infty}]}(t) \) where \( W_t \) is a two dimensional Brownian motion and

\[
\tau := \inf \left\{ t \in [T - \delta, \infty) : |W_t - W_{T-\delta}| \geq 1/\sqrt{2} \right\} \wedge T.
\]

Note that \( M \) satisfies (M) because of Lemma 2.1 of Peng (1991, [40]). Define the terminal condition \( \xi \) as

\[
\xi(M_T) := \begin{pmatrix}
\cos \theta_T \sin g_0(R_T) \\
\sin \theta_T \sin g_0(R_T) \\
\cos g_0(R_T)
\end{pmatrix}
\]

where \( g_0 : \mathbb{R}_+ \to \mathbb{R} \) is a smooth function with bounded derivatives of all order and \((R_s, \theta_s)\) is the polar coordinate of \( M_s \). Note that \( \xi \) is a smooth function with bounded derivative.

We let, for \( \varepsilon \in (0,1) \),

\[
\phi(r) := \arccos \left( \frac{\lambda^2 - r^2(1+\varepsilon)}{\lambda^2 + r^2(1+\varepsilon)} \right)
\]

where \( \lambda \) is big enough so that \( \cos \phi(r) \geq (1 + \varepsilon)^{-1} \) for \( r \in [0, 1] \). We choose a smooth function \( g_0 \) so that

\[
g_0(r) \geq \arccos \left( \frac{1 - r^2}{1 + r^2} \right) + \phi(r)
\]

for \( r \in [0, 1] \) and \( g_0(0) = 0 \). Note that \( g_0 \) has bounded derivatives all orders on \([0,1] \).

Let us show the following BSDE driven by \( M \) have a solution \((Y, Z)\) such that \( Z \) is bounded only when \([M, M]_T\) is small enough:

\[
Y_t = \xi(M_T) + \int_t^T \frac{1}{2} |Z_m|_s^2 \frac{Y_s}{|Y_s| + 1} dA_s - \int_t^T Z_s dM_s. \tag{5.4}
\]

Since we have \([M, M]_t := 2((s \wedge \tau - (T - \delta)) \vee 0)I\), we have \( m_s = \frac{1}{\sqrt{2}} I \) and \( A_s = 4(s \wedge \tau - (T - \delta)) \vee 0 \). Note that \( \sup_{\omega \in \Omega} A_T = 4\delta \). It is easy to check that (Loc) and (Diff') if we let \( \rho(x) = x + \frac{1}{2} |x|^2 \). Then, by Theorem 5.4 [5.4] has a unique solution \((Y, Z) \in \mathbb{H}^2 \times \mathbb{H}^2_m\) such that \( Z \) is bounded if \( \delta \) is small enough so that there exists \( R \) such that

\[
D_\xi e^{2\delta(R + R^2/2 + 1)^2} \leq R
\]

where \( D_\xi \) is the bound on the derivative of \( \xi \) with respect to \( M_T \). In order to prove the nonexistence of solution for large \( \delta \), we need the following proposition.

**Proposition 5.5.** Consider the PDE of \( g : [0, \infty) \times [0,1] \to \mathbb{R} \):

\[
\partial_t g = \partial_r g + \frac{1}{r} \partial_r g - \frac{\sin g \cos g}{r^2}; \quad g(0, r) = g_0(r), g(t, 0) = 0, \text{ and } g(t, 1) = 2\pi.
\]

This PDE admits a unique classical solution on \([0, T_0]\) for some \( T_0 \in \mathbb{R}_+ \) and \( \lim_{T \to T_0} \partial_r g(t, 0) = \infty \).

**Proof.** See the proof part (i) in Chang et al. (1992, [6]). \( \square \)
Now, for $T_0$ in Proposition 5.5 assume that $\delta > T_0$ and (5.4) has a solution $(Y, Z) \in \mathbb{H}^2 \times \mathbb{H}_m^2$ such that $Z$ is bounded. For $g(t, r)$ in Proposition 5.5, if we let $u : [0, T_0) \times \mathbb{R}^2 \to \mathbb{R}^3$ be
\[
u(t, x) = \begin{pmatrix}
\frac{\sin g(t, |x|)}{|x|} \\
\frac{\cos g(t, |x|)}{|x|}
\end{pmatrix},
\]
we can easily deduce that
\[
Y_t = u(T - t, M_t) \quad \text{and} \quad Z_t := \nabla u(T - t, M_t) \quad t \in (T - T_0, T]
\]
by using Itô formula and the uniqueness of solution for BSDE. Note that we can easily deduce that $Z$ is bounded. For $T$ starting at $|\nabla u|$, that by Proposition 5.5 and (5.4) is continuous in $(t, x)$. Therefore, for any large $L$, there exists $\varepsilon > 0$ such that $|\nabla u(T - t, x)| \geq L$ for all $(t, x) \in (T - T_0, T - T_0 + \varepsilon) \times (-\varepsilon, \varepsilon)^2$. Since $M$ is a scaled Brownian motion starting at $T - \delta$ and stopped at $\tau$,
\[
P(|Z_t| \geq L \text{ for all } t \in (T - T_0, T - T_0 + \varepsilon)) \geq P(M_t \in (-\varepsilon, \varepsilon)^2 \text{ for all } t \in (T - T_0, T - T_0 + \varepsilon)) > 0.
\]
This implies BSDE (5.4) cannot have a solution such that $Z$ is bounded when $\delta \geq T_0$.

**Remark 5.6.** Our counterexample above shows that there is no solution $(Y, Z)$ such that $Z$ is bounded and therefore, Theorem 5.4 is sharp in this sense. We do not exclude the possibility that there is a solution $(Y, Z) \in \mathbb{H}^2 \times \mathbb{H}_m^2$ such that $Z$ is not bounded. However, if $\delta = T_0$ in above example, $\lim_{t \downarrow T - T_0} Z_t = \infty$ almost surely.

**Remark 5.7.** Another counterexample for the existence of solution for multidimensional quadratic BSDE is given by Frei and dos Reis (2012, [21]). They proved the $Y$ part of solution explodes when the terminal condition is singular with respect to the perturbation of underlying martingale, i.e., Brownian motion. In our example, $Y$ is uniformly bounded and the terminal condition is smooth with bounded derivative. However, in our case, $Z$ explodes with positive probability.

### 5.4 Existence and uniqueness when $d = 1$.

When $d = 1$ and $m_t$ is invertible for all $t \in [0, T]$, we can remove the smallness condition on $K$ if we assume Lipschitzness of $f(t, \gamma_{[0, t]}, y, z)$ with respect to $y$.

**Theorem 5.8.** Assume that (M) and (Diff) hold, $d = 1$, and $m_t$ is invertible for all $t \in [0, T]$. In addition, assume that

\[(\text{Loc}') \text{ there exist } C_y \in \mathbb{R}_+ \text{ and a nondecreasing function } \rho : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that}
\]
\[
|f(t, \gamma_{[0, t]}, y, z) - f(t, \gamma_{[0, t]}, y', z')| \leq C_y|y - y'| + \rho(|z| \vee |z'|)|z - z'|
\]

for all $t \in [0, T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^{d \times n}$.

Then, the BSDE
\[
Y_t = \xi(M_{[0, T]}) + \int_t^T f(s, M_{[0, s]}, Y_s, Z_s m_s) dA_s - \int_t^T Z_s dM_s
\]
has a unique solution $(Y, Z) \in \mathbb{H}^2 \times \mathbb{H}_m^2$ such that $Z$ is bounded. Moreover,
\[
|Z_t| \leq \sqrt{n} \left[ \left( D_\xi + \frac{D_f}{C_y} \right) e^{C_y K} - \frac{D_f}{C_y} \right] dt \otimes d\mathbb{P} - \text{almost everywhere.}
\]

In the case where $C_y = 0$, the bound changes to $\sqrt{n} (D_\xi + D_f K)$. If we assume (D) and (Diff) as well, $Y$ and $Z$ are $\nabla$- and $\nabla^m$-differentiable, respectively, and $Z_u = \nabla_u Y_u$. 

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Proof. We already know that $(\xi, f)$ satisfies (S) by Lemma 5.3. Note that if we can prove the theorem under assumption (D), and (Diff), we can generalize it to (Diff') using the same argument in Theorem 5.1. Therefore, without loss of generality, we will assume (D) and (Diff), and moreover, $D_{\xi}, C_y > 0$. Let

$$R := \sqrt{n} \left[ \left( D_{\xi} + \frac{D_f}{C_y} \right) e^{C_y K} - \frac{D_f}{C_y} \right]$$

First let $(Y, Z)$ be the solution of

$$Y_t = \xi(M_{0,T}) + \int_t^T g(s, M_{[0,s]}, Y_s, Z_s m_s) dA_s - \int_t^T Z_s dM_s$$

where $g$ is a smooth extension of $f$ such that

$$g(t, \gamma_{[0,t]}, y, z) := \begin{cases} f(t, \gamma_{[0,t]}, y, z) & \text{if } |z| \leq R \\ f(t, \gamma_{[0,t]}, y, (R + 1)z/|z|) & \text{if } |z| \geq R + 1 \end{cases}$$

and $|\partial_z g| \leq \rho(R + 1)$. By our Proposition 4.3 and 5.1 for $t \geq u$,

$$\nabla_{\xi u} Y_t = \Xi + \int_t^T (\zeta_s + \eta_{\xi u} \nabla_{\xi u} Y + \theta_s \cdot (\nabla_{\xi u} Z)_{s m_s}) dA_s - \int_t^T (\nabla_{\xi u} Z)_{s m_s} dM_s \tag{5.5}$$

and $Z_t = \nabla_\xi Y_t$, where

$$\Xi := \nabla_{\xi u} \xi(M_{[0,T]})$$

$$\zeta := (\nabla_{\xi u} g)(\cdot, M_{[0,\cdot]}, y, z)_{y,z} = (Y, Z, m)$$

$$\eta := (\partial_y g)(\cdot, M_{[0,\cdot]}, Y, Z, m)$$

$$\theta := (\partial_z g)(\cdot, M_{[0,\cdot]}, Y, Z, m)$$

Let us compare (5.5) with

$$U_t = D_{\xi} + \int_t^T (D_f + C_y |U_s| + \rho(R + 1)|V_s m_s|) dA_s - \int_t^T V_s dM_s \tag{5.6}$$

$$\bar{U}_t = -D_{\xi} + \int_t^T (-D_f - C_y |U_s| - \rho(R + 1)|V_s m_s|) dA_s - \int_t^T \bar{V}_s dM_s$$

The BSDEs (5.6) have unique solutions in $\mathbb{H}^2 \times \mathbb{H}^2_m$ such that $U, \bar{U} \in \mathbb{S}^2$. Let us define

$$h(v) := \begin{cases} \frac{v^*}{|v|} & \text{if } v \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{d\Gamma_t}{\Gamma_t} = C_y h(U_s) dA_s + \rho(R + 1) h(V_s m_s)^* m^{-1}_s dM_s; \quad \Gamma_0 = 1.$$

Here, we use $h$ as defined on either $\mathbb{R}$ or $\mathbb{R}^{1 \times n}$ depending on the context. If we apply Itô formula to $\Gamma_t U_t$,

$$d(\Gamma_t U_t) = -D_f \Gamma_t dA_t + \left( \Gamma_t U_t \rho(R + 1) h(V_t m_t)^* m^{-1}_t + \Gamma_t \bar{V}_t \right) dM_t.$$
Since \( h(U_t), h(V_t m_t) \) are bounded by 1, by the same logic as in the proof of Theorem 3.5 \( \Gamma \in \mathbb{S}^2 \) and this implies that \( \int \Gamma_s \rho(R + 1)h(V_s m_s)^*m_s^{-1}dM_s \) and \( \Gamma U + \int D_f \Gamma_s dA_s \) are true martingales. Therefore,

\[
\Gamma_t = \mathcal{E} \left( \int_0^T C_y h(U_s) dA_s + \rho(R + 1) \int_0^T h(V_s m_s)^*m_s^{-1}dM_s \right)_t = \mathbb{E} \left[ \Gamma_T - C_y \int_t^T \Gamma_s h(U_s) dA_s \bigg| \mathcal{F}_t \right] > 0
\]

\[
U_t = \frac{1}{\Gamma_t} \mathbb{E} \left[ \Gamma_T D_t + \int_t^T D_f \Gamma_s dA_s \bigg| \mathcal{F}_t \right].
\]

Since \( U_t > 0 \), we have \( h(U_s) = 1 \) and the first part implies that, since \( \Gamma_t := e^{C_y A_t} \mathcal{E}(\rho(R + 1) \int_0^T h(V_s m_s)^*m_s^{-1}dM_s)_t \),

\[
\mathbb{E} \left[ \int_t^T \frac{\Gamma_s}{\Gamma_t} dA_s \bigg| \mathcal{F}_t \right] \leq \frac{1}{C_y} \mathbb{E} \left[ \Gamma_T \bigg| \mathcal{F}_t \right] - \frac{1}{C_y} \mathcal{E}(\rho(K - A_t) - \frac{1}{C_y}.
\]

Therefore, we get

\[
U_t \leq \left( D_t + \frac{D_f}{C_y} \right) e^{C_y(K - A_t)} - \frac{D_f}{C_y}.
\]

We can get the same upper bound for \( -\bar{U}_t \) using the same argument. By the comparison theorem 3.5 we know

\[
|\nabla^a u Y_t| \leq \left( D_t + \frac{D_f}{C_y} \right) e^{C_y(K - A_t)} - \frac{D_f}{C_y} \leq \frac{R}{\sqrt{n}}
\]

which implies \( |Z_t m_t| \leq R \). Therefore, \((Y, Z)\) is a solution of the original BSDE

\[
Y_t = \xi(M_{[0, T]}) + \int_t^T f(s, M_{[0, s]}, Y_s, Z_s m_s) dA_s - \int_t^T Z_s dM_s.
\]

Uniqueness can be easily checked by the same argument in the proof of Theorem 5.4. \( \square \)

## 6 Utility Maximization of Controlled SDE Driven by M

In this section, we apply the previous results to the utility maximization problem for controlled SDEs driven by \( M \). Our control is \( \Delta \) and we require

\[
\Delta \in \mathcal{A} := \left\{ X \in \mathbb{H}^2 : \esssup_{(t, \omega) \in [0, T] \times \Omega} |X_t(\omega)| < \infty \right\}
\]

For a given control \( \Delta \), consider one of the two SDEs driven by \( M \) where \( M \) satisfies (M):

\[
X_t^\Delta = x + \int_0^t X_s^\Delta b(s, M_{[0, s]}, \Delta_s) dA_s + \int_0^t X_s^\Delta \sigma(s, M_{[0, s]}, \Delta_s) dM_s \tag{6.7}
\]

\[
X_t^\Delta = x + \int_0^t b(s, M_{[0, s]}, \Delta_s) dA_s + \int_0^t \sigma(s, M_{[0, s]}, \Delta_s) dM_s \tag{6.8}
\]

Here, \( b : [0, T] \times D \times \mathbb{R}^{1 \times n} \to \mathbb{R} \) and \( \sigma : [0, T] \times D \times \mathbb{R}^{1 \times n} \to \mathbb{R}^{1 \times n} \) are jointly measurable functions such that \( b(\cdot, M_{[0, \cdot]}, \Delta_\cdot) \) and \( \sigma(\cdot, M_{[0, \cdot]}, \Delta_\cdot) \) are predictable for any \( \Delta \in \mathcal{A} \).
6.1 Power utility

Our objective in this subsection is to find a control $\Delta \in \mathcal{A}$ that maximize

$$
\mathbb{E}\left[ \frac{1}{\kappa} \left( X_T^\Delta e^{-\xi(M_{[0,T]})} \right)^\kappa \right]
$$

where $\kappa \in (-\infty, 0) \cup (0, 1]$ and $X^\Delta$ is given by (6.7).

**Theorem 6.1.** Assume that there exist an increasing continuous function $\rho: \mathbb{R}_+ \to \mathbb{R}$ and a $\mathcal{B}([0,T]) \otimes \mathcal{D} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable function $k : [0, T] \times D \times \mathbb{R}^{1 \times n} \to \mathbb{R}^{1 \times n}$ that satisfy the following conditions:

- $|b(s, M_{[0,s]}, \pi)| + |\sigma(s, M_{[0,s]}, \pi)| \leq \rho(|\pi|)$ for all $\pi \in \mathbb{R}^{1 \times n}$.
- $|k(s, \gamma, z)| \leq \rho(|z|)$ for all $(s, \gamma, z) \in [0, T] \times D \times \mathbb{R}^{1 \times n}$.
- For all $(s, \gamma, \pi, z) \in [0, T] \times D \times \mathcal{A} \times \mathbb{R}^{d \times n}$, the following inequality holds:

$$
G(s, \gamma, \pi, z) := -\frac{1}{2}\kappa|\sigma(s, \gamma_{[0,s]}, \pi)m_s - z|^2 - b(s, \gamma_{[0,s]}, \pi) + \frac{1}{2}|\sigma(s, \gamma_{[0,s]}, \pi)m_s|^2
$$

$$
\geq G(s, \gamma_{[0,s]}, k(s, \gamma_{[0,s]}, z), z) := f(s, \gamma_{[0,s]}, z)
$$

- The BSDE($\xi, f$) has a solution $(Y, Z) \in \mathbb{H}^2 \times \mathbb{H}_m^2$ such that $Z$ is bounded.

Then,

$$
\hat{\Delta}_s := k(s, M_{[0,s]}, Z_s m_s) \in \mathcal{A}
$$

is the optimal control and the optimal value is $\frac{x^n e^{-\kappa Y_0}}{\kappa}$.

**Proof.** First of all, since $M_{[0,\cdot]} : [0, T] \times \Omega \to D$ is an adapted continuous function, note that $\hat{\Delta}$ is a predictable process because it is a deterministic measurable function of predictable processes. Moreover, since $Z$ is bounded, $\hat{\Delta}$ is bounded by our assumption. Therefore, $\hat{\Delta} \in \mathcal{A}$.

As in Hu et al. (2005, [23]), we use the martingale technique to prove the theorem. Note that, since $b(s, M_{[0,s]}, \Delta_s)$ and $\sigma(s, M_{[0,s]}, \Delta_s)$ are bounded for $\Delta \in \mathcal{A}$, (6.7) has a unique strong solution

$$
X^\Delta_t = x \exp \left( \int_0^t \left( b(s, M_{[0,s]}, \Delta_s) - \frac{1}{2}|\sigma(s, M_{[0,s]}, \Delta_s)m_s|^2 \right) \, dA_s + \int_0^t \sigma(s, M_{[0,s]}, \Delta_s)dM_s \right).
$$

For notational convenience, let us use $b_s^\Delta := b(s, M_{[0,s]}, \Delta_s)$ and $\sigma_s^\Delta := \sigma(s, M_{[0,s]}, \Delta_s)$. Let us define a family of stochastic process $\{U^\Delta\}_{\Delta \in \mathcal{A}}$ given by

$$
U_t^\Delta = \frac{1}{\kappa}(X_t^\Delta e^{-Y_1})^\kappa
$$

$$
= \frac{x^\kappa}{\kappa} \exp \left( -\kappa Y_0 + \kappa \int_0^t \left( b_s^\Delta - \frac{1}{2}|\sigma_s^\Delta m_s|^2 + f(s, M_{[0,s]}, Z_s m_s) \right) \, dA_s + \kappa \int_0^t \sigma_s^\Delta \, dM_s \right)
$$

$$
= \frac{x^\kappa e^{-\kappa Y_0}}{\kappa} \mathbb{E} \left( \kappa \int_0^t (\sigma_s^\Delta - Z_s) \, dM_s \right) \exp \left( \kappa \int_0^t \left( b_s^\Delta - \frac{1}{2}|\sigma_s^\Delta m_s|^2 + \frac{\kappa}{2}|\sigma_s^\Delta m_s - Z_s m_s|^2 + f(s, M_{[0,s]}, Z_s m_s) \right) \, dA_s \right)
$$

$$
= \frac{x^\kappa e^{-\kappa Y_0}}{\kappa} \mathbb{E} \left( \kappa \int_0^t (\sigma_s^\Delta - Z_s) \, dM_s \right) \exp \left( \kappa \int_0^t \left( f(s, M_{[0,s]}, Z_s m_s) - G(s, M_{[0,s]}, \Delta_s, Z_s m_s) \right) \, dA_s \right)
$$

26
is a true martingale: see Kazamaki (1994, [31]). Moreover, note that
\[
\int_0^T \kappa (\sigma_s^\Delta - Z_s) dA_s
\]
and
\[
|\sigma_s^\Delta - Z_s| \leq \rho(|\Delta_s|) + |Z_s|,
\]
\[
\int_0^T \kappa (\sigma(s, M_{[0,s]}, \Delta_s) - Z_s) dM_s
\]
is a BMO martingale for each \( \Delta \in \mathcal{A} \). Therefore,
\[
\mathcal{E} \left( -\int_0^T \kappa (\sigma(s, M_{[0,s]}, \Delta_s) - Z_s) dM_s \right)
\]
is a true martingale: see Kazamaki (1994, [31]). Moreover, note that \( \frac{1}{\kappa} e^{\kappa x} \) is an increasing function and
\[
f(s, M_{[0,s]}, Z_s m_s) - G(s, M_{[0,s]}, \Delta_s, Z_s m_s) \leq 0
\]
where equality holds when \( \Delta_s = \bar{\Delta}_s = k(s, M_{[0,s]}, Z_s m_s) \). Therefore, \( U^\Delta \) are supermartingales and \( U^\Delta \) is an increasing function and \( \sigma(s, M_{[0,s]}, \pi) \leq \rho(\pi) \) for all \( \pi \in \mathbb{R}^{1 \times n} \).

Therefore, \( \bar{\Delta} \) is the optimal control and the claim is proved. \( \square \)

### 6.2 Exponential utility

In this subsection, our objective is to find a control \( \Delta \in \mathcal{A} \) that maximize
\[
\mathbb{E} \left[ -\exp(-\kappa(X_T^\Delta - \xi(M_{[0,T]}))) \right]
\]
where \( \kappa > 0 \) and \( X^\Delta \) is given by (6.8).

**Theorem 6.2.** Assume that there exist \( C \in \mathbb{R}_+ \), an increasing continuous function \( \rho : \mathbb{R}_+ \to \mathbb{R} \), and a \( \mathcal{B}([0,T]) \otimes \mathcal{D} \otimes \mathcal{B}(\mathbb{R}^n) \)-measurable function \( k : [0,T] \times D \times \mathbb{R}^{1 \times n} \to \mathbb{R}^{1 \times n} \) that satisfy the following conditions:

- \( X^\Delta \) is well-defined for any \( \Delta \in \mathcal{A} \).
- \( |\sigma(s, M_{[0,s]}, \pi)| \leq \rho(|\pi|) \) for all \( \pi \in \mathbb{R}^{1 \times n} \).
- \( |k(s, \gamma, z)| \leq \rho(|z|) \) for all \( (s, \gamma, z) \in [0,T] \times D \times \mathbb{R}^{1 \times n} \).
- For all \( (s, \gamma, \pi, z) \in [0,T] \times D \times \mathcal{A} \times \mathbb{R}^{d \times n} \), the following inequality holds:
  \[
  G(s, \gamma, \pi, z) := \frac{1}{2} \kappa |\sigma(s, \gamma_{[0,s]}, \pi)m_s - z|^2 - b(s, \gamma_{[0,s]}, \pi) \geq G(s, \gamma_{[0,s]}, k(s, \gamma_{[0,s]}, z), z) := f(s, \gamma_{[0,s]}, z)
  \]
- The BSDE(\( \xi, f \)) has a solution \( (Y, Z) \in \mathbb{H}^2 \times \mathbb{H}^2_{m_0} \) such that \( Z \) is bounded.

Then,
\[
\hat{\Delta}_s := k(s, M_{[0,s]}, Z_s m_s) \in \mathcal{A}
\]
is the optimal control and the optimal value is \( -e^{-\kappa(x-Y_0)} \).
Therefore, $\bar{\Delta}$ is the optimal control and the claim is proved. For $\Delta \in \mathcal{A}$, we define a family of stochastic process $\{U^\Delta\}$ given by

$$U^\Delta_t = -\exp\left(-\kappa(X^\Delta_t - Y_t)\right).$$

For notational convenience, let us use $b^\Delta_s := b(s, M_{[0,s]}, \Delta_s)$ and $\sigma^\Delta_s := \sigma(s, M_{[0,s]}, \Delta_s)$. Note that we have

$$U^\Delta_t = -\exp\left(-\kappa \left( x - Y_0 + \int_0^t \left( b^\Delta_s + f(s, M_{[0,s]}, Z_s m_s) \right) dA_s + \int_0^t \left( \sigma^\Delta_s - Z_s \right) dM_s \right) \right)
= -e^{-\kappa(x-Y_0)} \mathcal{E} \left( -\int_0^t \kappa \left( \sigma^\Delta_s - Z_s \right) dM_s \right) \exp \left( \kappa \int_0^t \left( G(s, M_{[0,s]}, \Delta_s, Z_s m_s) - f(s, M_{[0,s]}, Z_s m_s) \right) dA_s \right)
$$

By the same logic in the proof of Theorem 6.1, $\mathcal{E} \left( -\int_0^t \kappa \left( \sigma^\Delta_s - Z_s \right) dM_s \right)$ is a true martingale. Since $G(s, M_{[0,s]}, \Delta_s, Z_s m_s) \geq f(s, M_{[0,s]}, Z_s m_s)$ and equality holds when $\Delta_s = \bar{\Delta}_s = k(s, M_{[0,s]}, Z_s m_s)$, $U^\Delta$ are supermartingales and $U^\Delta_\bar{\Delta}$ is a true martingale. This implies that

$$\mathbb{E} \left[ -\exp \left( -\kappa(X^\Delta_T - Y_T) \right) \right] \leq U^\Delta_0 = -e^{-\kappa(x-Y_0)} = U^\Delta_\bar{\Delta} = \mathbb{E} \left[ -\exp \left( -\kappa(X^\Delta_T - Y_T) \right) \right].$$

Therefore, $\bar{\Delta}$ is the optimal control and the claim is proved. □

7 Application to Optimal Portfolio Selection

In this section, we apply above result to a market consists of 1 risk-free asset with no interest and $n$ risky assets $S^1, \ldots, S^n$ whose prices follow dynamics

$$\frac{dS^i_t}{S^i_t} = \theta^i dA_t + dM^i_t,$$

where $\theta \in \mathbb{R}^{n \times 1}$ and $M$ satisfies (M) and (Diff'). In addition we always assume that $m$ is invertible and deterministic and $\xi$ satisfies (Diff').

7.1 Power utility function

In this case, the $\Delta^i_t$ represent the portion of wealth invested on asset $S^i$ at time $t$. We also assume that, if the investor invest $\Delta$ on time $[t, t+dt]$, there is cost $X^\Delta_t p(\Delta_t)dt$ for some function $p : \mathbb{R} \to \mathbb{R}$ which we call a penalty function. Then, if the initial wealth is $x$, the wealth process associated with $\Delta$ is given by

$$dX^\Delta_t = X^\Delta_t (-p(\Delta_t) + \Delta_t \theta) dA_t + X^\Delta_t \Delta_t dM_t; \quad X^\Delta_0 = x.$$

The investor tries to maximize

$$\mathbb{E} \left[ \frac{1}{\kappa} \left( X^\Delta_T e^{-\xi(M_{[0,T]})} \right)^\kappa \right]$$

where $\kappa \in (-\infty, 0) \cup (0, 1]$. Then, the function $G$ and $k$ in Theorem 6.1 are given by

$$G(s, \gamma, \pi, z) := -\frac{1}{2} \kappa |\pi m_s - z|^2 + p(\pi) - \pi \theta + \frac{1}{2} |\pi m_s|^2$$

$$k(s, \gamma, \pi, z) \in \arg\min_{\gamma \in \mathcal{A}} G(s, \gamma, \pi, z).$$
Let us provides three specific examples. The first example is the case where the investment is restricted to some closed set. This is similar to Hu et al. (2005, [23]), but we are seeking the optimal investment that does not require infinite investment.

Example 7.1. [Incomplete Market] For an investor, there may be a restriction on the investment strategy \( \Delta \) so that \( \Delta_t(\omega) \in \mathcal{C} \) for some closed set \( \mathcal{C} \subset \mathbb{R}^{1\times n} \), then we can define \( p(\pi) = 0 \) when \( \pi \in \mathcal{C} \) and \( \infty \) otherwise. When \( \kappa < 1 \), the minimum of \( G \) is attained when

\[
\pi = \Pi_s \left( \frac{\theta^* - \kappa z}{1 - \kappa} m_s^{-1} \right) := k(s, \gamma, z)
\]

where \( \Pi_s \) is the projection to the set \( \mathcal{C} m_s := \{r m_s : r \in \mathcal{C}\} \). Since \( k \) does not depend on \( \gamma \) and Lipschitz in \( z \), \( f(s, z) := G(s, \gamma, k(s, \gamma, z), z) \) satisfies (Diff') and (Loc'), by Theorem 5.8, BSDE(\( \xi, f \)) has a unique solution \((Y, Z) \in \mathbb{H}^2 \times \mathbb{H}_m^2\) such that \( Z \) is bounded. Therefore, by Theorem 6.1, the optimal control is

\[
\bar{\Delta}_s := \Pi_s \left( \frac{\theta^* m_s^{-1} - \kappa Z_s}{1 - \kappa} \right)
\]

and the optimal value is \( \frac{\bar{\Delta}_s^e - \kappa Y_0}{\kappa} \).

On the other hand, if \( \kappa = 1 \) and \( \mathcal{C} \) is a bounded set,

\[
G(s, \gamma, \pi, z) := \pi m_s (z^* - m_s^{-1} \theta) - \frac{1}{2} |z|^2 + p(\pi)
\]

attains minimum \( f(s, z) \) when \( \pi m_s (z^* - m_s^{-1} \theta) \) is minimized over \( \pi \in \mathcal{C} \). Note that \( f(s, z) \) does not depend on \( \gamma \) and satisfies (Loc') where \( \rho \) has linear growth. Therefore, BSDE(\( \xi, f \)) has a unique solution \((Y, Z) \in \mathbb{H}^2 \times \mathbb{H}_m^2\) such that \( Z \) is bounded and the optimal control is

\[
\bar{\Delta}_s := \arg \min_{\pi \in \mathcal{C}} (\pi m_s (m_s^* Z^s - m_s^{-1} \theta))
\]

and the optimal value is \( \frac{\bar{\Delta}_s^e - \kappa Y_0}{\kappa} \).

The next example is the case where the penalty function encourage the diversification of portfolio among risky assets. The penalty function is given by \( p(\pi) := |\pi (\mathbb{I} - w)|^\beta \) where \( \beta > 1 \) and \( w \) is a \( n\)-by-\( n \) matrix where \( (\mathbb{I} - w)(\mathbb{I} - w)^* + \frac{1}{\kappa^2} m_s m_s^* \) is invertible. Note that \( p \) attains minimum when \( \pi = \pi w \), so we can think \( \pi w \) is the preferred distribution of wealth. If one want to encourage diversification of portfolio, one can set \( w_{ij} \in \left[ \frac{1}{n - \varepsilon}, \frac{1}{n + \varepsilon} \right], \forall i, j \), for small \( \varepsilon > 0 \). In the extreme case, if one set \( \varepsilon = 0 \), the penalty attains minimum when \( \pi_i = \pi_j \) for all \( i, j = 1, ..., n \). However, in this case, \( \mathbb{I} - w \) is not invertible so we encounter problem when \( \kappa = 1 \). The bigger \( \beta \) implies the stronger encouragement toward the preferred wealth distribution.

Example 7.2. [Diversification of Portfolio] Assume that

\[
p(\pi) := |\pi (\mathbb{I} - w)|^\beta.
\]

Note that \( p \) is a convex function which is differentiable everywhere and therefore, \( G \) has a minimum when it satisfies the first order condition:

\[
\nabla p(\pi) + (1 - \kappa) \pi m_s m_s^* = -\kappa z m_s^* + \theta^*.
\]
(Case $\kappa < 1$) To find an explicit solution, let us consider the case where $\beta = 2$ and $\kappa < 1$. Then, the first order condition becomes

$$2\pi (I - w) (I - w)^* + (1 - \kappa) \pi m_s m_s^* = - \kappa z m_s^* + \theta^*$$

and this implies

$$\pi = k(s, z) := \frac{1}{2} (- \kappa z m_s^* + \theta^*) \left( (I - w)(I - w)^* + \frac{1 - \kappa}{2} m_s m_s^* \right)^{-1}.$$ 

Let $f(s, z) := G(s, \gamma, k(s, z), z)$. Since $k$ is an affine function of $z$, $f$ satisfies (Loc') with $\rho$ that has linear growth, and the BSDE($\xi, f$) has a unique solution $(Y, Z) \in H^2 \times H^2_m$ such that $Z$ is bounded. Therefore, the optimal control is

$$\bar{\Delta}_s = \frac{1}{2} (- \kappa Z_s m_s m_s^* + \theta^*) \left( (I - w)(I - w)^* + \frac{1 - \kappa}{2} m_s m_s^* \right)^{-1}.$$ 

(Case $\kappa = 1$) On the other hand, let us assume $\kappa = 1$ which means that the investor is risk-neutral but there is a penalty if the portfolio is away from $w$. Then the first order condition becomes

$$\beta |\pi (I - w)|^{\beta - 2} \pi (I - w) (I - w)^* = - z m_s^* + \theta^*$$

and this implies

$$k(s, z) = \left( \frac{|(- z m_s^* + \theta^*)(I - w)^*)^{-1}|^{2 - \beta}}{\beta} \right)^{1/(\beta - 1)} (- z m_s^* + \theta^*) (I - w)(I - w)^*^{-1}.$$ 

Therefore,

$$f(s, z) := \left( \frac{|(- z m_s^* + \theta^*)(I - w)^*)^{-1}|^{\beta}}{\beta} - k(s, z)(\theta - m_s z^*) - \frac{1}{2} |z|^2.$$

Note that when $\beta \geq 2$, (Loc') is satisfied $\rho$ with linear growth. On the other hand, if $1 < \beta < 2$, (Loc') is satisfied with $\rho$ that has superlinear growth. Therefore, in any case, BSDE($\xi, f$) has a unique solution $(Y', Z')$ such that $Z'$ is bounded, and we have optimal control

$$\bar{\Delta} = \left( \frac{|(- Z'_s m_s m_s^* + \theta^*)(I - w)^*)^{-1}|^{2 - \beta}}{\beta} \right)^{1/(\beta - 1)} (Z'_s m_s m_s^* + \theta^*) (I - w)(I - w)^*^{-1}$$

Remark 7.3. In above example, if $\kappa = 1$ and $\beta < 2$, the driver $f(s, z)$ of BSDE has superquadratic growth in $z$. As in Delbaen et al. (2010, [16]), this BSDE may not have a solution or may have infinite number of solution if we do not assume (Diff'). Due to this difficulty, as far as the author knows, above example is the first financial application of superquadratic BSDE.

The last example is when there is information processing cost for trading assets. Any investment on an asset requires constant follow-up of information. It may be possible that this information processing cost is so high that it is more reasonable not to trade the asset at all.
Example 7.4. [Information Processing Cost] Assume that \( \theta = 0 \) and \( m_s := n^{-\frac{1}{2}} I \) where \( I \) is the \( n \)-by-\( n \) identity matrix. For each \( i = 1, \ldots, n \), risky assets \( S^i \) may require information processing cost \( C_i \) if the investor has nonzero amount of asset \( S^i \). This implies \( p(\pi) = \sum_{i=1}^{n} C_i \mathbb{1}_{\{\pi^i \neq 0\}} \). Then,

\[
G(s, \gamma, \pi, z) = \frac{1 - \kappa}{2n} |\pi|^2 + \sum_{i=1}^{n} C_i \mathbb{1}_{\{\pi^i \neq 0\}} + \frac{\kappa}{\sqrt{n}} \pi z^* - \frac{\kappa}{2} |z|^2
\]

\[
= \frac{1 - \kappa}{2n} \sum_{i=1}^{n} \left[ (\pi^i + \frac{\kappa}{1 - \kappa} z^i)^2 + C_i \mathbb{1}_{\{\pi^i \neq 0\}} - \frac{\kappa n}{(1 - \kappa)^2} |z|^2 \right]
\]

Therefore, \( G \) is minimized when

\[
\pi^i = k^i(s, z) := \begin{cases} 
- \frac{\kappa}{|z|} \frac{1}{1 - \kappa} z^i & \text{when } |z| > \frac{1 - \kappa}{|\kappa|} \sqrt{C_i} \\
0 & \text{otherwise}
\end{cases}
\]

for each \( i = 1, \ldots, n \). If we define

\[
f(s, z) := G(s, \gamma, k(s, z), z) = \frac{1 - \kappa}{2n} \sum_{i=1}^{n} \min \left( -\frac{\kappa n}{(1 - \kappa)} |z|^2, C_i - \frac{\kappa n}{(1 - \kappa)^2} |z|^2 \right)
\]

Since \( f \) satisfies (Loc') where \( \rho \) has linear growth, the BSDE(\( \xi, f \)) has a unique solution \((Y, Z)\) such that \( Z \) is bounded. Therefore, the optimal control is

\[
\Delta^i = \begin{cases} 
- \frac{\kappa}{1 - \kappa} Z^i & \text{when } |Z^i| > \frac{1 - \kappa}{|\kappa|} \sqrt{C_i} \\
0 & \text{otherwise}
\end{cases}
\]

7.2 Exponential utility function

At time \( t \in [0, T] \), an investor invest \( \Delta^i_t \) unit of currency on the asset \( S^i \). If the investor invest \( \Delta \) on time \([t, t + dt] \), there is cost \( p(\Delta_t)dt \) for some function \( p : \mathbb{R} \to \mathbb{R} \) which we call a penalty function. Then, if the initial wealth is \( x \), the wealth process associated with \( \Delta \) is given by

\[
dX^\Delta_t = (-p(\Delta_t) + \Delta_t \theta) dA_t + \Delta_t dM_t; \quad X^\Delta_0 = x.
\]

Assume that the investor trying to maximize

\[
\mathbb{E} \left[ -\exp \left( -\kappa(X_T^\Delta - \xi(M_{[0,T]})) \right) \right]
\]

for some \( \kappa > 0 \). Then, \( G \) and \( k \) in Theorem 6.2 are given by

\[
G(s, \gamma, \pi, z) := \frac{1}{2} \kappa |\pi m_s - z|^2 + p(\pi) - \pi \theta
\]

\[
k(s, \gamma, z) \in \arg \min_{\pi \in A} G(s, \gamma, \pi, z)
\]

and we can apply Theorem 6.2 if the assumptions are satisfied.

The followings are three examples which are analogous to the power utility examples in the previous sections.
Example 7.5. [Incomplete Market] We assume that $\theta = 0$ and the investment $\Delta_t(\omega)$ should be in some closed set $\mathcal{C} \subset \mathbb{R}^{1 \times n}$. In this case, we set $p(\pi) = 0$ when $\pi \in \mathcal{C}$ and $\infty$ otherwise. Then, the minimum of $G$ is attained when $\pi = (\Pi_s z)m_s^{-1}$ where $\Pi_s$ is the projection to the set $\mathcal{C}_m$. Since $f(s, z) := \frac{1}{\kappa} |\Pi_s z - z|^2$ satisfies (Diff') and (Loc'), by Theorem 5.8, BSDE($\xi, f$) has a unique solution $(Y, Z)$ such that $Z$ is bounded. Therefore, by Theorem 6.2, the optimal control is $\bar{\Delta}_s := \Pi_s (Z_m s) = \Pi (Z_s)$ where $\Pi$ is the projection to set $\mathcal{C}$.

Example 7.6. [Diversification of Portfolio] Let $w$ be an $n$-by-$n$ matrix and assume that $(I - w)(I - w)^* + \kappa m_s m_s^*/2$ is invertible for all $s \in [0, T]$. We set $p(\pi) := |\pi (I - w)|^2$. Note that $p$ is a convex function with differentiable everywhere and therefore, $G$ has a minimum when

$$\nabla p(\pi) + \kappa \pi m^*_s = \kappa z m^*_s + \theta^*.$$ 

Therefore, $G$ is minimized when

$$\pi = \frac{1}{2} (\kappa z m^*_s + \theta^*) \left[(I - w)(I - w)^* + \frac{1}{2} \kappa m_s m_s^*\right]^{-1} := k(s, z).$$

Let $f(s, z) := G(s, \gamma, k(s, z), z)$. Since $k$ is a linear function of $z$, the BSDE($\xi, f$) has a unique solution $(Y, Z)$ such that $Z$ is bounded. Therefore, the optimal control is

$$\Delta_s = \frac{1}{2} (\kappa Z_s m_s m_s^* + \theta^*) \left[(I - w)(I - w)^* + \frac{1}{2} \kappa m_s m_s^*\right]^{-1}.$$ 

Example 7.7. [Information Processing Cost] Assume that $\theta = 0$ and $m_s := n - \frac{1}{2} I$ where $I$ is the $n$-by-$n$ identity matrix. For each $i = 1, ..., n$, risky assets $S^i$ may require information processing cost $C_i$ if the investor decide to trade $S^i$. This implies $p(\pi) = \sum_{i=1}^{n} C_i 1_{\{\pi^i \neq 0\}}$. Then,

$$G(s, \gamma, \pi, z) = \frac{1}{2} \kappa \sum_{j=1}^{n} \left[ \frac{|\pi^j|}{\sqrt{n}} - z_j \right]^2 + \frac{2C_j}{\kappa} 1_{\{\pi^i \neq 0\}}$$

Therefore, $G$ is minimized when

$$\pi^j = k^j(s, z) := \begin{cases} \sqrt{n} z_j & \text{when } |z^j| > \sqrt{\frac{2C_j}{\kappa}} \\ 0 & \text{otherwise} \end{cases}$$

for each $j = 1, ..., n$. If we define

$$f(s, z) := G(s, \gamma, k(s, z), z) = \frac{1}{2} \kappa \sum_{j=1}^{n} \min \left(|z^j|^2, \frac{2C_j}{\kappa}\right)$$

Since $f$ is Lipschitz in $z$, the BSDE($\xi, f$) has a unique solution $(Y, Z)$ such that $Z$ is bounded. Therefore, the optimal control is

$$\bar{\Delta}_s^j = \begin{cases} Z^j & \text{when } |Z^j| > \sqrt{\frac{2nC_j}{\kappa}} \\ 0 & \text{otherwise} \end{cases}$$
A Appendix

Lemma A.1. Assume $Y \in \mathbb{S}^2$, $Z \in \mathbb{H}_m^2$, and $N \in \mathbb{M}^2$. Then, $\int Y^*_s Z_s dM_s$, $\int Y^*_s dN_s$ are true martingales.

Proof. Note that, by Cauchy-Schwartz inequality,

$$
\mathbb{E} \left[ \int_0^T Y^*_s Z_s dM_s, \int_0^T Y^*_s Z_s dM_s \right]^{\frac{1}{2}} \leq \mathbb{E} \left[ \int_0^T |Y^*_s Z_s m_s|^2 dA_s \right] \leq \mathbb{E} \sup_{t \in [0,T]} |Y_t| \left( \int_0^T |Z_s m_s|^2 dA_s \right) \leq \|Y\|_{\mathbb{S}^2} \|Z\|_{\mathbb{H}_m^2} < \infty
$$

and

$$
\mathbb{E} \left[ \int_0^T Y^*_s dN_s, \int_0^T Y^*_s dN_s \right]^{\frac{1}{2}} \leq \mathbb{E} \left[ \int_0^T |Y^*_s|^2 d\text{tr}[N,N]_s \right] \leq \mathbb{E} \sup_{t \in [0,T]} |Y_t| \sqrt{\text{tr}[N,N]_T} \leq \|Y\|_{\mathbb{S}^2} \|N\|_{\mathbb{M}^2} < \infty.
$$

Then, by Burkholder-Davis-Gundy inequality, $\sup_{t \in [0,T]} \int_0^T Y^*_s Z_s dM_s$ and $\sup_{t \in [0,T]} \int_0^T Y^*_s dN_s$ are integrable and therefore, $\int_0^T Y^*_s Z_s dM_s$ and $\int_0^T Y^*_s dN_s$ are martingales. \hfill \blacksquare

Lemma A.2. A $\mathbb{R}^k$-valued stochastic process $X$ is adapted to $\mathbb{F}^{M'}$ if and only if there exists a function $\mathcal{X} : [0,T] \times D \to \mathbb{R}^k$ such that

$$X_t = \mathcal{X}(t, M'_{[0,t]})$$

holds almost surely for each $t \in [0,T]$ and $\mathcal{X}(t, \cdot)$ is $\mathcal{D}$-measurable.

Proof. Note that if $X$ is adapted to $\mathbb{F}^{M'}$, there exists $X'$ which is adapted to the filtration generated by $M$ and $X = X'$ almost surely. Therefore, without loss of generality, we can prove the theorem under the filtration generated by $M'$.

$(\Rightarrow)$ By Cinlar (2010, [12], Chapter 2, Proposition 4.6), for all $t \in [0,\infty)$, there exists a sequence $(t_n)_{n=1,2,...} \subset [0,t]$ and a $\otimes_i \mathcal{B}(\mathbb{R}^{n_i})$-measurable function $\mathcal{X}^c_t : \prod_i \mathbb{R}^{n_i} \to \mathbb{R}^k$ such that

$$X_t = \mathcal{X}^c_t(M'_{t_1}, M'_{t_2}, ...).$$

Let us denote $\pi$ to be the coordinate map; that is, $\pi_{a,b,...}(\gamma) = (\gamma_a, \gamma_b, ...)$. Then,

$$X_t := (\mathcal{X}^c_t \circ \pi_{t_1,t_2,...})(M'_{[0,t]})$$

Since $\pi_{t_1,t_2,...}$ is $\mathcal{D}$-measurable, $\mathcal{X}(t, \cdot) := \mathcal{X}^c_t \circ \pi_{t_1,t_2,...}$ is the function we want.

$(\Leftarrow)$ Since $M'_{[0,t]}$ is $\mathcal{F}_t \setminus \mathcal{D}$-measurable and $\mathcal{X}$ is $\mathcal{D}$-measurable, the claim is obvious. \hfill \blacksquare

References


