Path Dependent Optimal Transport and Model Calibration on Exotic Derivatives

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Path Dependent Optimal Transport and Model Calibration on Exotic Derivatives

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Abstract

In this paper, we introduce and develop the theory of semimartingale optimal transport in a path dependent setting. Instead of the classical constraints on marginal distributions, we consider a general framework of path dependent constraints. Duality results are established, representing the solution in terms of path dependent partial differential equations (PPDEs). Moreover we provide a localisation result, which reduces the dimensionality of the solution by identifying appropriate state variables based on the constraints and the cost function. Our technique is then applied to the exact calibration of volatility models to the prices of general path dependent derivatives.

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Keywords: optimal transport, path dependent PDE, volatility calibration

1 Introduction

Inspired by the seminal work on optimal transport by Benamou and Brenier\textsuperscript{3}, and the duality theory developed in \textsuperscript{4} and later \textsuperscript{17}, we examine the problem of optimal transport by semimartingales. The \textit{semimartingale optimal transport problem} with constraint on marginals at given times has been studied by Tan and Touzi in \textsuperscript{20}, extending the work of \textsuperscript{18}. Other related works include \textsuperscript{21} \textsuperscript{16}. The main goal of our study is to extend this work by considering a much wider range of constraints. As shown in our abstract formulation, the constraints considered can be an arbitrary closed and convex set of probability measures. In particular, this encompasses the marginal constraints of the classical optimal transport problem. Furthermore, it can include constraints such as bounds on the distributions as well as expectations of path dependent functions.

One of the outcome of the duality techniques developed in \textsuperscript{4} \textsuperscript{17} is that it bypasses the need to establish the dynamic programming principle, as the Hamilton-Jacobi-Bellman equation arises directly from the dual formulation. Our study establishes a natural connection

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between this optimal transport problem and the recent theory of path dependent partial differential equations (PPDEs) as developed in [11, 12, 9, 6, 5]. Moreover, we introduce the notion of localisation, which reduces the complexity of the PPDE into a more tractable PDE by identifying relevant state variables from the cost function and constraints. The localisation applies to both the solution of the optimal transport problem as well as the optimal drift and diffusion functions.

Recently, optimal transport has found applications in mathematical finance, particularly in the areas of robust finance. For example, it is used to obtain model-free bounds for exotics derivatives (see [15]) and robust hedging (see [7]). There have also been recent applications in stochastic portfolio theory [19]. In this work, we study the application of optimal transport to model calibration, which is a crucial problem in financial modelling. The celebrated Dupire’s formula [10] provides a unique way to recover a local volatility model from the knowledge of vanilla options for all strikes and maturities. However, it requires some form of price interpolation as only a finite number of options are available. Beyond this analytical result, there are few theoretical advances that address the problem of calibration. Practitioners therefore often rely on parametric models, which they fine-tune to match observable instruments in the best possible way.

The duality theory developed in this paper is used to exactly calibrate to any number of path dependent derivatives, without the need to perform any price interpolation. The underlying idea for calibrating local volatility to European options was explored by the authors in [13], which is an adaptation of the numerical method of [3]. A similar numerical algorithm for discrete option prices was also studied in [2] in the context of entropy minimisation. In this paper, we extend the approach to the calibration of path dependent derivatives, resulting in a path dependent volatility function, a notion also explored in [14]. Despite the complex path dependent nature of the problem, efficient numerical algorithms are still possible via our localisation result, where the optimal volatility function “localises” to the state variables driving the derivatives. In fact, the term “localisation” is chosen in reference to the fact that European option prices can always be calibrated to local volatility models. Our method can also be used to calibrate stochastic volatility models to exactly match a set of given payoffs, while remaining close to a reference stochastic volatility model. This therefore extends the calibration of so-called LSV models (see [1] and references therein).

The paper is organised as follows. Section 2 introduces the basic notations used throughout the paper as well as the abstract formulation of the problem. Section 3 contains the main results and their proofs. A summary of the main results is found in Theorem 3.1. Section 4 includes the application to volatility calibration, demonstrating numerical examples that calibrate to a large number of European, barrier and lookback options.

2 Optimal transport under path dependent constraints

2.1 Preliminaries

Let \( \Omega := C([0,1], \mathbb{R}^d) \) be the set of continuous paths, \( X \) be the canonical process and \( \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq 1} \) be the canonical filtration generated by \( X \). For each \( t \in [0,1] \), let \( \Omega_t := \{ \omega_{\cdot \wedge t} : \omega \in \Omega_t \} \) be the set of paths stopped at time \( t \). In order to characterise path dependencies, let \( \Lambda := \{ (t, \omega) : t \in [0,1], \omega \in \Omega_t \} \). The spaces \( \Omega \) and \( \Omega_t \) are equipped with the norm \( \| \omega \|_{\infty} = \sup_{t \in [0,1]} | \omega_t | \), while the spaces \( \Lambda \) is equipped with the metric \( d_{\infty}((t, \omega), (t', \omega')) = |t - t'| + \| \omega - \omega' \|_{\infty} \).

Given a Polish space \( \mathcal{X} \) equipped with its Borel \( \sigma \)-algebra, let \( C(\mathcal{X}) \) be the set of con-
Now let us define the semimartingale optimal transport problem under path dependent constraints.

2.2 Problem formulation

For a function \( \phi : \Lambda \to \mathbb{R} \) it is said that \( \phi \in \mathcal{C}^{1,2}(\Lambda) \) if \( \phi \in \mathcal{C}(\Lambda) \) and there exist functions \((D_1\phi, \nabla_x \phi, \nabla_x^2 \phi) \in \mathcal{C}(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d)\) such that, for any \( \mathbb{P} \in \mathcal{P} \) and \( u \in [0,1] \), the following functional Itô formula holds:

\[
\phi(u, X) - \phi(0, X) = \int_0^u D_1\phi \, dt + \nabla_x \phi \cdot dX_t + \frac{1}{2} \nabla_x^2 \phi : d\langle X \rangle_t, \quad \mathbb{P}\text{-a.s.} \tag{2}
\]
Denote by $H : \Lambda \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ a cost function. Define $H^* : \Lambda \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R} \cup \{+\infty\}$ so that $H^*(t, \omega, \cdot)$ is the convex conjugate of $H(t, \omega, \cdot)$. When there is no ambiguity, we will simply write $H(\alpha, \beta) := H(t, \omega, \alpha, \beta)$ and $H^*(\alpha, \beta) := H^*(t, \omega, \alpha, \beta)$. We impose the following global assumption on $H$.

**Assumption 2.1.**
(i) For each $(t, \omega) \in \Lambda, H(t, \omega, \cdot)$ is a non-negative, lower semi-continuous, proper convex function with $\min_{\alpha, \beta} H(t, \omega, \alpha, \beta) = 0$.
(ii) If $\beta \in \mathbb{S}^d \setminus \mathbb{S}^d_+$ then $H(\cdot, \cdot, \cdot, \beta) = +\infty$.
(iii) The cost function $H$ is coercive in the sense that there exist constants $p > 1$ and $C > 0$ such that $|\alpha|^p + |\beta|^p \leq C(1 + H(t, \omega, \alpha, \beta))$, $\forall (t, \omega) \in \Lambda$.

In order to characterise the constraints, let $\mathcal{N} \subseteq \mathcal{P}$ be a convex set of probability measures. Assume that $\mathcal{N}$ is closed under the weak topology, that is, the coarsest topology under which $\mu \to \int_{\Omega} \psi d\mu$ is continuous for all $\psi \in C_b(\Omega)$. In our problem, we would like to restrict the probability measures to the set $\mathcal{N}$.

**Definition 2.2.** Given $t, \rho, H$ and $\mathcal{N}$, the semimartingale optimal transport problem under path dependent constraints refers to the following minimisation problem

$$
\inf_{\mathbb{P} \in \mathbb{P}^\rho_{\varnothing}(\rho_t)} \mathbb{E}^\mathbb{P} \int_0^1 H(\alpha_P^s, \beta_P^s) ds, \quad (3)
$$

subject to $\mathbb{P} \in \mathcal{N}$. \quad (4)

The problem is said to be *admissible* if $\mathbb{P}^\rho_{\varnothing}(\rho_t) \cap \mathcal{N} \neq \emptyset$ and the infimum above is finite.

To handle the constraint, consider the convex function $F : C_b(\Omega) \to \mathbb{R}$ defined by

$$
F(\psi) := \sup_{\mu \in \mathcal{N}} \int_{\Omega} \psi d\mu. \quad (5)
$$

Since $\mathcal{N}$ is a non-empty subset of $\mathcal{P}$, it is easy to verify that $F(0) = 0$ and $F(\psi + \kappa) = F(\psi) + \kappa$ for all $\kappa \in \mathbb{R}$. Next, since $\mathcal{N}$ is closed, the convex conjugate of $F$ has the following representation: For any $\mu \in \mathcal{M}(\Omega)$,

$$
F^*(\mu) = \sup_{\psi \in C_b(\Omega)} \int_{\Omega} \psi d\mu - F(\psi) = \begin{cases} 0, & \mu \in \mathcal{N}, \\ +\infty, & \mu \notin \mathcal{N}. \end{cases} \quad (6)
$$

Note that in general, the domain of the convex conjugate $F^*$ is $C_b(\Omega)^*$, which is isomorphic to the set of all regular, signed, finite and finitely additive Borel measures (Theorem IV.6.2). For our purposes, we only need to consider the restriction of $F^*$ on $\mathcal{M}(\Omega) \subseteq C_b(\Omega)^*$.

In many cases, it is possible to further restrict the choice of $\psi$ to some convex subset $\mathcal{N}^* \subseteq C_b(\Omega)$, so

$$
F^*(\mu) = \sup_{\psi \in \mathcal{N}^*} \int_{\Omega} \psi d\mu - F(\psi) = \begin{cases} 0, & \mu \in \mathcal{N}^*, \\ +\infty, & \mu \notin \mathcal{N}^*. \end{cases} \quad (7)
$$

For example, if $\mathcal{N}$ is a subspace (or its translation), then $\mathcal{N}^*$ can be chosen as the annihilator. In general, suitable choices of $\mathcal{N}^*$ cannot always be easily identified. However, when it is possible (as we will see in the later examples), the reduction of $C_b(\Omega)$ to $\mathcal{N}^*$ can greatly simplify the problem.
The formulation of $F^*$ in (7) indicates that it is, in fact, a suitable function for the penalisation of measures outside $N$. Hence, the problem can be reformulated as the following saddle point problem:

$$
\inf_{P \in P_t^{\rho}} \mathbb{E}^P \int_t^1 H(\alpha_s^P, \beta_s^P) \, ds,
\text{ subject to } P \in N
$$

$$
= \inf_{P \in P_t^{\rho}} F^*(P) + \mathbb{E}^P \int_t^1 H(\alpha_s^P, \beta_s^P) \, ds
$$

$$
= \inf_{P \in P_t^{\rho}} \sup_{\psi \in N^*} \mathbb{E}^P \psi - F(\psi) + \mathbb{E}^P \int_t^1 H(\alpha_s^P, \beta_s^P) \, ds.
$$

(8)

2.3 Examples

The constraint on measures in our formulation is quite general and allows for a wide range of problems, so let us give some examples below.

Example 2.3 (Benamou-Brenier optimal transport [3]). If the cost function satisfies $H(\alpha, \beta) = +\infty$ when $\beta \neq 0$, then we recover the classical deterministic optimal transport problem by Benamou-Brenier [3].

Example 2.4 (Tan and Touzi martingale optimal transport [20]). Let $\bar{\rho}$ be a probability measure on $\mathbb{R}^d$ and consider the constraint $P \circ X^{-1} = \bar{\rho}$. Then by setting

$$
N = \{ \mu \in \mathcal{P} : \mu \circ X^{-1} = \bar{\rho} \},
N^* = \{ \psi \in C_b(\Omega) : \psi = \lambda \circ X, \lambda \in C_b(\mathbb{R}^d) \},
$$

we recover the optimal transport problem addressed in [20]. In particular

$$
F(\psi) = \begin{cases} 
\int_{\mathbb{R}^d} \lambda \, d\bar{\rho}, & \lambda \in C_b(\mathbb{R}^d), \psi = \lambda \circ X, \\
+\infty, & \text{otherwise},
\end{cases}
$$

(9)

and the saddle point problem is given by

$$
\inf_{P \in P_t^{\rho}} \sup_{\lambda \in C_b(\mathbb{R}^d)} \mathbb{E}^P (\lambda(X_1)) - \int_{\mathbb{R}^d} \lambda \, d\bar{\rho} + \mathbb{E}^P \int_t^1 H(s, X, \lambda, v^P_s) \, ds.
$$

Example 2.5. Further generalising the previous example, let $G \in C(\Omega, \mathbb{R}^m)$ be any continuous function and $\bar{\rho} \in \mathcal{M}(\mathbb{R}^m)$ be a target distribution. We would like to impose the constraint $P \circ G^{-1} = \bar{\rho}$. In this case the constraint is characterised by

$$
N = \{ \mu \in \mathcal{P} : \mu \circ G^{-1} = \bar{\rho} \},
N^* = \{ \psi \in C_b(\Omega) : \psi = \lambda \circ G, \lambda \in C_b(\mathbb{R}^d) \},
$$

and

$$
F(\psi) = \begin{cases} 
\int_{\mathbb{R}^m} \lambda \, d\bar{\rho}, & \lambda \in C_b(\mathbb{R}^m), \psi = \lambda \circ G, \\
+\infty, & \text{otherwise}.
\end{cases}
$$

(10)

The saddle point problem is given by

$$
\inf_{P \in P_t^{\rho}} \sup_{\lambda \in C_b(\mathbb{R}^m)} \mathbb{E}^P (\lambda \circ G) - \int_{\mathbb{R}^d} \lambda \, d\rho_1 + \mathbb{E}^P \int_t^1 H(s, X, \lambda, v^P_s) \, ds.
$$
Example 2.6. Fix $G \in C_b(\Omega, \mathbb{R}^m)$ and consider the constraint $\mathbb{E}^P G = c$. This corresponds to

$$\mathcal{N} = \{ \mu \in \mathcal{P} : \int_{\Omega} G \, d\mu = 0 \}, \quad \mathcal{N}^* = \{ \psi \in C_b(\Omega) : \psi = \lambda \cdot G, \ \lambda \in \mathbb{R}^m \}.$$ 

In this case,

$$F(\psi) = \begin{cases} \lambda \cdot c, & \psi = \lambda \cdot G, \lambda \in \mathbb{R}^m, \\ +\infty, & \text{otherwise}. \end{cases} \quad (11)$$

Then the saddle point problem is given by

$$\inf_{P \in \mathcal{P}_0} \sup_{\lambda \in \mathbb{R}^m_+} \lambda \cdot (\mathbb{E}^P G - c) + \mathbb{E}^P \int_0^1 H(s, X_{\lambda \cdot s}, v^P_s) \, ds.$$ 

Example 2.7. Let $G : \Omega \to \mathbb{R}^m_+$ be a function that is lower semi-continuous in each component. Consider the constraint $\mathbb{E}^P G \leq c$ for some $c \in \mathbb{R}^m_+$, where the inequality is taken element-wise. This corresponds to $\mathcal{N} = \{ \mu \in \mathcal{P} : \int_{\Omega} G \, d\mu \leq c \}$. One can check that this set is in fact closed under the weak topology. In this case

$$F(\psi) = \inf\{ \lambda \cdot c : \lambda \in \mathbb{R}^m_+, \ \psi \leq \lambda \cdot G \}. \quad (12)$$

Using the density of $C_b(\Omega)$ in $L^1(\Omega, \mathcal{P})$, the saddle point problem is given by

$$\inf_{P \in \mathcal{P}_0} \sup_{\lambda \in \mathbb{R}^m_+} \int_0^1 \mathbb{E}^P \psi - \lambda \cdot c + \mathbb{E}^P \int_0^1 H(s, X_{\lambda \cdot s}, v^P_s) \, ds \quad (13)$$

Then the saddle point problem is given by

$$\inf_{P \in \mathcal{P}_0} \sup_{\lambda \in \mathbb{R}^m_+} \lambda \cdot (\mathbb{E}^P G - c) + \mathbb{E}^P \int_0^1 H(s, X_{\lambda \cdot s}, v^P_s) \, ds \quad (14)$$

Via a suitable translation, the inequality constraint $\mathbb{E}^P G \leq c$ can be relaxed so that $G$ is bounded from below.

Remark 2.8. Note that if $G$ is an arbitrary unbounded or discontinuous functions, the equality constraint $\mathbb{E}^P G = c$ does not immediately fall within our formulation since the set $\mathcal{N} = \{ \mu \in \mathcal{M}_+(\Omega) : \int_{\Omega} G \, d\mu = c \}$ is not closed. When $G$ is unbounded, this can be fixed in some cases by restricting the problem to a subspace of $\mathcal{M}(\Omega)$ in which $\mathcal{N}$ is closed (for example, the space of measures with finite first moment), and then proceed by identifying its dual space and so on.

The continuity of $G$ can also be weakened. Our main results can be extended to many piece-wise continuous functions by transforming the underlying Polish space. As a simple example, consider the indicator function $1(x > 0)$ which is discontinuous in $\mathbb{R}$. But if we instead consider the Polish space $\mathbb{R}^- \cup \{0\} \cup \mathbb{R}^+$ (with the discrete topology between the different subsets), the function becomes continuous and the duality analysis can proceed from there.

## 3 Main results

### 3.1 Summary of main results

**Theorem 3.1.** Let $\mathcal{N} \subseteq \mathcal{P}$ be a convex subset that is closed with respect to the weak topology and define $F : C_b(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by $F(\psi) := \sup_{\mu \in \mathcal{N}} \int_{\Omega} \psi \, d\mu$. Let $H : \Lambda \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$.
\( R \cup \{ +\infty \} \) be a function satisfying Assumption 2.4. Let \( F^* \) and \( H^*(t, \omega, \cdot) \) denote the convex conjugates of \( F \) and \( H(t, \omega, \cdot) \), respectively. Moreover, suppose that there exists a set \( N^* \subseteq C_b(\Omega) \) such that

\[
F^*(\mu) = \sup_{\psi \in N^*} \int_{\Omega} \psi \, d\mu - F(\psi) = \begin{cases} 0, & \mu \in N, \\ +\infty, & \mu \notin N. \end{cases} \tag{15}
\]

Given \( t \in [0, 1] \) and a probability measure \( \rho_t \) on \( \Omega_t \), recall that the semimartingale optimal transport problem with path dependent constraints refers to the following minimisation problem,

\[
V := \inf_{P \in P^0_t(\rho_t)} F^*(P) + \mathbb{E}^P \int_0^1 H(\alpha^P_s, \beta^P_s) \, ds. \tag{16}
\]

(i) **Duality:** If the problem is admissible, then the infimum in (16) is attained and it equals

\[
V = \mathcal{V} := \sup_{\psi \in N^*} -F(\psi) - J_{\psi, H}(t, \rho_t), \tag{17}
\]

where \( J \) satisfies the following equalities

\[
J_{\psi, H}(t, \rho_t) = \sup_{P \in P^0_t(\rho_t)} \mathbb{E}^P \psi - \mathbb{E}^P \int_0^1 H(\alpha^P_s, \beta^P_s) \, ds \tag{18}
\]

\[
= \inf_{\phi \in C^{1,2}_t(\Lambda)} \int_{\Omega_t} \phi(t, \cdot) \, d\rho_t, \tag{19}
\]

subject to \( \phi(1, \cdot) \geq -\psi \) and \( \mathcal{D}_t \phi + H^*(\nabla_x \phi, \frac{1}{2} \nabla^2_x \phi) \leq 0. \tag{20} \)

(ii) **PPDE characterisation:** Suppose that \((\psi, H)\) satisfies Assumption 3.6, then

\[
J_{\psi, H}(t, \rho_t) = \int_{\Omega_t} \phi_{\psi, H}(t, \cdot) \, d\rho_t,
\]

where \( \phi_{\psi, H} \) is the viscosity solution to the following path dependent PDE.

\[
\phi(1, \cdot) = -\psi \quad \text{and} \quad \mathcal{D}_t \phi + H^*(\nabla_x \phi, \frac{1}{2} \nabla^2_x \phi) = 0. \tag{21}
\]

Moreover, if \( \phi_{\psi, H} \in C^{1,2}_t(\Lambda) \), then the maximising probability measure \( P \) in (18) is characterised on \([t, 1]\) by

\[
(\alpha^P, \beta^P) = \nabla H^*(\nabla_x \phi_{\psi, H}, \frac{1}{2} \nabla^2_x \phi_{\psi, H}). \tag{22}
\]

(iii) **Localization:** Fix \( t \in [0, 1], \psi \in C_b(\Omega) \). Suppose that \( H \) also satisfies Assumption 3.8. Define the \( \sigma \)-algebra \( \mathcal{G}_t \) as in Definition 3.9. Then \( \mathcal{G}_t \subseteq \mathcal{F}_t \), and the map \( J(t, \cdot) : \Omega_t \to \mathbb{R} \) defined by

\[
J(t, x) := J_{\psi, H}(t, \delta(\omega, x = x))
\]

is \( \mathcal{G}_t \)-measurable. Moreover, if \((\psi, H)\) satisfies Assumption 3.6 and \( \phi_{\psi, H} \in C^{1,2}_t(\Lambda) \), then \( (\alpha^P, \beta^P) \) from (22) is also \( \mathcal{G}_t \)-measurable.

To avoid burdening readers with notation, the proof of the main duality result will focus on the case \( t = 0 \). In other words, the class of optimal transport problem starting at time 0 with initial distribution \( \rho_0 \). The general case can be dealt with using similar arguments.
3.2 Characterising suitable measures

The function $F^*$ was used to penalise measures outside of $\mathcal{N}$, a constraint of the problem. In this subsection, we aim to find a suitable function that penalises measures outside of $\mathcal{P}_0^1(\rho_0)$.

A key step in our argument is to extend measures in $\mathcal{M}(\Omega)$ to measures on the stopped paths $\mathcal{M}(\Lambda)$, then to utilise Fenchel’s duality theorem \[22\] on those measures. The following lemma provides a condition for the identification of measures in $\mathcal{P}_0^1(\rho_0)$ as well as a suitable corresponding measure in $\mathcal{M}(\Lambda)$.

**Lemma 3.2.** Let $\rho_0$ be a probability measure on $\Omega_0$. Suppose that $\mu \in \mathcal{M}_+(\Omega)$ and induces the measure $\hat{\mu} \in \mathcal{M}_+(\Lambda)$ via

$$\hat{\mu}(E) = \int_{\Omega} \int_0^1 1((t, \omega, \lambda) \in E) \, dt \, d\mu, \quad \forall E \subset \Lambda.$$  

Let $\nu \in \mathcal{M}_+(\Lambda)$ and $(\alpha, \beta) \in L^1(\Lambda, \nu, \mathbb{R}^d \times S^d_+)$. Moreover, assume $\alpha, \beta \leq 0$.

(i) The equality

$$\int_\Omega \phi(1, \cdot) \, d\mu - \int_{\Omega_b} \phi(0, \cdot) \, d\rho_0 = \int_\Lambda D_t \phi + \alpha \cdot \nabla_x \phi + \frac{1}{2} \beta : \nabla^2_x \phi \, d\nu$$  

holds for all $\phi \in C^{1,2}_0(\Lambda)$ if and only if $\hat{\mu} = \nu$, $\mu \in \mathcal{P}_0^1(\rho_0)$ and $\mu$ is characterised by $(\alpha, \beta)$.

(ii) The previous result can be strengthened as follows. Fix $\epsilon > 0$, the inequality

$$\int_\Omega \phi(1, \cdot) \, d\mu - \int_{\Omega_b} \phi(0, \cdot) \, d\rho_0 \leq \int_\Lambda D_t \phi + \alpha \cdot \nabla_x \phi + \frac{1}{2} \beta : \nabla^2_x \phi \, d\nu$$  

holds for all $\phi \in C^{1,2}_0(\Lambda)$ satisfying $\phi \geq -\epsilon$ if and only if $\hat{\mu} = \nu$, $\mu \in \mathcal{P}_0^1(\rho_0)$ and $\mu$ is characterised by $(\alpha, \beta)$.

**Proof.** See Appendix.  

3.3 Duality

Recall that, at time $t = 0$, the saddle point problem is given by

$$\inf_{P \in \mathcal{P}_0(\rho_0)} \sup_{\psi \in \mathcal{N}} F(\psi) - \mathbb{E}^P \psi + \mathbb{E}^P \int_0^1 H(s, X_{\Lambda_s}, v^P_s) \, ds.$$  

The aim is to establish duality (swapping the infimum and the supremum) and then characterise the solution in terms of PPDEs.

**Theorem 3.3.** Let $\mathcal{N} \subset \mathcal{P}$ be a convex set that is closed under the weak topology and define $F : C_b(\Omega) \to \mathbb{R}$ by $F(\psi) = \sup_{\mu \in \mathcal{N}} \int_\Omega \psi \, d\mu$. Let $H : \Lambda \times U \to \mathbb{R} \cup \{+\infty\}$ satisfy Assumption \[22\]. Let $F^*$ and $H^*(t, \omega, \cdot)$ be the convex conjugates of $F$ and $H(t, \omega, \cdot)$. Moreover, assume there exists a convex set $\mathcal{N}^* \subseteq C_b(\Omega)$ such that

$$F^*(\mu) = \sup_{\psi \in \mathcal{N}^*} \int_\Omega \psi \, d\mu - F(\psi).$$  

Define

$$V := \inf_{P \in \mathcal{P}_0(\rho_0)} F^*(P) + \mathbb{E}^P \int_0^1 H(\alpha^P_t, \beta^P_t) \, dt,$$  

and

$$V := \sup_{\psi \in \mathcal{N}^*, \phi \in C^{1,2}_0(\Lambda)} -F(\psi) - \int_{\Omega_b} \phi(0, \cdot) \, d\rho_0,$$  

subject to

$$\phi(1, \cdot) \geq -\psi \quad \text{and} \quad D_t \phi + H^* \left( \nabla_x \phi, \frac{1}{2} \nabla^2_x \phi \right) \leq 0.$$  

If $V \leq 0$, then $\mathcal{N}^*$ is closed and the dual problem is well-posed.
Then $V = \mathcal{V}$. Moreover, if $V < +\infty$, then the infimum in (26) is attained.

Proof. For convenience, let us write $\phi_\epsilon = \phi(t, \cdot)$. Recall that $\hat{P} \in \mathcal{M}(\Lambda)$ is induced by $P$ via $d\hat{P} = dt \times dP$. Under Assumption 2.1 it suffices to only consider the set of probability measures $P \in \mathcal{P}_d(\rho_0)$ in (26).

The direction $V \geq \mathcal{V}$ can be easily shown in the following way. Applying Fubini’s theorem and Lemma 3.2 (i), we have

$$ V = \inf_{P \in \mathcal{P}_d(\rho_0)} F^*(\hat{P}) + \int_{\Lambda} H(\nu^\Lambda) d\hat{P}. $$

Then $V \geq \mathcal{V}$. Throughout the remainder of the proof, let $\epsilon > 0$ be a fixed constant. First of all, since $F$ satisfies $F(\psi + \kappa) = F(\psi) + \kappa$ for $\kappa \in \mathbb{R}$, we note that for all $P \in \mathcal{P}$,

$$ F^*(\hat{P}) = \sup_{\psi \in \mathcal{N}^*, \psi \geq \gamma} \int_{\Omega} \psi d\hat{P} - F(\psi), $$

where $\mathcal{N}^* := \mathcal{N}^* + \mathbb{R} = \{\psi + \kappa : \psi \in \mathcal{N}, \kappa \in \mathbb{R}\}$.

Applying (29), we have

$$ V + 2\epsilon = \inf_{P \in \mathcal{P}_d(\rho_0)} F^*(\hat{P}) + \int_{\Omega} \epsilon d\hat{P} + \int_{\Lambda} (H(\alpha^\Lambda, \beta^\Lambda) + \epsilon) d\hat{P}. $$

### References

- Lemma 3.2 (i)
- Assumption 2.1
- Fubini’s theorem
- Lemma 3.2 (i)
- $V = \mathcal{V}$
Applying Fubini’s theorem and Lemma 3.2 (ii), we have

\[
V + 2\epsilon = \inf_{\mu \in M_+(\Omega), \nu \in M_+(\Lambda)} \sup_{(\alpha, \beta) \in L^1(\Lambda, \nu, \mathbb{R} \times \mathbb{R}^d)} -F(\psi) - \int_{\Omega_0} \phi_0 d\rho_0 + \int_{\Omega} (\psi + \phi_1 + \epsilon) d\mu
\]

\[
- \int_{\Lambda} (D_1 \phi + \alpha \cdot \nabla_x \phi + \frac{1}{2} \beta : \nabla^2_x \phi) d\nu + \int_{\Lambda} (H(\alpha, \beta) + \epsilon) d\nu
\]

\[
= \inf_{\mu \in M_+(\Omega), \nu \in M_+(\Lambda)} \sup_{(\alpha, \beta) \in L^1(\Lambda, \nu, \mathbb{R} \times \mathbb{R}^d)} -F(\psi) - \int_{\Omega_0} \phi_0 d\rho_0 + \int_{\Omega} (\epsilon - \varphi) d\mu
\]

\[
- \int_{\Lambda} (D_1 \phi + \alpha \cdot \nabla_x \phi + \frac{1}{2} \beta : \nabla^2_x \phi) d\nu + \int_{\Lambda} (H(\alpha, \beta) + \epsilon) d\nu
\]

\[
\leq \inf_{\mu \in M_+(\Omega), \nu \in M_+(\Lambda)} \sup_{(\alpha, \beta) \in L^1(\Lambda, \nu, \mathbb{R} \times \mathbb{R}^d)} -F(\psi) - \int_{\Omega_0} \phi_0 d\rho_0 + \int_{\Omega} (\epsilon - \varphi) d\mu
\]

\[
- \int_{\Lambda} (D_1 \phi + \alpha \cdot \nabla_x \phi + \frac{1}{2} \beta : \nabla^2_x \phi) d\nu + \int_{\Lambda} (H(\alpha, \beta) + \epsilon) d\nu.
\]

Note that we have used the fact that you can approximate, in the sense of \(\xi, \rho, \mu, \nu, \varphi, p, q, r\), any negative function in \(C_0\) from above by \(C_0\). Also at this point we may replace \(N^*\) with \(N^+\) since we can always replace \(\psi\) and \(\phi\) in opposite directions by the same constant without changing the values of \(\phi_1 + \psi\) or \(-F(\psi) - \int_{\Omega_0} \phi_0 d\rho_0\).

Now, introduce the measures \((\tilde{\nu}, \tilde{\varphi}) \in \mathcal{M}(\Lambda, \mathbb{R} \times \mathbb{S}^d)\) via

\[
d\tilde{\nu} = a d\nu, \quad d\tilde{\varphi} = \beta d\nu.
\]

So that we can write

\[
V + 2\epsilon \leq \inf_{(\xi, \rho, \mu, \nu, \tilde{\nu}, \tilde{\varphi}) \in A} \sup_{(\psi, \varphi, p, q, r) \in B} -F(\psi) + \int_{\Omega} \epsilon d\mu + \int_{\Lambda} \left( H \left( \frac{d\nu}{d\tilde{\varphi}}, \frac{d\tilde{\nu}}{d\tilde{\varphi}} \right) + \epsilon \right) d\nu
\]

\[
- \left( ((\xi, \rho, \mu, \nu, \tilde{\nu}, \tilde{\varphi}), (\psi, \varphi, p, q, r)) \right).
\]

where

\[A := \{ (\xi, \rho, \mu, \nu, \tilde{\nu}, \tilde{\varphi}) \in C_0(\Omega)^* \times C_0(\Omega_0)^* \times \mathcal{M}(\Omega) \times \mathcal{M}(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d) : \]

\[\xi = 0, \rho = \rho_0, \mu \geq 0, \nu \geq 0, (\tilde{\nu}, \tilde{\varphi}) \ll \nu \},\]

\[B := \{ (\psi, \varphi, p, q, r) \in C_0(\Omega) \times C_0(\Omega_0) \times C_0(\Omega) \times C_0(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d) : \]

\[\psi \in N^*, \exists \phi \in C_0^1(\Lambda) \text{ s.t. } \varphi = \phi_0, \varphi \geq -(\phi_1 + \psi), (p, q, r) = (D_1, \nabla_x, \frac{1}{2} \nabla^2_x) \phi \} \]

and the inner product is defined by

\[
\langle (\xi, \rho, \mu, \nu, \tilde{\nu}, \tilde{\varphi}), (\psi, \varphi, p, q, r) \rangle := \int_{\Omega} (\psi d\xi + \varphi d\mu) + \int_{\Omega_0} \varphi d\rho + \int_{\Lambda} (p d\nu + q \cdot d\tilde{\nu} + r : d\tilde{\varphi}).
\]

Note that both \(A\) and \(B\) are convex sets.

Next, define the convex function \(a : C_0(\Omega) \times C_0(\Omega_0) \times C_0(\Omega) \times C_0(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d) \to \mathbb{R}\) and its convex conjugate \(a^* : C_0(\Omega)^* \times C_0(\Omega_0)^* \times \mathcal{M}(\Omega) \times \mathcal{M}(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d) \to \mathbb{R}\) according
to Lemma A.1. They are given by the following expressions:

$$a(\psi, \bar{\varphi}, \varphi, p, q, r) := \begin{cases} \int_{\Omega_b} \bar{\varphi} \, d\rho_0, & \text{if } \varphi \leq \epsilon \text{ and } p + H^*(q, r) \leq \epsilon, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$a^*(\xi, \rho, \mu, \nu, \bar{\nu}, \bar{\nu}) := \begin{cases} \int_{\Omega} \epsilon \, d\mu + \int_{\Lambda} \left( H \left( \frac{d\nu}{d\nu'} \right) + \epsilon \right) \, d\nu, & \text{if } (\xi, \rho, \mu, \nu, \bar{\nu}, \bar{\nu}) \in \mathcal{A}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Furthermore define the concave function $b : C_b(\Omega) \times C_b(\Omega_b) \times C_0(\Omega) \times C_0(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d) \to \mathbb{R}$ and its concave conjugate $b^* : C_b(\Omega)^* \times C_b(\Omega_b)^* \times M(\Omega) \times M(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d) \to \mathbb{R}$ in the following way

$$b(\psi, \bar{\varphi}, \varphi, p, q, r) := \begin{cases} -F(\psi), & \text{if } (\psi, \bar{\varphi}, \varphi, p, q, r) \in \mathcal{B}, \\ -\infty, & \text{otherwise,} \end{cases}$$

$$b_*(\xi, \rho, \mu, \nu, \bar{\nu}, \bar{\nu}) := \inf_{(\xi, \rho, \mu, \nu, \bar{\nu}, \bar{\nu}) \in \mathcal{B}} \left( \langle \xi, \rho, \mu, \nu, \bar{\nu}, (\psi, \bar{\varphi}, \varphi, p, q, r) \rangle + F(\psi) \right).$$

Note that we do not need to compute $\beta_*$ explicitly.

Hence $V$ can be bounded above by

$$V + 2\epsilon \leq \inf_{(\xi, \rho, \mu, \nu, \bar{\nu}, \bar{\nu}) \in \mathcal{B}} \sup_{(\psi, \bar{\varphi}, \varphi, p, q, r)} a^*(\xi, \rho, \mu, \nu, \bar{\nu}, \bar{\nu}) + b(\psi, \bar{\varphi}, \varphi, p, q, r)$$

$$= \sup_{(\psi, \bar{\varphi}, \varphi, p, q, r) \in \mathcal{B}} -F(\psi) - \int_{\Omega_b} \bar{\varphi} \, d\rho_0, \quad \text{s.t. } \varphi \leq \epsilon \text{ and } p + H^*(q, r) \leq \epsilon,$n

$$= \sup_{\phi \in C_b^1(\Lambda), \phi \in \mathcal{N}^*} -F(\psi) - \int_{\Omega_b} \phi \, d\rho_0,$n

s.t. $-(\phi_1 + \psi) \leq \varphi \leq \epsilon$ for some $\varphi \in C_0(\Omega)$ and $D_t \phi + H^*(\nabla_x \phi, \frac{1}{2} \nabla_z^2 \phi) \leq \epsilon,$

$$\leq \sup_{\phi \in C_b^1(\Lambda), \phi \in \mathcal{N}^*} -F(\psi) - \int_{\Omega_b} \phi \, d\rho_0,$n

s.t. $\phi_1 + \psi \geq -\epsilon$ and $D_t \phi + H^*(\nabla_x \phi, \frac{1}{2} \nabla_z^2 \phi) \leq \epsilon.$

Translating $\phi$ by $2\epsilon$ yields

$$V \leq \sup_{\phi \in C_b^1(\Lambda), \phi \in \mathcal{N}^*} -F(\psi) - \int_{\Omega_b} \phi \, d\rho_0,$n

s.t. $\phi_1 + \psi \geq \epsilon$ and $D_t \phi + H^*(\nabla_x \phi, \frac{1}{2} \nabla_z^2 \phi) \leq \epsilon.$

(33)

(34)

For each $\phi \in C_b^1(\Lambda)$ satisfying (34), we can construct $\phi' \in C_b^1(\Lambda)$ via

$$\phi'(t, \cdot) = \phi(t, \cdot) - \int_0^t \left( D_t \phi(s, \cdot) + H^*(\nabla_x \phi(s, \cdot), \frac{1}{2} \nabla_z^2 \phi(s, \cdot)) \right) \, ds.$$
It is easy to check that
\[ \phi_0 = \phi_0, \quad \phi'_0 \geq \phi_1 - \epsilon, \quad (\nabla_x, \nabla^2_x)\phi' = (\nabla_x, \nabla^2_x)\phi, \]
\[ d_t \phi' = d_t \phi - \left( d_t \phi + H^*(\nabla_x \phi, \frac{1}{2} \nabla^2_x \phi) \right)^+ \leq -H^*(\nabla_x \phi', \frac{1}{2} \nabla^2_x \phi'). \]
Therefore
\[ V \leq \sup_{\phi \in C^{1,2}(\Lambda), \psi \in \nu^*} -F(\psi) - \int_{\Omega_0} \phi_0 d\rho_0, \]
\[ \text{s.t. } \phi_1 + \psi \geq 0 \text{ and } d_t \phi + H^*(\nabla_x \phi, \frac{1}{2} \nabla^2_x \phi) \leq 0. \]
Thus we may conclude \( V = V \). The fact that the infimum in (26) is attained if \( V < +\infty \) is a direct consequence of Fenchel’s duality theorem.

The analogous result for time \( t > 0 \) is given below.

**Corollary 3.4.** Fix \( \psi \in C_b(\Omega) \). Define \( J \) by
\[ J_{\psi,H}(t, \rho_t) := \sup_{P \in \mathcal{P}_t(\rho_t)} E^P \psi - E^P \int_1^t H(s, X_{\lambda, s}, \psi_s) ds. \]
Then
\[ J_{\psi,H}(t, \rho_t) = \inf_{\phi \in C^{1,2}(\Lambda)} \int_{\Omega_t} \phi(t, \cdot) d\rho_t, \]
\[ \text{s.t. } \phi(1, \cdot) \geq -\psi \text{ and } d_t \phi + H^*(\nabla_x \phi, \frac{1}{2} \nabla^2_x \phi) \leq 0. \]

**Proof.** By translating \( \psi \) and \( \phi \) by constants, we may assume, without loss of generality, that \( \psi > \epsilon \) for some constant \( \epsilon > 0 \).

Consider the function \( F : C_b(\Omega) \to \mathbb{R} \) and its convex conjugate \( F^* : \mathcal{M}(\Omega) \to \mathbb{R} \) defined by
\[ F(\psi') := \begin{cases} 0, & \psi' \leq \psi, \\ +\infty, & \text{otherwise}, \end{cases} \quad F^*(\mu) := \sup_{\psi' \in C_b(\Omega), \psi' \leq \psi} \int_{\Omega} \psi' d\mu = \begin{cases} -\int_{\Omega} \psi d\mu, & \mu \geq 0, \\ +\infty, & \text{otherwise}, \end{cases} \]
It is easy to see that \( F(0) = 0 \) and for all \( \mu \in \mathcal{P}, \)
\[ F^*(\mu) = \sup_{\psi' \in C_b(\Omega), \psi' \geq \epsilon} \int_{\Omega} \psi' d\mu - F(\psi'). \]
Since these are the only properties of \( F \) and \( F^* \) used in Theorems 3.3, we can apply the theorem to obtain the required result immediately. \( \square \)
3.4 Path dependent PDE

Assumption 3.6. Assume that $\psi$ and $H$ are chosen so that the following path dependent PDE,
\[ \phi(1, \cdot) = -\psi, \quad \mathcal{D}_t \phi + H^* \left( \nabla_x \phi, \frac{1}{2} \nabla^2_x \phi \right) = 0, \]  
has a viscosity solution $\phi_{\psi,H}$ and satisfies the partial comparison principle [11, 12].

Theorem 3.7. Suppose that Assumption 3.6 is satisfied, then
\[ J_{\psi,H}(t, \rho_t) = \sup_{P \in \mathcal{P}^2(\rho_t)} \mathbb{E}^P \psi - \mathbb{E}^P \int_1^t H(\alpha^P_s, \beta^P_s) \, ds = \int_{\Omega_t} \phi_{\psi,H}(t, \cdot) \, d\rho_t, \]  
where $\phi_{\psi,H}$ is the viscosity solution to the following path dependent PDE.
\[ \phi(1, \cdot) = -\psi \quad \text{and} \quad \mathcal{D}_t \phi + H^* \left( \nabla_x \phi, \frac{1}{2} \nabla^2_x \phi \right) = 0. \]

Moreover, if $\phi_{\psi,H} \in C^{1,2}_t(\Lambda)$, then the maximising probability measure $P$ in (40) is characterised on $[t, 1]$ by
\[ (\alpha^P, \beta^P) = \nabla H^* \left( \nabla_x \phi_{\psi,H}, \frac{1}{2} \nabla^2_x \phi_{\psi,H} \right). \]

3.5 Localisation

One particular interesting case is when $\rho_t$ is the Dirac mass $\rho_t = \delta_{t,x} = \delta(X_{\Lambda t} = x_{\Lambda t})$. Then
\[ J_{\psi,H}(t, \delta_{t,x}) = \sup_{P \in \mathcal{P}^2(\delta_{t,x})} \mathbb{E}^P \psi - \mathbb{E}^P \int_1^t H(\alpha^P_s, \beta^P_s) \, ds = \inf_{\phi \in C^{1,2}_t(\Lambda)} \phi(t, x_{\Lambda t}), \]
subject to $\phi(1, \cdot) \geq -\psi$ and $\mathcal{D}_t \phi + H^* \left( \nabla_x \phi, \frac{1}{2} \nabla^2_x \phi \right) \leq 0$. \[(43)\]

For $x \in \Omega_t$, let us use the shorthand $J(t, x) := J_{\phi_H}(t, \delta_{t,x})$.

In general, $J(t, \cdot)$ is a map of $\Omega_t$ and it can be shown to be $\mathcal{F}_t$-measurable. But in many relevant examples, both the payoff function $\psi$ and the cost function $H$ only depends on certain features of the path rather than the entire path, and can often be parametrised by a finite number of state variables. Intuitively, the solution $J(t, \cdot)$ should have a simpler representation in terms of those state variables, rather than the whole path to up to time $t$. In other words, $J(t, \cdot)$ is measurable with respect to a $\sigma$-algebra smaller than $\mathcal{F}_t$. This is useful in practice since it allows us identify the state variables driving the relevant features of the path, and reduces an infinite dimensional PPDE to a finite dimensional PDE.

For this subsection, we impose an additional assumption on the cost function $H$.

Assumption 3.8. The effective domain of $H$ is given by $\text{dom } H = \Lambda \times U$, where $U \subseteq \mathbb{R}^d \times \mathbb{S}_+^d$ is a convex and closed subset. In other words, for all $(t, \omega) \in \Lambda$, $H(t, \omega, \alpha, \beta) < +\infty$ if and only if $(\alpha, \beta) \in U$. Furthermore, $H \in C^0_b(\Lambda \times U)$, so it is bounded within its effective domain.

First, let us introduce some new notations for the set of paths that stay at zero on $[0, t]$,
\[ \Omega^t = \{ \omega \in C([0,1], \mathbb{R}^d) : \omega_x = 0, x \in [0, t] \}, \]
\[ \Lambda^t = \{ (u, \omega_{\Lambda u}) : u \in [t, 1], \omega \in \Omega^t \}. \]
Note that we have the relation \( \Omega = \Omega_t + \Omega' \).

Fix the functions \( \psi \) and \( H \). The idea is to reinterpret them as maps from \( \Omega_t \) to the possible functions on \( \Omega' \). For all \( x \in \Omega_t \), we begin by defining \( \psi_{t,x} \in C_b(\Omega') \) and \( H_{t,x} \in C_b(\Lambda' \times U) \) as follows:

\[
\psi_{t,x}(\omega) = \psi(x + \omega), \quad H_{t,x}(u, \omega, \alpha, \beta) = H(u, x + \omega, \alpha, \beta).
\]  

Both \( C_b(\Omega') \) and \( C_b(\Lambda' \times U) \) are endowed with the norm \( \| \cdot \|_\infty \) and the associated Borel \( \sigma \)-algebras. Next define the maps \( \Psi_t : \Omega_t \to C_b(\Omega') \) and \( \mathcal{H}_t : \Omega_t \to C_b(\Lambda' \times U) \) by

\[
\Psi_t(x) = \psi_{t,x}, \quad \mathcal{H}_t(x) = H_{t,x}.
\]  

Now, we are in the position to describe the candidate \( \sigma \)-algebra for our localisation of \( J(t, \cdot) \).

**Definition 3.9.** Fix \( t \in [0, 1] \), \( \psi \) and \( H \) and assume \( H \) satisfies Assumption 3.8. Define the maps \( \Psi_t \) and \( \mathcal{H}_t \) by (44) and (45). Define \( \mathcal{G}_t \) to be the \( \sigma \)-algebra on \( \Omega_t \) generated by \( (\Psi_t, \mathcal{H}_t) \). In other words \( \mathcal{G}_t = \sigma(\Psi_t) \cap \sigma(\mathcal{H}_t) \).

**Remark 3.10.** In general, \( \{\mathcal{G}_t\}_{t \in [0, 1]} \) is not a filtration.

We claim that \( J(t, \cdot) : \Omega_t \to \mathbb{R} \) is \( \mathcal{G}_t \)-measurable.

**Theorem 3.11.** Suppose Assumption 3.8 holds.

(a) \( \mathcal{G}_t \subseteq \mathcal{F}_t \);

(b) The map \( J(t, \cdot) : \Omega_t \to \mathbb{R} \) is \( \mathcal{G}_t \)-measurable;

(c) If Assumption 3.8 holds, then \( \phi_{\psi, H}(t, \cdot) \) is \( \mathcal{G}_t \)-measurable. Furthermore, if \( \phi_{\psi, H} \in C^{1,2}_t(\Lambda) \), then \( (\alpha, \beta) = \nabla H^t(\nabla_x \phi_{\psi, H}, \frac{1}{2} \nabla_x^2 \phi_{\psi, H}) \) is \( \mathcal{G}_t \)-measurable.

**Proof.** (a) We begin by noting that, for each \( u \in [t, 1] \), \( \omega \in \Omega' \) and \( (\alpha, \beta) \in U \), the map \( \Omega_t \to C_b(\Omega') \times C_b(\Lambda' \times U) \) given by

\[
x \mapsto (\psi_{t,x}(\omega), H_{t,x}(u, x, \alpha, \beta))
\]

is \( \mathcal{F}_t \)-measurable. Thus the \( \mathcal{F}_t \)-measurability of \( (\Psi_t, \mathcal{H}_t) \) follows from the continuity of \( (\psi_{t,x}, H_{t,x}) \) and the separability of \( \Omega' \) and \( \Lambda' \times U \).

(b) First, recall that \( J(t, x) = J_{\psi, H}(t, x) \) depends on \( (\psi, H) \). After a suitable translation, it can be represented as a mapping of the pair \( (\psi_{t,x}, H_{t,x}) \).

\[
J_{\psi, H}(t, x) = \sup_{P_t(\delta_t, \cdot)} E^P \psi - E^P \int_t^1 H(s, X, \alpha, \beta) \, ds
\]

\[
= \sup_{P_t(\delta_t, \cdot)} E^P \psi_{t,x} - E^P \int_t^1 H_{t,x}(s, X, \alpha, \beta) \, ds.
\]

Now consider the map:

\[
\mathcal{J}_t : C_b(\Omega') \times C_b(\Lambda' \times U) \to \mathbb{R},
\]

\[
\mathcal{J}_t(\psi', H') = \sup_{P_t(\delta_t, \cdot)} E^P \psi' - E^P \int_t^1 H'(s, X, \alpha, \beta) \, ds.
\]

The map \( \mathcal{J}_t \) is continuous, since for \( \| \psi' - \psi'' \| \leq \epsilon \),

\[
|\mathcal{J}_t(\psi', H') - \mathcal{J}_t(\psi'', H'')| \leq \sup_{P_t(\delta_t, \cdot)} \left| E^P(\psi' - \psi'') - E^P \int_t^1 (H'(s, X, \alpha, \beta) - H''(s, X, \alpha, \beta)) \, ds \right|
\]

\[
\leq (2 - t) \epsilon.
\]
Therefore the map
\[ J(t, x) = J_t(\Psi_t(x), \mathcal{H}_t(x)) \]
is \( \mathcal{G}_t \)-measurable.

(c) The \( \mathcal{G}_t \)-measurability of \( \phi_{\psi, H} \) follows immediately from Theorem 3.7.

4 Volatility calibration

The main results of this paper can be applied to the problem of calibrating volatility to path dependent derivatives. Suppose that some derivative prices are known, our goal is to find a martingale diffusion for the underlying asset which attains those prices. Conventional calibration methods only work on vanilla products, or European options. Due to the path dependent nature of our results, we can use it to further include path dependent derivatives such as Asian options, barrier options and lookback options.

Let the canonical process \( X \) be the logarithm of the discounted underlying stock price.

Note that the logarithm transform is purely chosen for notational and numerical convenience, and is not at all necessary. We are interested in finding a probability measure \( P \in \mathcal{P}_0 \) characterised by \((\frac{1}{2} \sigma^2, \sigma^2)\) where \( \sigma \) is some \( \mathcal{F} \)-adapted process. In other words, we want \( X \) to be an \((\mathcal{F}, P)\)-semimartingale in the form of
\[ dX_t = -\frac{1}{2} \sigma^2 dt + \sigma dW^P_t. \]

Next let \( G : \Omega \to \mathbb{R}^m \) denote a vector of \( m \) (possibly path dependent) discounted option payoff functions. We further restrict \( \mathcal{P} \) so that the options have known prices \( \mathbb{E}^P G = c \) for some \( c \in \mathbb{R}^m \). It is immediate that this problem is a special case of the general problem we introduced in Section 2, specifically in Example 2.6. In particular, we want to solve
\[ \inf_{P \in \mathcal{P}_0} \mathbb{E}^P \int_0^1 H(\alpha_s^P, \beta_s^P) ds, \quad (46) \]
subject to \( \mathbb{E}^P G = c, \quad (47) \)

where \( H \) is any suitable convex cost function whose effective domain is within the set \( 2\alpha = \beta \).

In the examples of this Section, we consider a cost function of the form
\[ H(\alpha, \beta) = \begin{cases} a(\beta/\bar{\sigma}^2)^p + b(\beta/\bar{\sigma}^2)^{-q} + c, & 2\alpha = \beta, \\ \infty, & 2\alpha \neq \beta, \end{cases} \]
where \( \bar{\sigma} \) is some reference volatility level, \( p, q \) are constants greater than 1, and \( a, b, c \) are constants chosen so that the function minimises at \( \beta = \bar{\sigma}^2 \) with \( \min H = 0 \). The basic idea is to keep \( \beta \) positive and punish any large deviations from \( \bar{\sigma}^2 \).

As mentioned in Example 2.6, the corresponding saddle point problem is
\[ V = \inf_{P \in \mathcal{P}_0} \sup_{\lambda \in \mathbb{R}^m} \lambda \cdot (\mathbb{E}^P G - c) + \mathbb{E}^P \int_0^1 H(\alpha_s^P, \beta_s^P) ds, \quad (48) \]

Applying Theorem 3.1 and assuming sufficient regularity on the payoff functions, the dual formulation of the problem is
\[ V = \mathcal{V} = \sup_{\lambda \in \mathbb{R}^m} -\lambda \cdot c - \phi(0, X_0), \quad (49) \]
where \( \phi \) is the viscosity solution to the PPDE

\[
\phi(1, \cdot) = -\lambda \cdot G \quad \text{and} \quad \mathcal{D}_t \phi + H^* \left( \nabla_x \phi, \frac{1}{2} \nabla^2_x \phi \right) = 0. \tag{50}
\]

Numerically, the difficult part is to solve the PPDE (50) to find \( \phi(0, X_0) \). The key idea is to use the last part of Theorem 3.1 to effectively reduce the dimensionality of \( \phi \) into a manageable size. For many examples, instead of \( \phi \) being dependent on the whole path of \( X \), we can “localise” the solution so that it is Markovian with respect to a few state variables.

The term “localisation” is chosen in reference to the fact that European option prices can always be calibrated to local volatility models. The general abstract result is described in Definition 3.9 and Theorem 3.8. But for most practical cases, it is straightforward to identify the relevant state variables by inspection and they are the familiar variable from standard option pricing techniques. Here are some examples of relevant state variables:

- **European options**: the spot price \( X_t \);
- **Asian options**: the spot price \( X_t \) and the running average \( \frac{1}{t} \int_0^t X_s \, ds \);
- **Barrier options**: the spot price \( X_t \) and the indicator variable \( 1 \{ X_s > B, s \in [0, t] \} \) for lower barriers or \( 1 \{ X_s < B, s \in [0, t] \} \) for upper barriers;
- **Lookback options**: the spot price \( X_t \) and the running minimum \( \min_{s \in [0, t]} X_s \) or running maximum \( \max_{s \in [0, t]} X_s \).

In these cases, the PPDE reduces to a PDE and can be solved via conventional methods.

The supremum in (49) over \( \lambda \in \mathbb{R}^m \) can then be found by a standard optimisation routine. This process can be further aided by numerically computing the gradient of the objective with respect to \( \lambda \) in the following way. Consider the perturbation \( \phi + \epsilon \phi' \). Differentiating yields

\[
\mathcal{D}_t \phi' + \alpha \nabla_x \phi' + \frac{1}{2} \beta \nabla^2_x \phi' = 0. \tag{51}
\]

where \((\alpha, \beta) = \nabla H^* (\nabla_x \phi, \frac{1}{2} \nabla^2_x \phi)\). Hence \( \nabla_\lambda (-\lambda \cdot c - \phi(0, X_0)) = -c - \phi'(0, X_0) \) where \( \phi' \) satisfies \( \phi'(1, \cdot) = -G \) and (51). This also has a nice interpretation, since \( \phi'(0, X_0) = \mathbb{E}^P (-G) \) where \( P \) is characterised by \((\alpha, \beta)\), the gradient is in fact \( \mathbb{E}^P G - c \), or the difference between the option prices given by the current optimisation iteration and the true option prices.

Once the optimal \( \lambda \) and \( \phi \) has been found, the optimal volatility is given by \( \sigma^2 = \beta = 2\alpha \) where \((\alpha, \beta) = \nabla H^* (\nabla_x \phi, \frac{1}{2} \nabla^2_x \phi)\).

**Remark 4.1.** (i) Recall that in our original formulation, \( G \) was required to be a bounded and continuous function. In practice, many options do not have bounded payoffs (e.g., call options). This can be fixed by either turning them into bounded options via arbitrage arguments (e.g., put options via put-call parity), or to truncate the domain at some extremely large value. For option that do not have continuous payoffs (e.g., digital options, barrier options), it is worth mentioning that our main results can be extended to many piece-wise continuous functions as mentioned in Remark 2.8.

(ii) By increasing the dimension of the canonical process to include other features such as the variance process, the same technique can be used to calibrate local stochastic volatility (LSV) models.

(iii) Our results here are limited to non-callable products, hence excluding the calibration of Bermudan and American options. Callable products requires incorporating stopping times into the duality spaces, which requires additional techniques beyond the scope of the current work. This problem is under current research and will be addressed in a forthcoming paper.

In the following subsections, we will demonstrate a few numerical examples.
4.1 European options

European options have payoffs of the form $G(X_t)$ where each option depends on the value of the underlying at a fixed maturity $t$. In this case, the function $\phi$ and the optimal volatility $\sigma$ only depend on the state variable $t$ and $X_t$. In other words, we recover a local volatility model. This is consistent with classical approaches such as Dupire’s formula [10]. In some sense, local volatility models are the “simplest” models that can calibrate to all European products. Unlike Dupire’s formula, our approach does not require the interpolation of option prices between discrete strikes and maturities. The functional derivatives in the PPDE simply reduces to the usual partial derivatives,

$$\partial_t \phi + H^* \left( \partial_x \phi, \frac{1}{2} \partial^2_x \phi \right) = 0.$$ (52)

This case is similar to the approach used in [2] in the context of calibrating an entropy minimising local volatility.

Figure 1 shows an example of a volatility calibrated to European options at all strikes and four different maturities.

![Volatility surface calibrated to European put options at all strikes and four different maturities](image)

Figure 1: Volatility surface calibrated to European put options at all strikes and four different maturities

4.2 Barrier options

Formally speaking, a barrier is a closed subset $B \subset [0,1] \times \mathbb{R}$ whose complement is a connected region containing $(0,X_0)$. A barrier product depends on the value of the underlying $X_t$ at some fixed maturity $t$, as well as the indicator variable $1_t := 1(X_s \in B, s \in [0,t])$, checking whether the path of the underlying has hit the barrier. When calibrating to a collection of barrier products with a single fixed barrier, the required state variables are $t, X_t$ and $1_t$. Then the function $\phi$ can be effectively split into two functions, $\phi_0(t, x)$ and $\phi_1(t, x)$, corresponding
to $t = 0$ and $t = 1$, respectively. The PDE is then given by

$$\partial_t \phi_1 + H^* \left( \partial_x \phi_1, \frac{1}{2} \partial_x^2 \phi_1 \right) = 0, \quad (t, x) \in [0, 1] \times \mathbb{R},$$

(53)

$$\partial_t \phi_0 + H^* \left( \partial_x \phi_0, \frac{1}{2} \partial_x^2 \phi_0 \right) = 0, \quad (t, x) \notin \mathbb{R},$$

(54)

$$\phi_0 = \phi_1, \quad (t, x) \in \partial \mathbb{R}.$$  

(55)

Similarly, the optimal volatility will be switching between two local volatilities $\sigma_0(t, x)$ and $\sigma_1(t, x)$, conditioned on whether the underlying has hit $B$. If we calibrate to options with $l$ distinct barriers, then a similar approach applies but with $l$ indicator variables. In this case $\phi$ and $\sigma$ will be split into $2^l$ functions, conditioning on the subset of the barriers that has been reached. Although the number $2^l$ can be reduced in many cases (e.g., if some barriers are nested) if some subsets are not reachable.
As an example, let us consider barrier products with respect to a continuous lower barrier \( \{ x \leq b \} \) where \( b < X_0 \) is a constant. In particular, we will be calibrating to all down-and-in and down-and-out puts at all strikes and four different maturities. The top half of Figure 2 shows the calibrated volatility function \( \sigma_0 \) (before hitting the barrier) and the bottom half shows \( \sigma_1 \) (after hitting the barrier). Even though \( \sigma_0 \) is only defined for \( x \geq b \), for the purpose of visualisation, we set \( \sigma_0 = \sigma_1 \) for \( x < b \).

### 4.3 Lookback options

Lookback products have payoff that depends on \( X_t \) as well as either of the extrema \( \max_{s \in [0,t]} X_s \) or \( \min_{s \in [0,t]} X_s \). Here we will focus on cases involving the minimum \( Y_t := \)
min_{x \in [0,t]} X_s. The state variables for \( \phi \) and \( \sigma \) will be \( t, X_t \) and \( Y_t \). The PDE is then given by

\[\partial_t \phi + H^* \left( \partial_x \phi, \frac{1}{2} \partial_x^2 \phi \right) = 0, \quad x \geq y, \quad (56)\]

\[\partial_y \phi = 0, \quad x = y. \quad (57)\]

Note that European options and barrier options with lower barriers are also special cases of lookback products.

In Figure 3 we show the resulting volatility function \( \sigma(t,x,y) \) calibrated to European puts, all lower barrier down-and-out puts and fixed strike lookback puts at all strikes and four different maturities. The top half of the figure shows cross sections at different values of \( t \) while the bottom half shows cross sections at different values of \( y \). Even though \( \sigma \) is only defined for \( x \geq y \), for the purpose of visualisation, we set \( \sigma(t,x,y) = \sigma(t,x,x) \) for \( x < y \).

Figure 4: The top half shows a volatility function \( \sigma(t,x,y) \) calibrated to European options only. The bottom half is calibrated to European options and barrier options at two different barriers.
To further demonstrate the effect of “localisation”, we repeat the same computation but removing some of the options. In the first test, we only calibrate to European options, while in the second test we calibrate to European options and barrier options at two different barriers $b_1 < b_2 < X_0$. The results are shown in Figure [4]. When only European options are used, $\sigma$ only depends on $(t,x)$ but not $y$. In the cases where some barrier options are added, the dependence of $\sigma$ on $y$ can be divided into three regions, $y > b_2$, $b_2 \geq y > b_1$ and $b_1 \geq y$, corresponding to the number of barriers the underlying has hit. This behaviour is consistent with our localisation results.

A Appendix

A.1 Proof of Lemma 3.2

Proof of Lemma 3.2

(i) The “if” direction follows immediately from the functional Itô formula, so we will focus only on the “only if” direction. First note that we can translate $\phi$ by any $\phi(0, \cdot) \in C_0[\Omega_0]$ without altering the right hand side. This yields

$$\int_{\Omega_0} \phi(0, \cdot) \, d(\mu \circ X_0^{-1} - \rho_0) = 0$$

for all $\phi(0, \cdot) \in C_0[\Omega_0]$. Therefore $\mu$ is a probability measure with $\mu \circ X_0^{-1} = \rho_0$.

Let $f \in C_0(\Lambda)$ be any function. Consider

$$\phi(t, X) = \int_0^t f(s, X) \, ds.$$ 

Then we have $\nabla_x \phi = \nabla_x \phi = 0$ and $D_t \phi = f$. Applying Fubini’s theorem, we obtain

$$\int_\Lambda f(\nu - d\hat{\mu}) = 0,$$

Since $f$ is an arbitrary continuous function on $\Lambda$, this implies that $\nu = \hat{\mu}$ and we can rewrite our condition as

$$E^\mu(\phi(1, X) - \phi(0, X)) = E^\mu \int_0^1 D_t \phi + \alpha_t \cdot \nabla_x \phi + \frac{1}{2} \beta_t : \nabla_x^2 \phi \, dt. \quad (58)$$

Fix $u \in [0, 1]$ and let $I^n \in C^1([0, 1])$ be a sequence of increasing functions with $I^n(t) = 0$ for $t \in [0, u]$ and $I^n(t) = 1$ for $t \in [u + 1/n, 1]$. Define the function $P : \mathbb{R}^d \rightarrow \mathbb{R}$ by $P(x) = \sqrt{1 + x \cdot x} \geq |x|_\infty$ and let $P^n \in C^2_b(\mathbb{R}^d)$ be a sequence of positive, bounded functions with uniformly bounded derivatives such that $P^n(x) = P(x)$ if $|x|_\infty \leq n$ and $P^n(x) = 0$ if $|x|_\infty \geq 2n$. Let $Q^n \in C_0(\Omega_u, [0, 1])$ be a sequence of cutoff functions with $Q^n(x) = 1$ if $|x|_\infty \leq 2n$. Set

$$\phi^n(t, X) = P^n(X_t - X_u)Q^n(\|X_{\Lambda_t} - X_{\Lambda_u}\|_\infty)I^n(t)Q^n(\|X_{\Lambda_u}\|_\infty).$$

It is clear that $\phi^n \in C^{1,2}_0(\Lambda)$ since $\|X\|_\infty \leq \|X_{\Lambda_u}\|_\infty + \|X - X_{\Lambda_u}\|_\infty$. Then the integral of $D_t \phi^n$ can be bounded by

$$\left| \int_0^1 D_t \phi^n \, dt \right| \leq \int_u^{u + \frac{1}{2}} P^n(X_t - X_u) \partial_t I^n(t) \, dt \quad (59)$$

$$\leq \max_{t \in [u, u + \frac{1}{2}]} |P^n(X_t - X_u)| \int_u^{u + \frac{1}{2}} \partial_t I^n(t) \, dt \quad (60)$$

$$= \max_{t \in [u, u + \frac{1}{2}]} |P^n(X_t - X_u)|, \quad (61)$$

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which vanishes as \( n \to +\infty \). The space derivatives of \( \phi^n \) are uniformly bounded and satisfy
\[
\lim_{n \to +\infty} \nabla_x \phi^n = \mathds{1}(t > u), \quad \lim_{n \to +\infty} \nabla^2_x \phi^n = 0.
\]

Using Fatou’s lemma and the integrability of \((\alpha, \beta)\), we have
\[
\mathbb{E}^\mu|X_1 - X_u|_\infty \leq \mathbb{E}^\mu P(X_1 - X_u) \leq \lim_{n \to +\infty} \mathbb{E}^\mu P^n(X_1 - X_0) < +\infty,
\]
Hence \( X_1 - X_u \) is \( \mu \)-integrable for all \( u \in [0, 1] \).

Next, fix \( u \in [0, 1] \) and let \( K^n \in C^2_b(\mathbb{R}^d, \mathbb{R}^d) \) be a sequence of bounded functions with uniformly bounded derivatives satisfying \( |K^n(x)| < |x| \), \( K^n(x) = x \) if \( |x| \leq n \) and \( K^n(x) = 0 \) if \( |x| \geq 2n \). Let \( g \in C_0(\Omega_u, \mathbb{R}^d) \) be an arbitrary function and consider
\[
\phi^n(t, X) = K^n(X_t - X_u) \cdot g(X \wedge u) I^n(t) Q^n(\|X \wedge t - X \wedge u\|_\infty).
\]

Using arguments similar to before, \( \phi^n \in C^{1, 2}_b(\Lambda) \). Furthermore, the space derivatives of \( \phi^n \) are uniformly bounded and we have
\[
\lim_{n \to +\infty} D_t \phi^n = 0, \quad \lim_{n \to +\infty} \nabla_x \phi^n = g(X \wedge u) \mathds{1}(t > u), \quad \lim_{n \to +\infty} \nabla^2_x \phi^n = 0.
\]

Thus by integrability of \( \alpha \) and \( X_1 - X_u \) as well as the dominated convergence theorem, as \( n \to +\infty \),
\[
\mathbb{E}^\mu((X_1 - X_u) \cdot g(X \wedge u)) = \mathbb{E}^\mu\left( \int_u^1 \alpha_t \ dt \cdot g(X \wedge u) \right).
\]
Recall that \( g \) is arbitrary, so this implies
\[
\mathbb{E}^\mu(X_1 - X_u - \int_u^1 \alpha_t \ dt \mid \mathcal{F}_u) = 0.
\]

From the integrability of \( \alpha \) and \( X_1 - X_u \), and the fact that the above equality holds for all \( u \in [0, 1] \), \( M := X - \int_0^t \alpha_s \ ds \) must be a continuous \((\mathbb{F}, \mu)\)-martingale and \( X \) is a \((\mathbb{F}, \mu)\)-semimartingale characterised by \((\alpha, \beta)\) for some \( \tilde{\beta} \).

Applying the functional Ito’s formula, our condition reduces to
\[
\mathbb{E}^\mu \int_0^1 (\beta - \tilde{\beta}) : \nabla^2_x \phi \ dt = 0.
\]

Fix \( u \in [0, 1] \), let \( h \in C_0(\Omega_u, \mathcal{S}_d^2) \) and consider
\[
\phi^n(t, X) = K^n(X_t - X_u) I^n(t) Q^n(\|X \wedge t - X \wedge u\|_\infty),
\]
where \( K^n \) and \( I^n \) are defined as above. Once again, we have \( \phi^n \in C^{1, 2}_b(\Lambda) \). In particular, \( \nabla^2_x \phi^n \) takes value in \( \mathcal{S}_d^2 \), is uniformly bounded and satisfies
\[
\lim_{n \to +\infty} \nabla^2_x \phi^n(t, X) = h(X \wedge u) \mathds{1}(t > u).
\]

By the integrability of \( \beta \), Fatou’s lemma and the dominated convergence theorem, we have
\[
\mathbb{E}^\mu \int_u^1 \tilde{\beta} : h(X \wedge u) \ dt \leq \lim_{n \to +\infty} \mathbb{E}^\mu \int_0^1 \tilde{\beta} : \nabla^2_x \phi^n \ dt = \lim_{n \to +\infty} \mathbb{E}^\mu \int_0^1 \beta : h(X \wedge u) \ dt < +\infty,
\]

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Thus \( \mathbb{E}^\mu \int_0^1 \tilde{\beta} : h(X_{\lambda t}) \, dt < +\infty \) and we may apply the dominated convergence theorem again to change the first inequality to an equality, which yields
\[
\mathbb{E}^\mu \int_0^1 (\beta - \tilde{\beta}) : h(X_{\lambda t}) \, dt = 0,
\]
for all \( h \in C_0(\Omega, \mathbb{S}_+^d) \). Recall the notation \( M = X - \int_0^1 \alpha_t \, dt \) and that \( \tilde{\beta} = \langle M \rangle \). Thus
\[
\mathbb{E}^\mu ((M_1 - M_a)(M_1 - M_a)^\top | \mathcal{F}_a) = \mathbb{E}^\mu \left( \int_0^1 \tilde{\beta}_t \, dt \right) | \mathcal{F}_a) = \mathbb{E}^\mu \left( \int_0^1 \beta_t \, dt \right) | \mathcal{F}_a),
\]
and so \( \beta = \langle M \rangle = \tilde{\beta}, \mu \text{-a.s.} \), completing the proof.

(ii) For the strengthened version, by setting \( \phi = \epsilon \) and \( \phi = -\epsilon \), we quickly obtain that \( \mu(\Omega) = \rho_0(\Omega_0) = 1 \). Thus we can ignore the constraint \( \phi \geq -\epsilon \) since \( \phi \) can be translated by a constant without altering the result. Finally the inequality can be changed to an equality by substituting \( -\phi \) and the required result is thus reduced to part (i). \( \square \)

### A.2 Lemma [A.1]

**Lemma A.1.** Define \( a : C_b(\Omega) \times C_b(\Omega_0) \times C_0(\Omega) \times C_0(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^d) \to \mathbb{R} \) by
\[
a(\psi, \varphi, p, q, r) := \int_{\Omega_0} \varphi \, d\rho_0, \quad \text{if } \varphi \leq \epsilon \text{ and } p + H^*(q, r) \leq \epsilon,
\]
\[
+\infty, \quad \text{otherwise.}
\]

Its convex conjugate \( a^* : C_b(\Omega)^* \times C_b(\Omega_0)^* \times \mathcal{M}(\Omega) \times \mathcal{M}(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^d) \to \mathbb{R} \) is given by
\[
a^*(\xi, \rho, \mu, \nu, \tilde{\nu}) := \left\{ \begin{array}{ll}
\int_{\Omega} \epsilon \, d\mu + \int_{\Lambda} \left( H \left( \frac{d\nu}{d\tilde{\nu}} \right) + \epsilon \right) \, d\nu, & \text{if } (\xi, \rho, \mu, \nu, \tilde{\nu}, \tilde{\nu}) \in \mathcal{A},
+\infty, & \text{otherwise},
\end{array} \right.
\]

where
\[
\mathcal{A} := \{ (\xi, \rho, \mu, \nu, \tilde{\nu}, \tilde{\nu}) \in C_b(\Omega)^* \times C_b(\Omega_0)^* \times \mathcal{M}(\Omega) \times \mathcal{M}(\Lambda, \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^d) : \xi = 0, \ \rho = \rho_0, \ \mu \geq 0, \ \nu \geq 0, \ (\tilde{\nu}, \tilde{\nu}) \ll \nu \}.
\]

**Proof.** Throughout the proof, we will use the fact that \( C_b \) is dense in \( L^1 \). Let us identify the cases where \( a^* < +\infty \). Using the definition of convex conjugates, we have
\[
a^*(\xi, \rho, \mu, \nu, \tilde{\nu}, \tilde{\nu}) = \sup_{(\psi, \varphi, p, q, r), \varphi \leq \epsilon, p + H^*(q, r) \leq \epsilon} \int_{\Omega} (\psi \, d\mu + \varphi \, d\mu) + \int_{\Omega_0} \varphi \, d\rho_0
\]
\[
+ \int_{\Lambda} (p \, d\nu + q \cdot d\tilde{\nu} + r : d\tilde{\nu}).
\]

If \( a^* < +\infty \), then \( \xi = 0, \ \rho = \rho_0, \ \mu \geq 0 \) and \( \nu \geq 0 \). To see why the \( \mu \geq 0 \) is true, suppose \( \mu(E) < 0 \) for some measurable set \( E \subset \Omega \). Then there exists a sequence of negative functions \( \varphi_n \in C_0(\Omega) \) that converge to \( -1 \) in \( L^1(\Omega, \mu) \). By scaling \( \varphi_n \) arbitrarily and adding them to \( \varphi \), the function \( a^* \) becomes unbounded. A similar argument shows that \( \nu \geq 0 \). So our function reduces to
\[
a^*(\xi, \rho, \mu, \nu, \tilde{\nu}, \tilde{\nu}) = \int_{\Omega} \epsilon \, d\mu + \sup_{p + H^*(q, r) \leq \epsilon} \int_{\Lambda} p \, d\nu + q \cdot d\tilde{\nu} + r : d\tilde{\nu}.
\]
Next, since the function is linear in \((p, q, r)\), if \(a^*\) is finite, then the supremum must occur at the boundary

\[
a^*(\xi, \rho, \mu, \nu, \bar{\nu}, \tilde{\nu}) = \int_\Omega \epsilon \, d\mu + \sup_{p + H^*(q, r) = \epsilon} \int_\Lambda p \, d\nu + q \cdot d\bar{\nu} + r : d\tilde{\nu}
= \int_\Omega \epsilon \, d\mu + \int_\Lambda \epsilon \, d\nu + \sup_{(q, r)} \int_\Lambda q \cdot d\bar{\nu} + r : d\tilde{\nu} - H^*(q, r) \, d\nu.
\]

We claim that it is necessary to have \((\bar{\nu}, \tilde{\nu}) \ll \nu\). Suppose that there exists a measurable set \(E\) such that \((\bar{\nu}, \tilde{\nu})(E) \neq 0\) but \(\nu(E) = 0\). Once again let us construct a sequence of continuous function in \(C_0(\Lambda)\) converging to \(\mathbb{1}(E)\) in \(L^1(\Lambda)\) and add multiples of it (depending on the sign of \(\nu(E)\)) to \((q, r)\), which would allow \(a^*\) to grow arbitrarily. Thus we may write and bound \(a^*\) in the following way,

\[
a^*(\xi, \rho, \mu, \nu, \bar{\nu}, \tilde{\nu}) = \int_\Omega \epsilon \, d\mu + \int_\Lambda \epsilon \, d\nu + \sup_{(q, r)} \int_\Lambda \left( q \cdot \frac{d\bar{\nu}}{d\nu} + r : \frac{d\tilde{\nu}}{d\nu} - H^*(q, r) \right) d\nu
\leq \int_\Omega \epsilon \, d\mu + \int_\Lambda \epsilon \, d\nu + \int_\Lambda \sup_{(q, r)} \left( q \cdot \frac{d\bar{\nu}}{d\nu} + r : \frac{d\tilde{\nu}}{d\nu} - H^*(q, r) \right) d\nu
= \int_\Omega \epsilon \, d\mu + \int_\Lambda \epsilon \, d\nu + \int_\Lambda H \left( \frac{d\bar{\nu}}{d\nu} \frac{d\tilde{\nu}}{d\nu} \right) d\nu.
\]

Note that we have used the lower-semicontinuity of \(H\). Equality can be shown by choosing \((q, r)_n\) to be a sequence of continuous functions converging to \(\nabla H(\frac{d\bar{\nu}}{d\nu}, \frac{d\tilde{\nu}}{d\nu})\), then applying the dominated convergence theorem and the fact that \(H^*\) is continuous in \(\text{dom}(H^*)\).

Finally, we see that the conditions \(\xi = 0\), \(\rho = \rho_0\), \(\mu \geq 0\), \(\nu \geq 0\) and \((\bar{\nu}, \tilde{\nu}) \ll \nu\) are necessary for \(a^* < +\infty\). Thus the claim is proven.

\[\square\]

References


