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The Power Principle and Tail-Fatness Uncertainty

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The power principle and tail-fatness uncertainty

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Abstract: When insurance claims are governed by fat-tailed distributions, gross uncertainty about the value of the tail-fatness index is virtually inescapable.
In this paper a new premium principle (the power principle) analogous to the exponential principle for thin-tailed claims, is discussed.
Pareto premiums determined under the principle have a transparent ratio structure, cater convincingly for uncertainty in the tail-fatness index, and *are applicable in passage to the extremal limit, to all fat-tailed distributions in the domain of attraction of the (Frechet) extreme-value distribution.* Cover can be provided for part claims if existence of the claims mean is in doubt. Stop-loss premiums are also discussed. Mathematical requirements are very modest.

KEYWORDS: Exponential principle, power principle, constant risk aversion, ratio premium, stop-loss insurance

JEL Classification G22, IM Classification 30

1.0 Introduction

1.1 Objectives and structure of the paper

The aim is to describe and demonstrate the application of a new premium principle (the power principle) for fat-tailed risk as described in Section 1.3. For this type of risk, the precise value of the tail-fatness index is of first importance in determination of claim severity. But the index is very difficult to estimate precisely, particularly with the sorts of sample sizes likely to be available to insurers in practice.
The premium principle can be used to provide a coherent framework in which to determine premiums in a transparent form, when uncertainty in the tail-fatness index is suitably modelled.
In the remainder of the paper, the exponential principle for thin-tailed risk is described. The power principle is introduced as the analogous principle for fat-tailed risk. Risk-neutral and risk-averse premiums for Pareto claims are shown to take the form of ratios of expected values of the largest order statistics independently of sample size. In passage to the extremal limit, the premiums predicated on Pareto are shown to apply to all fat-tailed distribution types in the domain of attraction of the fat-tailed extreme value distribution.
New results start with the Power Principle in Section 1.3

1.2 Thin-tailed risk and the exponential principle

When general insurance arrival claims are thin-tailed (possessing moments of all orders) *the exponential principle* (see Rolski *et al*, 1999, p.80), predicated on constant absolute risk aversion (see Pratt, (1964) and Arrow (1971)) applies.

Under the exponential principle, premium P for claim X is set as

$$P = \ln\{M_X(s)\} \tag{1.1}$$

$$\approx \mu + ½s\sigma^2 + o(s)$$

where $M_X(\cdot)$, μ, σ^2 are respectively the moment generating function, mean and variance of X. Parameter s measures the constant absolute risk aversion of the insurer. See for instance Bowers *et al*, (1986).
Indeed the exponential principle applies *to annual aggregate claims* Y where

$$Y = X_1+X_2+ \cdots + X_N, \tag{1.2}$$

the $\{X_i\}$ are identically independently distributed, and N is the claim number distribution (frequently Poisson or negative binomial), since

$$M_Y(\theta) = M_N[\log\{M_X(\theta)\}] \tag{1.3}$$

so that

$$P \approx E[X]\times E[N] + ½s\{ var(X)\times E[N]+var(N)\times E[X]^2\} \tag{1.4}$$

The exponential principle is conveniently established using the *pricing function*

$$m(x) = \exp(sx) \tag{1.5}$$

and the *pricing rule* (analogous to the expected utility principle, see Appendix; Note 1)

$$m(P) = E[m(X)] \tag{1.6}$$

1.3 Fat-tailed risk and the power principle

Individual claims $\{X_i\}$ of (1.2) are governed by a fat-tailed distribution F(x) when

$$1-F(x) = L(x)x^{-\delta}, \ (x > 0, \delta > 0) \tag{1.7}$$

where L(x) is 'slowly varying at infinity', i.e. for any $\varepsilon > 0$ and sufficiently large x

$$x^{-\varepsilon} < L(x) < x^{\varepsilon}.$$

Thus the claim random variable X has moments only up to order $k < \delta$.

Much of our discussion centres on Pareto claims with

(1.8) $1\text{-}F(x) = (x+1)^{-\delta}$, $(x > 0, \delta > 0)$

It is easy to show (and *is* shown in the sequel) that relevant results for Pareto claims, extend in passage to the extremal limit to all claims with distributions in the domain of attraction of the Type I (Frechet) extreme value distribution, i.e. certainly to the class of distributions comprehended by (1.7).
Exposure to the extremal process would imply that the general insurer is indefinitely continuing, in business in fact as a *going concern* (the 'going concern' concept is first of four fundamental accounting principles; assumptions on which the statutory reports of all companies are compiled; see Holmes and Sugden, 1990, Chapter 1).

A Pareto claim X under (1.8) is equivalent to claim V measured in terms of a scale parameter λ (i.e. $V = X\lambda$) when V has the Pareto form

(1.9) $$1\text{-}F(x) = \left(\frac{\lambda}{\lambda + x}\right)^{\delta} \quad (x > 0, \delta > 0, \lambda > 0)$$

For example λ might be \$100,000 or \$1m.

Fat-tailed claims (having limited moments) have no moment generating functions, hence no constant absolute risk-averse premiums.

The Power Principle

For fat-tailed risk premium P can be set using the power principle, predicated on constant relative risk-aversion measured by parameter α, via pricing function

(1.10) $$m(x) = x^{\alpha+1} \ (x > 0, \alpha \geq 0)$$

and pricing rule (1.6); (see Appendix; Note 1).

For any claims random variable X, the power principle leads to a premium P determined by the risk-aversion coefficient α as

(1.11) $$P = \{\mu'_{\alpha+1}\}^{\frac{1}{\alpha+1}}$$

where μ_k' is the kth raw moment of X.

Thus P is a non-decreasing function of $\alpha+1 \geq 0$ (see for instance, Puri and Sen, (1973, p.12).

<u>1.4 Risk-averse Pareto premiums</u>

When the power principle is applied to Pareto (1.8) we have for claim (1+X):

$$P^{\alpha+1} = E[(1+X)^{(\alpha+1)}]$$

$$= \int_0^\infty (1+x)^{(\alpha+1)} \delta (1+x)^{-\delta-1} dx$$

$$= \delta\{\delta-(\alpha+1)\}^{-1}$$

So $$P = \{1-\frac{(\alpha+1)}{\delta}\}^{-1/(\alpha+1)}$$

In terms of $\rho = 1/\delta$, the constant relative risk-averse premium P is given by

(1.12) $$P = (1-\rho/\beta)^{-\beta}$$

where $\beta = (\alpha+1)^{-1}$ is *the maximum tail-fatness index considered operative by the insurer with constant relative risk-aversion* α.

1.4 Remarks:

(i) The insurer's level of constant relative risk-aversion α is manifested by its choice of maximum tail-fatness index $(\alpha+1)^{-1} = \beta = \rho_{max} = 1/\delta_{min}$.

(ii) Non-existence of the mean

The foregoing argument assumes $\delta > 1$ ($\rho < 1$) and the existence of $E[1+X]$.
If $\delta \leq 1$, ($\rho \geq 1$) then cover can be offered on part of the claim $(1+X)^{\varphi}$ where $\varphi\rho < 1$.
In this case, the pricing rule is applied to claim $(1+X)^{\varphi}$ resulting in premium

(1.13) $$P = (1-\rho/\beta)^{-\varphi\beta}$$

where now $\beta^{-1} = \varphi(\alpha+1)$.

(iii) The risk-neutral premium

For Pareto (1.8) $E[X] = (\delta-1)^{-1}$, $\delta > 1$.
If $\alpha = 0$, $\beta = 1$, equation (1.13) leads to $(1-\rho)^{-1} = \delta/(\delta-1) = E[1+X]$.

We show in Appendix: Note 3 that

$$(1.14) \qquad (1-\rho)^{-1} = \frac{E[1+X^{(n)}]}{E[1+X^{(n-1)}]}$$

where $X^{(k)}$ is the kth Pareto order statistic, the ratio being independent of n.

As $\rho \uparrow 1$, the ratio becomes very large. The expected value of the second largest order statistic (claim) is small compared with that of the largest. This suggests annual aggregate claims Y of (1.2) will be largely attributable to the largest or few largest claims.

Corollary: For thin-tailed claims $\rho = 0$. Heuristally from (1.14) we expect the ratio $\frac{E[1+X^{(n)}]}{E[1+X^{(n-1)}]}$ to be 1 *for all thin-tailed distributions*.

In fact it is not difficult to prove this if $1-F(x) = O(\exp(-\lambda x^{\gamma}))$, $(x > 0, \lambda > 0, \gamma > 0)$.

2.0 Risk-averse premiums when tail-fatness index is uncertain

2.1 Fat-tail claims models

The main reason for using fat-tailed arrival claims is to model classes of insurance with potentially a few very large individual claims. We have in mind catastrophe, public liability, professional indemnity, industrial fire and the like.
For (1.2) to provide a useful model for very large claims, *the variance of individual claims must be infinite*; otherwise aggregate claims will be governed by central limit theorems.
The principal cost of annual aggregate claims in the classes of insurance involving fat-tailed claim is usually attributable to one or two large claims (see Mikosch, 1997 for argument, Feller (1971, p.289, for theoretical justification; also Appendix: Note 2 (quote from Mikosch, 1997). Remark (iii) of Section 1.4 above is also relevant).

This means that the tail-fatness index δ of (1.7) or (1.8) does not exceed 2.

If in addition, typical large claims from (1.8) are large multiples of the (unit) scale parameter, and $E[X]$ is supposed to exist, δ must be close to 1.

It is probably worth pointing out that for Pareto claims X under (1.8), the mean $\mu = (\delta\text{-}1^{-1}$ is 'a rare event' as $\delta \downarrow 1$, in the sense that $\Pr(X > \mu) \to 0$.
Thus the mean will be greater than any quantile of the distribution as δ approaches 1.
For δ in the range (0, 2], the tail-fatness index is notoriously difficult to estimate, even when quasi-parametric or parametric assumptions are made.

Empirical evidence is provided in Appendix: Note 2.

Since δ (and so μ) is never known precisely, there is some danger in attempting to use 'the quantile principle' (Rolski *et al*, 1999, p.83) to set premiums.

In practical terms this difficulty can be side-stepped by modelling tail-fatness uncertainty as we now outline. Unless otherwise stated, it is assumed that the claims mean E[X] exists, so that $\delta > 1$.

2.2 Model for tail-index uncertainty

Uncertainty about the precise value of ρ (= $1/\delta$) is modelled using a transformed beta density on $(0, \beta)$ where $\beta = 1/(\alpha+1)$, the maximum tail-fatness ρ_{max} deemed operable by an insurer with constant relative risk-aversion α.

The risk-averse premium $P = (1-x/\beta)^{-\beta}$ from (1.12) is averaged over $(0, \beta)$ for weights given by

(2.1) $$f_\rho(x) = \nu x^{\nu-1}/\beta^\nu \quad (0 \le x < \beta, \nu \ge 1)$$

the premium being calculated as a mean value.

Alternatively, distribution (2.1) can be regarded as a Bayesian prior for ρ (with E[P] the resultant posterior mean).

The reasons for using ρ weights in the form (2.1) are:

(i) *Prudence*: Insurers would doubtless prefer ρ to be small, near zero (if $\rho = 0$ risk arises from a thin-tailed, or a mixture of thin-tailed distributions), but financial prudence suggests that values near 1 should be most heavily weighted, since it is these values which generate greatest claim severity. The functional form (2.1) does not absolutely preclude the possibility that claims arise from a mixture of thin-tailed distributions (when $\nu = 1$, 'law of equal ignorance' for ρ on $[0,\beta)$, the value $\rho = 0$ 'is as likely as' any specific non-zero value, in that its probability density is β^{-1}), however as ν is increased, ρ values near 1 are given heavier weightings.
Even so, this is a little misleading.
Even if $\nu = 1$, $f_\rho(x) = 1/\beta$, the 'law of equal ignorance' for ρ on $(0, \beta)$, implies that *the distribution in terms of δ weights is* $f_\delta(x) = \beta^{-1}/x^2$ $(x > \delta_{min} = 1/\beta)$. That is, the δ weights are Pareto with a mode at $x = \delta_{min}$.

(ii) *Mathematical tractability*: The form $P = (1-\rho/\beta)^{-\beta}$ of the risk-averse premium, with an integrable pole at $\rho = \beta$, $(\beta < 1)$, suggests the functional form (2.1) to generate risk-weighted premiums involving the Beta function.
This indeed happens, but the interpretation in terms of ratios of expected values of higher order statistics (discussed below) is indeed fortuitous.

2.3 The nature of the uncertainty premiums

Denote by P_v the premium obtained as mean value of $\int_0^\beta \nu\beta^{-\nu} x^{\nu-1}(1-x/\beta)^{-\beta}\,dx$.
Substitution u = x/β in the integral leads immediately to

(2.2) $$P_v = \nu B(\nu, 1-\beta)$$

where B(· , ·) is the beta function. We call P_v the uncertainty premium.

2.4 Ratio structure of the uncertainty premium P_v

If ν = k an integer,

(2.3a) $$P_k = kB(k,1-\beta)$$

but also,

(2.3b) $$P_k = \frac{E[1+X^{(n)}]}{E[1+X^{(n-k)}]}$$

where $X^{(k)}$ is the kth order statistic of the claim distribution. If E[X] exists, so does $E[X^{(n)}]$.

The ratio is independent of n (Appendix: Note 3)

In passage to the extremal limit (as n becomes large; the insurer is supposed to be an indefinitely continuing entity) *the value P_k remains constant*.

However the *character of the ratio* $\frac{E[1+X^{(n)}]}{E[1+X^{(n-k)}]}$ *changes to that of the ratio of expected values of extremes of the distribution*; i.e.

$$P_k \to P_k^*$$

where

(2.4) $$P_k^* = \frac{E[X^{*(1)}]}{E[X^{*(1+k)}]}$$

and where $X^{*(m)}$ is the kth extreme of the distribution. i.e. *it is the limiting value for large n, of $X^{(n-m+1)}$*.

Aside: An interesting independent check of this result is available.

The limiting density of $X^{(n-k+1)}$ is the density of the kth extreme of any distribution in the domain of attraction of the Frechet extreme value distribution and is

(2.5) $$f_k(x) = \delta x^{-k\delta-1}(v_n)^{k\delta}\exp\{-(v_n)^{\delta}x^{-\delta}\}/(k-1)!$$

where $n(1-F(v_n)) = 1$, i.e v_n is the 'tail-quantile' function.

(See for instance David and Nagaraja, (2003, p.306).

It follows that $E[X^{*(k)}] = v_n\Gamma(k-1/\delta)/\Gamma(k)$

$= v_n\Gamma(k-\rho)/\Gamma(k)$

So for instance $E[X^{*(1)}]/E[X^{*(1+k)}] = \Gamma(k+1)\Gamma(1-\rho)/\Gamma(k+1-\rho)$

$= kB(1-\rho,k)$

$=kB(1-\beta,k)$

as for the ratio

$$\frac{E[1+X^{(n)}]}{E[1+X^{(n-k)}]}$$

in equation (2.3) with $\rho = \beta$. <u>End: Aside</u>

<u>2.5 Remarks:</u>

(i) Universal fat-tailed premiums

The premium $P_k = P^*_k$ is independent of n.

The alternative derivation based on (2.5) *shows that it is also independent of the set of distribution-specific normalizing constants $\{v_n\}$ required to ensure convergence of extremes from individual parent type distributions to the extreme value distribution.*

For instance. for Pareto (1.8) , $v_n = n^{\rho}$

For loggamma, $f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}(\log(x))^{\alpha-1}x^{-\lambda-1}$, $(x > 1,\ \alpha,\lambda > 0)$,

$v_n = \left(\frac{n}{\Gamma(\alpha)}\right)^{\rho}(\log(n))^{\rho(\alpha-1)}$ *; (see Teugels and VanRoelen, 2004)*

Its independence of v_n shows that the ratio premium <u>is independent of distribution type</u>.

In the extremal limit (for ongoing insurers) it provides a universal premium for all fat-tailed distribution types in the domain of attraction of the Frechet extreme value distribution.

(ii) Continuous premiums

When only integer order statistics are used, the premiums available have large gaps. For instance when β is set at 1/1.1 ($\delta = 1.1$), premiums increase with $\nu = k$ as shown in Table 1 below. The premiums are for claim (1+X)

Value of $\nu = k$	1	2	3	4
Premium	11	20.1667	28.9347	37.4449

Table 1 Constant relative risk averse premiums for (1+X) when $\beta = 1/1.1$ when $\nu = k$ an integer

By using values of ν *between integers* we can obtain premiums to fill in these gaps. One way of preserving the structural (ratio) form of the premium is as follows.

Define *the fractional order statistic* $X^{(\nu)}$ *interpolating between* $X^{(k)}$ *and* $X^{(k+1)}$.
For any value ν such that $1 \le k \le \nu < k+1 \le n$, the νth Pareto order statistic $X^{(\nu)}$ is defined via its density on $(0, \infty)$ given by

(2.6) $$f_\nu(x) = \delta[1-(x+1)^{-\delta}]^{\nu-1}(x+1)^{-(n-\nu)\delta}(x+1)^{-\delta-1}/B(\nu,n+1-\nu)$$

by analogy with the density of the kth Pareto order statistic

(2.7) $$f_k(x) = \delta[1-(x+1)^{-\delta}]^{k-1}(x+1)^{-(n-k)\delta}(x+1)^{-\delta-1}/B(k,n+1-k)$$

Here $B(\nu, n+1-\nu) = \Gamma(\nu)\Gamma(n+1-\nu)/\Gamma(n+1)$.

The ν*th extreme* (the limiting value of $X^{(n-\nu+1)}$, $\nu \ge 1$) is similarly defined via its density , denoted by $f_\nu(x)$ where

(2.8) $$f_\nu(x) = \delta(\nu_n)^{\nu\delta}x^{-\nu\delta-1}\exp\{-(\nu_n)^{\delta}x^{-\delta}\}/\Gamma(\nu) \quad (\nu \ge 1, x > 0, \delta > 0)$$

Then

(2.9a) $$P_\nu = \nu B(\nu, 1-\beta)$$

and

(2.9b) $$P_\nu = \frac{E[1+X^{(n)}]}{E[1+X^{(n-\nu)}]}$$

while

(2.10)

$$P_\nu^* = \frac{E[X^{*(1)}]}{E[X^{*(1+\nu)}]}$$

(These results are established in Appendix: Note 3).

Illustration

For instance for $\beta = 1/1.1$ and ν between 0 and 1 we obtain for claim (1+X):

Value of nu	Premium
1.0	11.0000
1.1	11.9421
1.2	12.8774
1.3	13.8064
1.4	14.7297
1.5	15.6476
1.6	16.5604
1.7	17.4685
1.8	18.3720
1.9	19.2714
2.0	20.1667

Table 2 Flexible premiums obtained using fractional order statistics and $\beta = 1/1.1$

(iii) Insurer's pragmatic strategy ($\nu = 1$; the implications)

In Appendix: Note 2 evidence it was intimated that whatever information can be gained from sample data, is best gained via direct estimation of the tail-fatness index. Nevertheless, precise knowledge about the operative value is just not available. What the insurer must do effectively is choose a value of the tail-fatness δ or ρ which it considers financially prudent and use it as a basis to set premiums.

This is equivalent to making a choice for $\beta = \rho_{max}$ and using a ratio premium based on $\nu = 1$; i.e. setting $P = (1-\beta)^{-1}$ the 'risk-neutral premium' based on the fattest tail, $\beta = \rho_{max}$ using the law of equal ignorance for ρ on $(0, \beta)$. This choice nevertheless implies the fattest tails obtain heaviest weights as explained in Section 2.1.

2.5 More general ratio premiums

If values of $\delta \leq 1$ are in play ($\rho \geq 1$) the existence of E[X] is not guaranteed. The most general premium modelled under uncertainty (which includes the cases already discussed above) is given by the following theorem

Theorem When claims have a Pareto distribution with $1-F(x) = (x+1)^{-\delta}$, $(x > 0, \delta > 0)$ the ratio premium for claim $(1+X)^{\varphi}$ ($\varphi\rho < 1$ if $\rho \geq 1$, $\varphi = 1$ otherwise) calculated as a mean value of constant relative risk-averse premiums $P = (1-\rho/\beta)^{-\varphi\beta}$ with weights $f_\rho(x) = \nu(x-\beta_0)^{\nu-1}/(\beta-\beta_0)^{\nu}$, $(\beta_0 < x < \beta)$ is:

(2.10) $$P_\nu = (1-\beta_0/\beta)^{-\varphi\beta}\nu B(\nu,1-\varphi\beta)$$

or

(2.11) $$P_\nu = (1-\beta_0/\beta)^{-\varphi\beta}\frac{E[(1+X^{(n)})^{\varphi}]}{E[(1+X^{(n-\nu)})^{\varphi}]}$$

The premium is independent of n, and of standardizing constants.
For an insurer exposed to the extremal process it is thus also expressible as

(2.13) $$P_\nu^* = (1-\beta_0/\beta)^{-\varphi\beta}\frac{E[(X^{*(1)})^{\varphi}]}{E[(X^{*(1+\nu)})^{\varphi}]}$$

applicable to claims from every parent in the domain of attraction of the fat-tailed extreme value (Frechet) distribution. See Appendix, Note 4

3.0 Stop-loss insurance

3.1 Constant relative risk-averse stop-loss premiums

In this section the emphasis is on premiums for claims against the extreme-value distribution (i.e. insurer is a going concern, and n is large).

We want a premium $P_\nu^*(K)$ for $(X-K)^+ = X-K$ $(X \geq K)$

$= 0$ otherwise.

(The risk-neutral premium based on the fattest tail index can then be found by putting $\nu = 1$).

Notice that $(X+1) = \{(X-K)^+ + K+1\}$ on (K, ∞)

$$= (K+1)\{\frac{(X-K)^+}{K+1} + 1\} \text{ on } (K, \infty)$$

has the distribution $(K+1)(U+1)$ on where U is a positive fat-tailed random variable on $(0,\infty)$.
Hence for large n its risk-averse stop loss premium is $P_\nu^*(K) = E[U^{*(1)}]/E[U^{*(1+\nu)}]$ where U is positive and fat-tailed on $(0,\infty)$; i.e.

(3.1) $$P_v^*(K) \sim P_v^*$$

Stop-loss constant relative risk-averse premiums for the extremal process are the same as the risk-averse premiums for the entire claim.

Summary

For thin-tailed distributions the exponential principle is used to set premiums in the form $P = \ln\{M_X(s)\}$ based on constant absolute risk-aversion s.

For fat-tailed claims distributions there is a corresponding power principle

$$P = E[X^{\alpha+1}]^{1/(\alpha+1)}$$

based on constant relative risk-aversion α.

The risk-neutral Pareto premium for claim (1+X) is

$$P_1 = (1-\rho)^{-1}$$

$$= B(1,1-\rho)$$

$$= \frac{E[1+X^{(n)}]}{E[1+X^{(n-1)}]}$$

which is independent of n.

For large n the ratio changes in character (i.e. meaning) without changing value.
It is expressible as the ratio of the two largest extremes; i.e.

$$P_1^* = \frac{E[X^{*(1)}]}{E[X^{*(2)}]}$$

$$= 1B(1, 1-\rho)$$

Hence premiums initially predicated on Pareto claims are applicable for an ongoing insurer to all fat-tailed claim distribution types in the domain of attraction of the extreme value distribution.

The risk-neutral premium is a special case of the risk-averse uncertainty premium, obtained when uncertainty in ρ is modelled by transformed beta density $f_\rho(x) = \nu x^{\nu-1}\beta^{-\nu}$ for $0 \le x < \beta$, which is

$$P_\nu^* = \frac{E[X^{*(1)}]}{E[X^{*(1+\nu)}]}$$

$$= \nu B(\nu, 1-\rho)$$

where $X^{*(\nu)}$ $(\nu \geq 1)$ is a 'fractional extreme'.

For the extremal process, the same premium can be used for stop-loss insurance

Appendix

<u>Note 1. (Pricing functions and a pricing rule deriving from constant risk aversion)</u>

The ‘no-arbitrage’ principle widely used in pricing risk is inappropriate for pricing general insurance premiums; insurers are price-setters (see Albrecht, 1992)

Premiums can be set using the classic theory of risk-aversion.

Arrow (1971) defined two measures of risk-aversion described below:

Absolute risk aversion R_A where

(A.1) $$R_A = -\frac{U''(x)}{U'(x)}$$

and

Relative risk aversion R_R where

(A.2) $$R_R = -x\frac{U''(x)}{U'(x)}$$

If $R_A = s$ (>0, constant), the utility function

(A.3) $$U(x) = -\exp(-sx)$$

obtains.

The *exponential principle* which determines the risk-averse premium P for claim X as

(A.4) $$P = \ln\{M_X(s)\}$$

(where $M_X(s)$ is the moment generating function of X) follows by using *a version* of the expected utility principle (see for instance Bowers *et. al*. 1986, p.9); i.e. a version of

(A.5) $$U(P) = E[U(X)]$$

For instance, the minimum acceptable premium P to an insurer with utility function U and wealth W for insurance against random claim X is given as in (A.4) by using utility function (A.3) and

(A.6) $$U(W) = E[U(W+P-X)])$$

Or it derives from the 'zero utility principle',

(A.7) $$E[U(P-X)] = U(0)$$

(see for instance, Rolski et al. 1999. p.91), in conjunction with utility function(A.3).

Equivalently, the exponential function m(x) = exp(sx) could be regarded as the *pricing function* of an insurer with constant absolute risk-aversion (see Gay, 2004) and the principle follows directly from a *pricing rule*.

(A.8) $$m(P) = E[m(X)]$$

analogous to (A.5).

This is just manipulating the expected utility principle to apply to premium determination rather than asset pricing. Think of the graph of U(x) = -exp(-sx) in the third Cartesian quadrant ($x<0$, $U < 0$).
Now transfer the shape back to the first quadrant ($x>0$, $U>0$) i.e. measure loss positively and negative utility positively (where –U(x) is "degree of discomfort" m(x), say).
Then m(x) = exp(sx) is used to price insurance premiums in the face of constant absolute risk aversion.

For *constant relative risk aversion*, the process requires amendment of the definition of R_R for pricing insurance premiums to

(A.9) $$R_R = x\frac{U''(x)}{U'(x)}$$

resulting in pricing function

(A.10) $$m(x) = x^{\alpha+1}$$

when R_R is set to constant α.

Evidently $\alpha = 0$ provides for risk-neutrality, $\alpha>0$ for positive the risk-aversion.

However it is derived, *the exponential principle*:

- has its roots in constant absolute risk-aversion,
- is applicable to thin-tailed distributions of exponential order, and
- results in setting premiums with reference to mean and standard deviation of claims.

Since the random sums (1.2) are subject to central limit theorems there are no serious complications of a theoretical nature in respect of thin-tailed claims distributions.

The power principle derives from constant relative risk-aversion, is applicable to fat-tailed (power) distributions. Premiums are determined in terms of ratios of expected values of the largest order statistics of the claims distribution as demonstrated in Note 3.

Note 2: (Some empirical results on estimation of tail-fatness index)

When δ is in the range (0,2], the tail-fatness index δ is notoriously difficult to estimate from sample sizes likely to be available to a general insurer, even using information across the whole national industry.

Example A.1

For Pareto (1.8) with $\delta = 1.1$, i.e. $\mu = (\delta-1)^{-1} = 10$, an insurer might hope to determine μ in the range (9,11) with 100 or so observations. This would seem reasonable. The reality is as follows:

Non-parametric estimation

When a large number of samples of size n = 100 from Pareto (1.8) with $\delta = 1.1$ is generated (the present author used one million several times), *typically only about 3.1% of sample means are in the range (9,11).*

When sample size is increased to n = 200, the proportion increases to about 3.3%.

The figures can be checked on a standard laptop.

Generally speaking direct estimators of δ, which however, require at least partial parametrisation, are more efficient.

Quasi- and maximum likelihood estimation

Under assumption, Pareto (1.8) admits a maximum likelihood estimator (MLE).

For Pareto, when the MLE, Hill's (1975) estimator, and method of moments estimators with Box-Cox transformations (see Teugels and Vanroelen, 2004 for the justification) were used, the following results, based on one million samples of size n = 100 and n = 200 were obtained.
The proportions below are for sample values of δ in the range (1.0909, 1.1111); i.e. indicating that $\mu = (\delta-1)^{-1}$ is in the range (9,11).

sample size	MLE	Moments (Box-Cox)	Hill
100	0.0733	0.0736	0.0724
200	0.1028	0.1025	0.1007

Table A.1 Proportions of estimates of $\delta = 1.1$ from one million Pareto samples of n = 100 and n = 200 observations which imply μ in the range (9,11).

Note: About 16% of samples gave an estimate of $\delta < 1$.

These figures do represent a considerable improvement on use of the mean.

The imprecision of sample information available on δ prompted one experienced investigator to comment:

"*Statistical analyses of large claim data are based on extreme value theory and related methods. These are known to be very sensitive with respect to the tails of the distributions, and therefore the existence of one very large claim may justify a fit of a Pareto instead of a lognormal distribution, say*" (Mikosch,1997).

Note 3: (The ratio premiums)

Using (2.5) it is easy to show that for Pareto (1.8)

$$E[1+X^{(\nu)}] = B(\nu, n+1-\nu-\rho)/B(\nu,n+1-\nu)$$

$$= \{\Gamma(\nu)\Gamma(n+1-\nu-\rho)/\Gamma(n+1-\rho)\}\times\Gamma(n+1)/\Gamma(\nu)/\Gamma(n+1-\nu)$$

$$= \Gamma(n+1-\nu-\rho)\Gamma(n+1)/\{\Gamma(n+1-\rho)\Gamma(n+1-\nu)\}$$

Then

$$\frac{E[1+X^{(n)}]}{E[1+X^{(n-\nu)}]} = \{\Gamma(1-\rho)\Gamma(n+1)/\Gamma(n+1-\rho)\}\div[\Gamma(\nu+1-\rho)\Gamma(n+1)/\{\Gamma(n+1-\rho)\Gamma(\nu+1)\}]$$

$$= \Gamma(1-\rho)\Gamma(\nu+1)\}/\Gamma(\nu+1-\rho)$$

$= \nu B(1-\rho,\nu)$

$= P_\nu$

which is independent of n.

Similarly, $E[(1+X^{(\nu)})^\varphi] = B(\nu, n+1-\nu-\rho\varphi)/B(\nu,n+1-\nu)$

and

$$\frac{E[(1+X^{(n)})^\varphi]}{E[(1+X^{(n-\nu)})^\varphi]} = \nu B(1-\rho\varphi, \nu) \text{ for } \rho\varphi < 1.$$

The universal premium $P_\nu^* = \dfrac{E[X^{*(1)}]}{E[X^{*(1+\nu)}]}$ is consequence of passage to the extremal limit.

Note 4: (Proof of Theorem)

Denote by P_ν the premium obtained as mean value of

$$\int_0^\beta \nu(\beta-\beta_0)^{-\nu}(x-\beta_0)^{\nu-1}(1-x/\beta)^{-\varphi\beta}\,dx.$$

Substitution $u = (x-\beta_0)/(\beta-\beta_0)$ in the integral leads immediately to

$$P_\nu = (1-\beta_0/\beta)^{-\varphi\beta}\nu B(\nu,1-\beta\varphi)$$

$$= (1-\beta_0/\beta)^{-\varphi\beta}\frac{E[(1+X^{(n)})^\varphi]}{E[(1+X^{(n-\nu)})^\varphi]}$$

and thence to P_ν^* on passage to extremal limit.

References

Albrecht, P. (1992), Premium calculation without arbitrage, *ASTIN Bulletin*, **22**, 247-256

Arrow, K.J. (1971) Theory of risk aversion in *Essays on the theory of risk bearing*, 90-120, Markham, Chicago

Bowers, N.L. Gerber, H.U. Hickman, J.C. Jones, D.A. and Nesbitt, C.J. (1986) *Risk Theory* Society of Actuaries, USA

David, H.A. and Nagaraja, H.N. (2003) *Order Statistics*, 3rd Edition, J.Wiley

Hill, B.M. (1975), A simple general approach to inference about the tail of a distribution, *Ann. Statist*. **3**, 1163-1174

Feller, W. (1970) An Introduction to Probability Theory and Its Applications, 2nd Edn. J. Wiley

Holmes, G. and Sugden, A. (1990), *Interpreting company reports and accounts*, 3rd Edition, Woodhead-Faulker

Mikosch, T. (1997), Heavy-tailed modelling in insurance, *Commun. Statist. - Stochastic models*, **13**, 799-815

Pratt, J.W. (1964) Risk aversion in the small and in the large, *Econometrica,* **32**, 122-136.

Rolski, T, Schmidli, H. Schmidt, V. and Teugels, J. (1999), *Stochastic processes for insurance and finance,* Wiley