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EXPONENTIAL SMOOTHING FOR INVENTORY CONTROL:

MEANS AND VARIANCES OF LEAD-TIME DEMAND

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ABSTRACT

Exponential smoothing is often used to forecast lead-time demand for inventory control. In this paper, formulae are provided for calculating means and variances of lead-time demand for a wide variety of exponential smoothing methods. A feature of many of the formulae is that variances, as well as the means, depend on trends and seasonal effects. Thus, these formulae provide the opportunity to implement methods that ensure that safety stocks adjust to changes in trend or changes in season.

KEYWORDS

Forecasting; inventory control; lead-time demand; exponential smoothing; forecast variance.

1. INTRODUCTION

Inventory control software typically contains a forecasting module based on exponential smoothing. The purpose of such a module is to feed means and variances of lead-time demand to an inventory control module for the determination of ordering parameters such as reorder levels, order-up-to levels and reorder quantities. Typically, exponential smoothing is chosen because it has a proven record for generating sensible point forecasts (Gardner, 1985).

To be more specific, consider the typical situation where a replenishment decision is to be made at the beginning of period n+1. Any order placed at this time is assumed to arrive a lead-time later at the start of period $n+\lambda$. Inventory theory dictates that the primary focus should be on lead-time demand, an aggregate of unknown future values y_{n+j} defined by

$$Y_n(\lambda) = \sum_{j=1}^{\lambda} y_{n+j} \quad . \tag{1}$$

The problem is to make inferences about the distribution of lead-time demand. Typically an appropriate form of exponential smoothing is applied to past demand data $y_1, \dots y_n$, the results being used to predict the mean of the lead-time demand distribution.

Variances of lead-time demand are also needed for the implementation of inventory strategies that provide a protection against the worst effects of uncertain customer demand. Until Johnston and Harrison (1986) derived a variance formula for use with simple exponential smoothing, rather ad-hoc formulae were the vogue in inventory control software. Using a simple state space model, Johnston and Harrison utilized the fact that simple exponential smoothing emerges as the steady state form of the associated Kalman filter in large samples. Adopting a different model, Snyder, Koehler and Ord (1999) were able to obtain the same formula without recourse to the Kalman filter strategy. The advantage of their approach is that no restrictive large sample assumption is needed. Johnston and Harrison (1986) also obtained a variance formula for trend corrected exponential smoothing. Yar and Chatfield (1990), however, have suggested a slightly different formula. They also provide a formula that incorporates seasonal effects for use with the additive Winters (1960) method.

The purpose of this paper is to take a fresh look at the problem of deriving formulae for forecast variances of lead-time demand. We use the linear version of the single source of error model from Ord, Koehler and Snyder (1997) to unify the derivations. We also provide useful extensions to accommodate errors that depend on trend and seasonal effects. The model and its special cases are introduced in Section 2. Associated formulae for means and variances of lead-time demand are presented in Section 3. General principles used in their derivation are presented in the Appendix. Throughout the paper, we adopt a convention concerning the sum operator Σ. In those cases where the lower limit is less than the upper limit, the sum should be equated to zero.

2. MODELS FOR EXPONENTIAL SMOOTHING

Future values of a time series are unknown and must be treated as random variables. Their behavior must be linked to a statistical model in order to derive prediction distributions. A model should have the potential to include unobserved components such as levels, growth rates and seasonal effects, because various forms of exponential smoothing are based on these concepts. Common cases of exponential smoothing and their models are shown in Table 1. The column marked 'Code' uses nomenclature from Hyndman et al (2001). Here N designates 'None', 'A' designates 'Additive' and D designates 'Damped'. All codes involve two letters. The first letter is used to describe the trend. The second letter describes the seasonal component. The various components are ℓ_t for local level, b_t for local growth rate, s_t for local seasonal effect and e_t for a random variable designating the irregular component. The α, β, γ are so-called smoothing parameters. The ϕ, another parameter, is a damping factor. The purpose of the caret symbol is outlined later.

Case	Code	Model	Smoothing Method	Description
1	NN	$y_t = \ell_{t-1} + e_t$ $\ell_t = \ell_{t-1} + \alpha e_t$	$\hat{y}_t = \hat{\ell}_{t-1}$ $\hat{\ell}_t = \hat{\ell}_{t-1} + \alpha(y_t - \hat{y}_t)$	Simple exponential smoothing (Brown, 1959)
2	AN	$y_t = \ell_{t-1} + b_{t-1} + e_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t$ $b_t = b_{t-1} + \alpha\beta e_t$	$\hat{y}_t = \hat{\ell}_{t-1} + \hat{b}_{t-1}$ $\hat{\ell}_t = \hat{\ell}_{t-1} + \hat{b}_{t-1} + \alpha(y_t - \hat{y}_t)$ $\hat{b}_t = \hat{b}_{t-1} + \alpha\beta(y_t - \hat{y}_t)$	Trend-corrected exponential smoothing (Holt, 1957)

3	AD	$y_t = \ell_{t-1} + b_{t-1} + e_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t$ $b_t = \phi b_{t-1} + \alpha\beta e_t$	$\hat{y}_t = \hat{\ell}_{t-1} + \hat{b}_{t-1}$ $\hat{\ell}_t = \hat{\ell}_{t-1} + \hat{b}_{t-1} + \alpha(y_t - \hat{y}_t)$ $\hat{b}_t = \phi\hat{b}_{t-1} + \alpha\beta(y_t - \hat{y}_t)$	Damped trend (Gardner and McKenzie, 1985)
4		$y_t = s_{t-m} + e_t$ $s_t = s_{t-m} + \gamma e_t$	$\hat{y}_t = \hat{s}_{t-m}$ $\hat{s}_t = \hat{s}_{t-m} + \gamma(y_t - \hat{y}_t)$	Elementary seasonal case
5	AA	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + e_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t$ $b_t = b_{t-1} + \alpha\beta e_t$ $s_t = s_{t-1} + \gamma e_t$	$\hat{y}_t = \hat{\ell}_{t-1} + \hat{b}_{t-1} + \hat{s}_{t-m}$ $\hat{\ell}_t = \hat{\ell}_{t-1} + \hat{b}_{t-1} + \alpha(y_t - \hat{y}_t)$ $\hat{b}_t = \hat{b}_{t-1} + \alpha\beta(y_t - \hat{y}_t)$ $\hat{s}_t = \hat{s}_{t-m} + \gamma(y_t - \hat{y}_t)$	Winters additive method (Winters, 1960)
6	DA	$y_t = \ell_{t-1} + b_{t-1} + c_{t-m} + e_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha e_t$ $b_t = \phi b_{t-1} + \alpha\beta e_t$ $s_t = s_{t-1} + \gamma e_t$	$\hat{y}_t = \hat{\ell}_{t-1} + \hat{b}_{t-1} + \hat{s}_{t-m}$ $\hat{\ell}_t = \hat{\ell}_{t-1} + \hat{b}_{t-1} + \alpha(y_t - \hat{y}_t)$ $\hat{b}_t = \phi\hat{b}_{t-1} + \alpha\beta(y_t - \hat{y}_t)$ $\hat{s}_t = \hat{s}_{t-m} + \gamma(y_t - \hat{y}_t)$	Damped trend with seasonal effects

Table 1. Models for Common Linear Forms of Exponential Smoothing.

Each model in Table 1 contains a measurement equation that specifies how a series value is built from unobserved components. It contains transition equations that describe how the unobserved components change over time in response to the effects of structural change. It involves a random variable representing the irregular component.

All the models in Table 1 are special cases of what is best called a single source of error state space model. The unobserved components are stacked to give a vector x_t. It is assumed that all components combine linearly to give the series value, so the measurement equation is specified as

$$y_t = h' x_{t-1} + e_t \tag{1}$$

where *h* is a fixed vector of coefficients. The lag on x_t is used to reflect the assumption that the conditions at time *t*-1 determine what happens during the period *t*. The evolution of the unobserved components is governed by the first-order transition relationship

$$x_t = F x_{t-1} + g e_t \tag{2}$$

where F is a fixed matrix and g is a fixed vector that reflects the impact of structural change.

It is possible to think of the first component of (1) as an underlying level and to designate it by $m_t = h'x_{t-1}$. It is possible that the disturbance is independent of this level. It is also possible that its variance increases with this level. Both possibilities are captured by the assumption that the disturbance is governed by the relationship

$$e_t = m_t^r \varepsilon_t \quad \text{for} \quad r = 0,1 \tag{3}$$

where ε_t is a member of a $\text{NID}\left(0,\sigma^2\right)$ series? The measurement equation may now be written as $y_t = m_t + \varepsilon_t$ when $r = 0$ or $y_t = m_t\left(1+\varepsilon_t\right)$ when $r = 1$. In the latter case, the ε_t is a unit-less quantity, conveniently thought of as a relative error. It means that the irregular component potentially depends on the other components of a time series, something that can be very important in practice. The elements h, F, g potentially depend on a vector of parameters designated by ω.

It is assumed that the same model governs both past and future values of a time series. Past values are known, in which case it is possible to make a pass through the data, applying a compatible form of exponential smoothing in each period. Suppose, at the beginning of typical period *t*, past applications of exponential smoothing have yielded the value $\hat{x}_{t-1}$ for the state vector x_{t-1}. After observing y_t at the end of period *t*, it is possible to calculate the error $\hat{e}_t = y_t - h'\hat{x}_{t-1}$. The error can be substituted into the transition equation to give $\hat{x}_t = F\hat{x}_{t-1} + g\left(y_t - h'\hat{x}_{t-1}\right)$ for the value of the state vector x_t. Given the progressive nature of this algorithm, it is clear that $\hat{x}_t = x_t \mid y_1, \dots y_t, x_0, \omega$. Induction may be used to confirm that $\hat{x}_t$ is a fixed value.

A special case of the above model, best termed a composite model, is now considered. The state vector x_t is partitioned into random sub-vectors designated by $x_{1,t}$ and $x_{2,t}$. The measurement equation has the form

$$y_t = h_1'x_{1,t-1} + h_2'x_{2,t-1} + e_t \tag{4}$$

where h_1 and h_2 are sub-vectors of h. The sub-vectors of the state vector are governed by transition equations

$$x_{k,t} = F_k x_{k,t-1} + g_k e_t \quad (k = 1, 2) \tag{5}$$

where F_1, F_2 are transition matrices and g_1, g_2 are sub-vectors of g. The special feature of this composite model is that the transition equation for $x_{1,t}$ does not contain $x_{2,t}$ and vice versa. It is shown in the Appendix that the results for a composite model can be built directly from those of its constituent models.

All the models in Table 1 are special cases of the single source of error model or the composite model. The links with these general models are provided in Table 2. Here 0_k refers to a *k*-vector of zeros and I_k refers to a $k \times k$ identity matrix. Note that although the seasonal cases are governed by *m*th-order recurrence relationships, they are converted to equivalent first-order relationships. Also note that ω is a vector formed from some or all of the parameters $\alpha, \beta, \gamma, \phi$.

Case	x_t	h	F	g
1	$x_t = \ell_t$	$h = 1$	$F = 1$	$g = \alpha$
2	$x_t = [\ell_t \quad b_t]'$	$h' = [1 \quad 1]$	$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$g = [\alpha \quad \alpha\beta]'$
3	$x_t = [\ell_t \quad b_t]'$	$h' = [1 \quad 1]$	$F = \begin{bmatrix} 1 & 1 \\ 0 & \phi \end{bmatrix}$	$g = [\alpha \quad \alpha\beta]'$
4	$x_t = [s_t \quad \cdots \quad s_{t-m+1}]'$	$h' = [0'_{m-1} \quad 1]$	$F = \begin{bmatrix} 0'_{m-1} & 1 \\ I_{m-1} & 0_{m-1} \end{bmatrix}$	$g = [\gamma \quad 0'_{m-1}]'$
5	$x_{1,t} = [\ell_t \quad b_t]'$ $x_{2,t} = [s_t \quad \cdots \quad s_{t-m+1}]$	$h'_1 = [1 \quad 1]$ $h'_2 = [0'_{m-1} \quad 1]$	$F_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $F_2 = \begin{bmatrix} 0'_{m-1} & 1 \\ I_{m-1} & 0_{m-1} \end{bmatrix}$	$g_1 = [\alpha \quad \alpha\beta]'$ $g_2 = [\gamma \quad 0'_{m-1}]'$

6	$x_{1,t} = [\ell_t \quad b_t]'$ $x_{2,t} = [s_t \quad \cdots \quad s_{t-m+1}]$	$h_1' = [1 \quad 1]$ $h_2' = [0_{m-1}' \quad 1]$	$F_1 = \begin{bmatrix} 1 & 1 \\ 0 & \phi \end{bmatrix}$ $F_2 = \begin{bmatrix} 0_{m-1}' & 1 \\ I_{m-1} & 0_{m-1} \end{bmatrix}$	$g_1 = [\alpha \quad \alpha\beta]'$ $g_2 = [\gamma \quad 0_{m-1}']'$

Table 2. Conformity of Special Cases to the General Model or Composite Model.

An intriguing insight from Table 2 is that each smoothing method applies for both a homoscedastic and a heteroscedastic model. Now, each homoscedastic case is equivalent to an ARIMA process (Box, Jenkins and Reinsel, 1994). However, no heteroscedastic case is equivalent to an ARIMA process. Thus, exponential smoothing applies for a wider class of models than the ARIMA class (Ord, Koehler and Snyder, 1997).

In the homoscedastic cases, only the mean potentially depends on trend and seasonal effects. However, in the heteroscedastic cases, both the mean and the variance of the irregular component depend on trend and seasonal effects. Thus, prediction variances reflect trend and seasonal effects in the heteroscedastic case, a feature that is potentially quite useful in practice.

Many other cases are conceivable when addition operators are replaced in the measurement equation by multiplications. Examples of such cases are presented in Hyndman, Koehler, Snyder and Grose (2002). A variety of models underlying the multiplicative version of Winters multiplicative method have been introduced in Koehler, Snyder and Ord (2001). The complexity of these non-linear possibilities precludes the derivation of results using the methodology of this paper.

3. MEANS AND VARIANCES OF LEAD TIME DEMAND

It is assumed that methods similar to those described in Ord, Koehler and Snyder (1997) have been applied to past demand data to estimate the parameters of an appropriate model The problem is now to find the moments of the lead-time demand (1). Our analysis is built, in part, on prediction variance results from Hyndman, Koehler, Ord and Snyder (2001) for conventional prediction distributions.

It is shown in the Appendix that lead-time demand can be resolved into a linear function of the uncorrelated irregular components:

$$Y_n(\lambda) = \sum_{j=1}^{\lambda} \mu_{n+j} + \sum_{j=1}^{\lambda} C_j e_{n+j} \ . \tag{6}$$

where

$$\mu_{n+j} = h' F^{j-1} x_n \tag{7}$$

is the mean of the *j*-step prediction distribution. It is further established that the coefficients of the errors in (6) are given by

$$C_j = 1 + \sum_{i=1}^{\lambda-j} c_i \quad \text{for} \quad j = 1, \ldots, \lambda \ . \tag{8}$$

where

$$c_i = h' F^{i-1} g \ . \tag{9}$$

Particular cases of the formulae for the means μ_{n+j} and the coefficients C_j are shown in Table 3. Note that $\phi_j = \sum_{i=0}^{j-1} \phi^i$; $\phi_j^{(2)} = \sum_{i=1}^{j-1} i\phi^i$; $p = \left\lceil \frac{j+m-1}{m} \right\rceil$; $d_{j,m} = 1$ if j is a multiple of m and $d_{j,m} = 0$ otherwise. The results for Case 5 and Case 6 are constructed by adding the corresponding results for constituent basic models, an approach that is also rationalized in the Appendix.

Case	μ_{n+j}	c_j	C_j
1	$\hat{\ell}_n$	α	$1+(\lambda-j)\alpha$
2	$\hat{\ell}_n + j\hat{b}_n$	$\alpha(1+j\beta)$	$1+(\lambda-j)\alpha+\frac{(\lambda-j)(\lambda-j+1)}{2}\alpha\beta$
3	$\hat{\ell}_n + \phi_j \hat{b}_n$	$\alpha(1+\beta\phi_j)$	$1+(\lambda-j)\alpha+(\lambda-j)\alpha\beta\phi_{\lambda-j}-\alpha\beta\phi_{\lambda-j}^{(2)}$
4	$\hat{s}_{n+j-pm}$	$d_{j,m}\gamma$	$1+\gamma\sum_{i=1}^{\lambda-j} d_{i,m}$
5	$\hat{\ell}_n + j\hat{b}_n + \hat{s}_{n+j-pm}$	$\alpha(1+j\beta)+d_{j,m}\gamma$	$1+(\lambda-j)\alpha+\frac{(\lambda-j)(\lambda-j+1)}{2}\alpha\beta+\gamma\sum_{i=1}^{\lambda-j} d_{i,m}$
6	$\hat{\ell}_n + \phi_j \hat{b}_n + \hat{s}_{n+j-pm}$	$\alpha(1+\beta\phi_j)+d_{j,m}\gamma$	$1+(\lambda-j)\alpha+(\lambda-j)\alpha\beta\phi_{\lambda-j}-\alpha\beta\phi_{\lambda-j}^{(2)}+\gamma\sum_{i=1}^{\lambda-j} d_{i,m}$

Table 3. Key Results for Basic models.

From (6), the conditional variance is given by

$$\operatorname{var}\left(Y_n(\lambda) \mid x_n, \omega\right) = \sigma^2 \sum_{j=1}^{\lambda} C_j^2 . \tag{10}$$

in the homoscedastic case. All the information needed to evaluate the grand mean and the grand variance is available in Table 3. In the heteroscedastic case the grand variance is

$$\operatorname{var}\left(Y_n(\lambda) \mid x_n, \omega\right) = \sigma^2 \sum_{j=1}^{\lambda} C_j^2 \theta_{n+j} \tag{11}$$

where $\theta_{n+j} = E\left(m_{n+j}^2 \mid x_n, \omega\right)$. It is established, in the Appendix, that the heteroscedastic formulae may be computed using the recurrence relationship

$$\theta_{n+j} = \mu_{n+j}^2 + \sum_{i=1}^{j-1} c_{j-i}^2 \theta_{n+i} \sigma^2 \tag{12}$$

where the c_j are also given in Table 3.

4. CONCLUSIONS

Formulae for calculating the mean and variance of lead-time demand have been derived for many common forms of exponential smoothing in this paper. For the homoscedastic cases, the prediction distributions are Gaussian, so the means and variances provide all the information required to make probabilistic statements about future lead-time demand. In theory, the prediction distributions for the heteroscedastic cases are not Gaussian. However, a numerical study in Hyndman, Koehler, Ord and Snyder (2001) indicates that there is little error involved in approximating them by a Gaussian distribution. The same conclusion must apply to lead-time distributions where aggregation must help to further reduce the approximation error.

By using the single source of error state space model, we have unified the derivation of the formulae. In the homoscedastic cases, many of the formulae obtained in this paper agree with those found in earlier work (Johnston and Harrison, 1986; Yar and Chatfield, 1990; Snyder, Koehler and Ord, 1999). A small advance was obtained in relation to Winters additive seasonal method in that the recursive variance formulae in Yar and Chatfield (1990) has been replaced by a closed counterpart. Furthermore, we have obtained, for the first time, formulae for the variance of lead-time demand for the damped trend cases.

It has been argued in the paper that the irregular component of a demand series can depend on trend and seasonal effects. Thus, a major part of our contribution has been the provision of lead-time demand variance formulae for heteroscedastic extensions to exponential smoothing. Such formulae admit the possibility of smarter approaches to safety stock determination. It is now possible to implement schemes that tailor levels of safety stock to changes in trend or changes in season.

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APPENDIX

General results governing the formulae in Table 3 are derived in this Appendix. To get the formulae governing Cases 1-4, back solve the transition equation (2) from period $n+j$ to period *n*, to give

$$x_{n+j} = F^j x_n + \sum_{i=1}^{j} F^{j-i} g e_{n+i} \quad \text{(A1)}$$

Lag (A1) by one period, pre-multiply the result by h', and use the definitions (7) and (9) to get

$$m_{n+j} = \mu_{n+j} + \sum_{i=1}^{j-1} c_{j-i} e_{n+i} \,. \tag{A2}$$

Recall that e_t is given by (3) so that $E(e_{n+i}^2 \mid) = \sigma^2 E(m_{n+i}^2)$. Then we may square (A2) and take expectations to give the recurrence relationship (12) for the heteroscedastic factors. Substitute (A2) into (1) to give $y_{n+j} = \mu_{n+j} + \sum_{i=1}^{j-1} c_{j-i} e_{n+i} + e_{n+j}$. Substitute this into (1) to give $Y_n(j) = \sum_{j=1}^{\lambda} \left(\mu_{n+j} + \sum_{i=1}^{j-1} c_{j-i} e_{n+i} + e_{n+j} \right)$. Rearrange terms to yield the required result (6) where the C_j are defined by (8). Note that the derivation of the C_j is expedited using the following equations: $C_\lambda = 1$ and $C_j = C_{j+1} + c_{\lambda - j}$ for $j = \lambda - 1, \ldots, 1$.

Cases 5 and 6 are composite models. Each transition equation (5), for a composite model, has the same structure as (2). Thus,

$$x_{k,n+j} = F_k^j x_{k,n} + \sum_{i=1}^{j} F_k^{j-i} g_k e_{n+i} \,. \tag{A3}$$

Lag (11) by one period and pre-multiply the result by h_k' to give

$$m_{k,n+j} = \mu_{k,n+j} + \sum_{i=1}^{i-1} c_{k,j-i} e_{n+i} \tag{A4}$$

where

$$\mu_{k,n+j} = h_k' F_k^{j-1} x_{k,n} \tag{A5}$$

and

$$c_{k,i} = h_k' F_k^{i-1} g_k \,. \tag{A6}$$

Substitute (A4) into $m_{n+j} = m_{1,n+j} + m_{2,n+j}$ to yield the earlier equation (A2) where

$$\mu_{n+j} = \mu_{1,n+j} + \mu_{2,n+j} \tag{A7}$$

and

$$c_i = c_{1,i} + c_{2,i} \, . \qquad \text{(A8)}$$

Thus, the formula $C_i = C_{1,i} + C_{2,i} - 1$ may be used to derive the results for Case 5 and Case 6 from their constituent basic cases. In the heteroscedastic cases, the appropriate factors are still derived with the relationship (12).