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Abstract: In most finance papers and textbooks mean-variance preferences are usually introduced and motivated as a special case of expected utility theory. In general, the two sufficient conditions to allow this are either quadratic preferences with an arbitrary distribution of stochastic assets, or arbitrary preferences with Normally distributed assets. In the first case, the specific functional form of mean-variance preferences follows naturally. In the second case, the only specific functional form usually provided is the case of negative exponential preferences. In this note, the specific functional form for mean-variance preferences is derived for the much more realistic example of lognormally distributed assets, and constant relative risk aversion (CRRA) preferences.

Keywords: Mean-variance preferences; expected utility; lognormal assets; risk aversion.

JEL Classification: D81, G11.

1. Introduction

Mean-variance preferences are usually introduced and motivated by an argument something like the following. Let A_0 represent (non-random) initial assets and let A represent (random) end of period assets. Under certain axioms there exists a von-Neumann-Morgenstern utility function $\tilde{U}(A)$ such that, when evaluating uncertain prospects, the decision maker acts as if he/she maximizes $E\left[\tilde{U}(A)\right]$. It is usual to assume that $\tilde{U}(\ \ )$ is:

non-decreasing i.e. $\tilde{U}'(A) \geq 0$

concave i.e. $\tilde{U}''(A) \leq 0$

so the decision maker prefers higher wealth to less, but is averse to risk.

Associated with $\tilde{U}(A)$ are two measures of risk aversion (Arrow-Pratt):

Absolute Risk Aversion $\quad ARA(A) = -\frac{\tilde{U}''(A)}{\tilde{U}'(A)}$

Relative Risk Aversion $\quad RRA(A) = -\frac{\tilde{U}''(A).A}{\tilde{U}'(A)}$

and recall that behaviour toward risk is preserved under linear transformations of $\tilde{U}(\ \ )$ (because of the linearity of the expectation operator). In general, it is argued that *ARA* should be decreasing with the level of assets, but *RRA* could be close to constant, with reasonable values satisfying $0 \le RRA \le 6$. (see, for example, Lengwiler (2004))

Three simple but popular examples used to illustrate preferences based on von-Neumann-Morgenstern utility functions are:

(a) $\tilde{U}(\ \ )$ quadratic:

$$\tilde{U}(A) = aA - \frac{b}{2}A^2 \qquad (a, b > 0,\ A < \frac{a}{b}).$$

In this case $ARA = \frac{b}{a - bA}$ and $RRA = \frac{bA}{a - bA}$ and increasing *ARA* would seem to be counterintuitive.

(b) $\tilde{U}(\ \ )$ negative exponential:

$$\tilde{U}(A) = \gamma - \delta e^{-\eta A} \qquad \gamma, \delta, \eta, > 0.$$

In this case $ARA = \eta$ and $RRA = \eta A$, and constant *ARA* appears counterintuitive. Note that the parameters δ and γ are redundant, being the parameters of a positive linear transformation.

(c) $\tilde{U}(\ \ )$ Constant Relative Risk Aversion (CRRA):

$$\tilde{U}(A) = \frac{A^{1-\gamma}}{1-\gamma} \qquad \gamma > 0.$$

In this case $ARA = \gamma / A$ and $RRA = \gamma$, and decreasing *ARA*, but constant *RRA*, are appealing as an illustrative example.

It is then usually argued that in two special cases the expected value of the von-Neumann-Morgenstern utility function can be replaced by a function of mean and variance alone, the mean-variance function, i.e. replace $E[\tilde{U}(A)]$ by $U(\mu_A, \sigma_A^2)$, $U_1 > 0,\ U_2 < 0$.

U_1 denotes the partial derivative with respect to μ_A, and U_2 the partial derivative with respect to σ_A^2. These two special cases are:

(i) For <u>arbitrary</u> probability distributions on A, let $\tilde{U}(\ \ )$ be quadratic:

$$\tilde{U}(A) = aA - \frac{b}{2}A^2; \quad a, b > 0, A < \frac{a}{b}$$

Then $E\left[\tilde{U}(A)\right] = aE(A) - \frac{b}{2}E(A^2)$

$$= a\mu_A - \frac{b}{2}(\mu_A^2 + \sigma_A^2)$$

$$= U(\mu_A, \sigma_A^2)$$

which is clearly a mean-variance function, quadratic in mean and linear in variance. In this case the existence of a mean-variance function follows by construction. But as noted above, the underlying quadratic von-Neumann-Morgenstern utility function is unattractive, and thus this mean-variance function is also unattractive.

(ii) For arbitrary <u>preferences</u>, assume $A \sim N(\mu_A, \sigma_A^2)$ which, being Normally distributed, is characterised solely by its mean and variance. Hence it must be the case that $E[\tilde{U}(A)] = U(\mu_A, \sigma_A^2)$, an implication of Normality. However, the explicit functional form of $U(\mu_A, \sigma_A^2)$ will not in general be known. An exception is the popular illustrative example of function (b) above. Assume $\tilde{U}(\ \ )$ is negative exponential, $\tilde{U}(A) = -e^{-\eta A}$, and A is Normally distributed, $A \sim N(\mu_A, \sigma_A^2)$ .Appealing to results on the lognormal distribution, if $X \sim N(\mu_X, \sigma_X^2)$ and $Y = e^X$, then Y is lognormally distributed (i.e. the natural logarithm of Y is Normal) and $\mu_Y = e^{\mu_X + \frac{1}{2}\sigma_X^2}$. Note that this implies that $Y \geq 0$, so this is an attractive distribution to model asset prices. In the negative exponential example
$-\eta A \sim N(-\eta\mu_A, \eta^2\sigma_A^2)$

and hence by this basic result

$$E\left[\tilde{U}(A)\right] = -E(e^{-\eta A})$$

$$= -e^{[-\eta\mu_A + \frac{1}{2}\eta^2\sigma_A^2]} = -e^{-\eta[\mu_A - \frac{1}{2}\eta\sigma_A^2]}.$$

Hence, by monotonicity, maximizing $E\left[\tilde{U}(A)\right]$ is equivalent to maximizing the function

$$U(\mu_A, \sigma_A^2) = \mu_A - \frac{\eta}{2}\sigma_A^2$$

with respect to mean and variance. Note that this function is linear in mean and variance. But again,as noted above, the underlying negative exponential von-Neumann-Morgenstern utility function is unattractive, and thus this mean-variance function is also an unattractive illustrative example.

In the above, preferences over risky outcomes have been motivated in terms of preferences over the uncertain levels of *assets*. In portfolio analysis preferences are often expressed in terms of the uncertain levels of *returns* on the portfolio. Given the fixed initial level of assets A_0, the two ideas are equivalent, provided that returns are expressed as ordinary compound returns. Assume that the uncertainty is over the return R on a given portfolio. Then $A = A_0(1+R)$ for A_0 fixed, and hence

$$\mu_A = A_0(1+\mu_R) = A_0 + A_0\mu_R$$

$$\sigma_A^2 = A_0^2\sigma_R^2$$

and so means and variances of assets are positively linearly related to the means and variances of returns, and it is easy to translate mean-variance preferences over assets to mean -variance preferences over compound returns (plus the initial level of assets A_0). This is not the case for continuously compounded returns, which are used below.

2. A New Result

A third, much more realistic, example that does not seem to appear in standard text books or journal articles, is the following. Assume that the von-Neumann-Morgenstern utility function is example (c) above, CRRA, and that it is the continuously compounding rate of return, *r*, that is Normally distributed. (CRRA preferences are used for von-Neumann-Morgenstern preferences in Courakis (1989), but the corresponding mean-variance function is derived as an approximation using a quadratic approximation.) In the derivation, more general results from the lognormal distribution will be used. These results are that, if $X \sim N(\mu_X, \sigma_X^2)$ and $Y = e^X$, then

$$\mu_Y = e^{\mu_X + \frac{1}{2}\sigma_X^2}; \sigma_Y^2 = \left(e^{\sigma_X^2} - 1\right)e^{2\mu_X + \frac{1}{2}\sigma_X^2}$$

and, conversely, that

$$\sigma_X^2 = \ln\left(1 + \frac{\sigma_Y^2}{\mu_Y^2}\right)$$

$$\mu_X = \ln\mu_Y - \frac{1}{2}\sigma_X^2 = \ln\mu_Y - \frac{1}{2}\ln\left(1+\frac{\sigma_Y^2}{\mu_Y^2}\right).$$

Now consider the CRRA $\tilde{U}(A) = \frac{A^{1-\gamma}}{1-\gamma}$ and set $A = A_0(1+R) = A_0 e^r$ with $r \sim N\left(\mu_r, \sigma_r^2\right)$. Then

$$\begin{aligned}\tilde{U}(A) &= \frac{A^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma}e^{(1-\gamma)\ln A} \\ &= \frac{1}{1-\gamma}e^{(1-\gamma)\ln A_0 + (1-\gamma)r} \\ &= Ke^{(1-\gamma)r} \quad \text{with } K = \frac{1}{1-\gamma}e^{(1-\gamma)\ln A_0}.\end{aligned}$$

Since $(1-\gamma)r$ is also Normal, $\tilde{U}(A)$ is scaled lognormal, and hence $E\left[\tilde{U}(A)\right]$ has the closed form expression

$$E\left[\tilde{U}(A)\right] = Ke^{(1-\gamma)\mu_r + \frac{1}{2}(1-\gamma)^2\sigma_r^2} \tag{1}$$

a function of the parameter of the CRRA utility function and the mean and variance of r. Interestingly, in this case, the slope of expected utility with respect to mean return is positive, but the slope with respect to variance of return is positive or negative according to whether $\gamma < 1$ or $\gamma > 1$. This is because the variance of *r* directly affects the mean <u>and</u> the variance of *A*, and illustrates the fact noted above that mean-variance analysis should be expressed in terms of the parameters of the distribution of *A*., and this can be also expressed in terms of returns only in the ordinary compound return case. Thus the above result (1) is *not* a mean-variance representation. A simple transformation allows the application of the extended lognormal results. Define the gross rate of return $Y = (1+R) = e^r$, and since $A = A_0(1+R) = A_0 Y = A_0 e^r$ then $\mu_A = A_0\mu_Y; \sigma_A^2 = A_0^2\sigma_Y^2$, and hence

$$\sigma_r^2 = \ln\left(1+\frac{\sigma_A^2 / A_0^2}{\mu_A^2 / A_0^2}\right) = \ln\left(1+\frac{\sigma_A^2}{\mu_A^2}\right)$$

$$\mu_r = \ln\mu_Y - \ln A_0 - \frac{1}{2}\ln\left(1+\frac{\sigma_A^2}{\mu_A^2}\right).$$

Substituting these in the expression for expected utility then gives

(2) $$E\left[\tilde{U}(A)\right]=\frac{1}{1-\gamma}e^{(1-\gamma)\ln\mu_A+\frac{1}{2}\gamma(1-\gamma)\ln\left(1+\frac{\sigma_A^2}{\mu_A^2}\right)}=U(\mu_A,\sigma_A^2)$$

which is a closed form, and a much more realistic, illustrative (or even empirically applicable) example of a mean-variance utility function. It can be seen that the slope of this function with respect to μ_A is positive, and the slope with respect to σ_A^2 is negative, consistent with the usual assumptions of mean-variance preferences. (Although this is not necessary – see Bigelow (1993), Hadar and Russell(1969), Meyer (1987).) In terms of the ordinary compound return *R* the corresponding expression is

$$E\left[\tilde{U}(A)\right]=\frac{1}{1-\gamma}e^{(1-\gamma)\ln(A_0+A_0\mu_R)+\frac{1}{2}\gamma(1-\gamma)\ln\left(1+\frac{\sigma_R^2}{1+2\mu_R+\mu_R^2}\right)}=U(\mu_R,\sigma_R^2)\,.$$

3. Conclusion

The closed form mean-variance expression (2) is based on a reasonably realistic CRRA von-Neumann Morgenstern utility function, and exploits the properties of the Normal distribution by associating Normality more realistically with the continuously compounded rate of return, rather than with the actual distribution of assets. Thus it is a far more appealing specific example of a mean-variance preference function derived from maximizing behaviour than the examples based on either quadratic preferences or negative exponential preferences.

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