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Bias Reduction of Long Memory Parameter Estimators via the Pre-filtered Sieve Bootstrap*

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Abstract

This paper investigates the use of bootstrap-based bias correction of semi-parametric estimators of the long memory parameter in fractionally integrated processes. The re-sampling method involves the application of the sieve bootstrap to data pre-filtered by a preliminary semi-parametric estimate of the long memory parameter. Theoretical justification for using the bootstrap techniques to bias adjust log-periodogram and semi-parametric local Whittle estimators of the memory parameter is provided. Simulation evidence comparing the performance of the bootstrap bias correction with analytical bias correction techniques is also presented. The bootstrap method is shown to produce notable bias reductions, in particular when applied to an estimator for which analytical adjustments have already been used. The empirical coverage of confidence intervals based on the bias-adjusted estimators is very close to the nominal, for a reasonably large sample size, more so than for the comparable analytically adjusted estimators. The precision of inferences (as measured by interval length) is also greater when the bootstrap is used to bias correct rather than analytical adjustments.

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1 Introduction

The so-called long memory, or strongly dependent, processes have come to play an important role in time series analysis. Long range dependence, observed in a very wide range of empirical applications, is characterized by an autocovariance structure that decays too slowly to be absolutely summable. Specifically, rather than the autocovariance function declining at the exponential rate characteristic of a stable and invertible *ARMA* process, it declines at a hyperbolic rate dependent on a “long memory” parameter $\alpha \in (0, 1)$; i.e.,

$$\gamma(\tau) \sim C\tau^{-\alpha}, C \neq 0, \text{ as } \tau \rightarrow \infty. \quad (1.1)$$

A detailed description of the properties of such processes can be found in [Beran \(1994\)](#).

Perhaps the most popular model of a long memory process is the fractionally integrated ($I(d)$) process introduced by [Granger and Joyeux \(1980\)](#) and [Hosking \(1980\)](#). This class of processes can be characterized by the specification,

$$y(t) = \sum_{j=0}^{\infty} k(j)\varepsilon(t-j) = \frac{\kappa(z)}{(1-z)^d} \varepsilon(t), \quad (1.2)$$

where $\varepsilon(t)$ is zero-mean white noise, z is here interpreted as the lag operator ($z^j y(t) = y(t-j)$), and $\kappa(z) = \sum_{j \geq 0} \kappa(j)z^j$, $\kappa(0) = 1$. For any $d > -1$ the operator $(1-z)^d$ is defined via the binomial expansion

$$(1-z)^d = 1 - dz + \frac{d(d-1)z^2}{2!} - \frac{d(d-1)(d-2)z^3}{3!} + \dots, \quad (1.3)$$

and if the “short memory” component $\kappa(z)$ is the transfer function of a stable, invertible ARMA process and $|d| < 0.5$, then the coefficients of $k(z)$ are square-summable ($\sum_{j \geq 0} |k(j)|^2 < \infty$). In this case $y(t)$ is well-defined as the limit in mean square of a covariance-stationary process and the model is essentially a generalization of the classic Box-Jenkins ARIMA model ([Box and Jenkins, 1970](#)),

$$(1-z)^d \Phi(z)y(t) = \Theta(z)\varepsilon(t), \quad (1.4)$$

in which we now allow non-integer values of the integrating parameter d and $\kappa(z) = \Theta(z)/\Phi(z)$. The long run behaviour of this process naturally depends on the fractional integration parameter d . In particular, for any $d > 0$ the impulse response coefficients of the Wold representation in (1.2) are not absolutely summable and, for $0 < d < 0.5$, the autocovariances decline at the rate $\gamma(\tau) \sim C\tau^{2d-1}$ (i.e. with reference to (1.1), $\alpha = 1 - 2d$). Such processes have been found to exhibit dynamic behaviour very similar to that observed in many empirical time series. See [Robinson \(2003\)](#) for a collection of the seminal articles in the area and [Doukhan, Oppenheim and Taqqu \(2003\)](#)

for a thorough review of theory and applications. The role played by fractional processes in finance, most notably in the modelling of the variance of financial returns, is highlighted in [Andersen, Bollerslev, Christoffersen and Diebold \(2006\)](#) and in multiple papers published in a special issue of *Econometric Reviews* (2008, 27, Issue 1-3).

Statistical procedures for analyzing long memory processes have ranged from the likelihood-based methods of [Fox and Taqqu \(1986\)](#), [Dahlhaus \(1989\)](#), [Sowell \(1992\)](#) and [Beran \(1995\)](#), to the semi-parametric techniques advanced by [Geweke and Porter-Hudak \(1983\)](#) and [Robinson \(1995a,b\)](#), among others. The asymptotic theory for maximum likelihood estimation (MLE) of the parameters of such processes is well established, at least under the assumption of Gaussian errors. In particular, we have consistency, asymptotic efficiency, and asymptotic normality for the MLE of the fractional differencing parameter, so providing a basis for large sample inference in the usual manner. Such asymptotic results are, however, conditional on correct model specification, with the MLE of d typically inconsistent if either or both the autoregressive and moving average operators in (1.4) (or, alternatively, the operator $\kappa(z)$ in (1.2)) are incorrectly specified. The semi-parametric methods aim to produce consistent estimators of d whilst placing only very mild restrictions on the behaviour of $\kappa(e^{i\lambda})$ for frequency values λ near zero. The semi-parametric estimators are therefore robust to different forms of short run dynamics and offer broader applicability than a fully parametric method. They are also asymptotically pivotal and have particularly simple asymptotic normal distributions.

Whilst such features place the semi-parametric methods at the forefront for use in conducting inference on d , the price paid for their application is a reduction in asymptotic efficiency (relative to exact MLE) and a slower rate of convergence to the true parameter ([Giraitis, Robinson and Samarov, 1997](#)). Also, despite asymptotic robustness to the short run dynamics, semi-parametric estimators have been shown to exhibit large finite sample bias in the presence (in particular) of a substantial autoregressive component – see [Agiakloglou, Newbold and Wohar \(1993\)](#) and [Lieberman \(2001\)](#) for example. Hence, bias correction of semi-parametric estimators is an important area to explore.

In this paper we focus on bias correction of the following two semi-parametric estimators \hat{d}_T of d :

1. The [Geweke and Porter-Hudak \(1983\)/Robinson \(1995b\)](#) log-periodogram regression estimator (referred to hereafter as LPR): The ordinary least squares (OLS) slope coefficient in a regression of $\log I_T(\lambda_j)$ on a constant and $-2 \log \lambda_j$, $j = 1, \dots, N$,

$$\hat{d}_T = \arg \min_{|d| < \frac{1}{2}} \sum_{j=1}^N (\log I_T(\lambda_j) + 2d(\log \lambda_j - \overline{\log \lambda}))^2,$$

where $I_T(\lambda) = (2\pi T)^{-1} |\sum_{t=1}^T y(t)e^{-i\lambda t}|^2$, the periodogram, $\lambda_j = 2\pi j/T$, $j = 1, \dots, N$, are the first N fundamental frequencies, and $\overline{\log \lambda} = N^{-1} \sum_{j=1}^N \log \lambda_j$.¹

2. The semi-parametric Gaussian (local Whittle) estimator of [Robinson \(1995a\)](#) (SPLW):

$$\hat{d}_T = \arg \min_{|d| < \frac{1}{2}} \left(\log(N^{-1} \sum_{j=1}^N \lambda_j^{2d} I_T(\lambda_j)) - 2d \overline{\log \lambda} \right).$$

Both \hat{d}_T are \sqrt{N} -CAN estimators of d , by which we mean that \hat{d}_T is consistent ($|\hat{d}_T - d| = o_p(1)$) and asymptotically normal, $N^{\frac{1}{2}}(\hat{d}_T - d)/v \xrightarrow{D} G(x)$, where $G(x)$ denotes the standard normal cumulative distribution function. For the LPR estimator $v^2 = \pi/24$, and $v^2 = 1/4$ for the SPLW estimator. For both estimators the bandwidth parameter N , denoting the number of periodogram ordinates employed, is chosen as a monotonically increasing function of sample size T , and because $\kappa(z)$ is only specified locally, N must be assigned such that $N/T \rightarrow 0$ as $T \rightarrow \infty$. Too small a choice of N may prompt concern about the accuracy of the normal approximation, whereas too large a value for N entails an element of non-local averaging and is a source of bias. In brief, although $\lim_{T \rightarrow \infty} E[\hat{d}_T - d] = 0$ the finite sample bias in such estimators can, as previously observed, present problems.

One approach to the problem of bias is to seek an analytical solution that will reduce the first-order bias. [Moulines and Soulier \(1999\)](#), for example, reduce bias by fitting a finite number of Fourier coefficients to the logarithm of the short memory spectrum and constructing a broad-band LPR estimator of d that uses all of the frequencies in the range $(0, \pi]$, not just those in a neighborhood of zero. [Andrews and Guggenberger \(2003\)](#) consider a bias-adjusted estimator of d obtained by including even powers of frequency as additional regressors in the log-periodogram pseudo regression defined in 1. above, and [Andrews and Sun \(2004\)](#) adapt this approach to the SPLW estimator. Monte-Carlo evidence presented in [Nielsen and Frederiksen \(2005\)](#) demonstrates the usefulness of these bias-adjusted versions of the LPR and SPLW estimators. In particular, the bias-corrected semi-parametric estimators are shown to outperform correctly specified parametric estimators, although at the expense of an increase in mean squared error.

An alternative methodological approach to bias correction, and the one that we examine here, is to use the bootstrap. Bootstrap methodology may be thought of as coming in two ‘‘flavours’’: the parametric, or model-based, bootstrap, and a variety of non- or semi-parametric schemes. The parametric bootstrap relies on having a full, correct parametric specification for the process and is therefore at odds with the semi-parametric approach to estimation being considered here. A less model-dependent

¹ We have written the estimator in the form given by [Robinson \(1995b\)](#). [Geweke and Porter-Hudak \(1983\)](#) use the regressor $-2 \log |1 - e^{-i\lambda}|$. The two are equivalent because $|1 - e^{-i\lambda}|^{2d} = |\lambda|^{2d}(1 + o(1))$ as $\lambda \rightarrow 0$.

approach nonetheless requires a re-sampling scheme that is able to capture the salient features of the data generating process, the dependence structure of the process being of prime importance in the time series context. While the block bootstrap of [Künsch \(1989\)](#) has traditionally been employed for this purpose, blocking has been found to suffer from relatively poor convergence rates. For instance, the error in the coverage probability of a one-sided confidence interval derived from the block bootstrap is $O(T^{-3/4})$, compared to the $O(T^{-1})$ rate achieved with simple random samples. An attractive alternative is the “sieve” bootstrap. This works by “pre-whitening” the data using an autoregressive approximation, with the dynamics of the process captured in a fitted autoregression (See [Politis, 2003](#)). Provided the order, h , of the autoregression increases at a suitable rate with T , the convergence rates for the sieve bootstrap are much closer (in fact arbitrarily close) to those for simple random samples. [Choi and Hall \(2000\)](#) demonstrate the superior convergence performance of the sieve bootstrap (over the block bootstrap) for linear short memory processes, whilst [Poskitt, Grose and Martin \(2013\)](#) build on the results of [Poskitt \(2008\)](#) to show that under regularity conditions that allow for fractionally integrated $I(d)$ processes, the sieve bootstrap achieves an error rate of $O(T^{-(1-\max\{0,d\})+\beta})$ for all $\beta > 0$, for a general class of statistics.

The current paper uses a modified sieve bootstrap to bias-correct the LPR and SPLW estimators of the memory parameter in fractionally integrated $I(d)$ processes. The bootstrap method uses a consistent semi-parametric estimator of the long memory parameter to pre-filter the raw data, *prior to* the use of a long autoregressive approximation as the ‘sieve’ from which bootstrap samples are produced. The bias correction proceeds in an iterative fashion, with a stochastic stopping rule invoked to produce the final estimator. Starting with \sqrt{N} -CAN estimators that satisfy a requisite Edgeworth expansion and large-deviation properties we derive error rates for estimating the bias of both the LPR and SPLW estimators, with the accuracy with which the bootstrap method estimates the bias in finite samples then documented in a simulation setting. We also use the bootstrap method to bias-adjust the (already) analytically bias-adjusted versions of these two estimators. The analytically bias-adjusted LPR estimator of [Andrews and Guggenberger \(2003\)](#) (LPR-BA hereafter) is produced as the OLS coefficient of the regressor $-2 \log \lambda_j$ in the regression of $\log I_T(\lambda_j)$ on a constant, $-2 \log \lambda_j$, and λ_j^{2p} , $p = 1, \dots, P$, $j = 1, \dots, N$. The analytically bias-adjusted SPLW estimator of [Andrews and Sun \(2004\)](#) (SPLW-BA hereafter) is produced as the first

element of $(\widehat{d}_T, \widehat{\theta}_1, \dots, \widehat{\theta}_P) = \arg \min LW(d, \theta_1, \dots, \theta_P)$ where

$$LW(d, \theta_1, \dots, \theta_P) = \log \left(N^{-1} \sum_{j=1}^N \lambda_j^{2d} I_T(\lambda_j) \exp \left\{ \sum_{p=1}^P \theta_p \lambda_j^{2p} \right\} \right) - N^{-1} \sum_{j=1}^N \left\{ \sum_{p=1}^P \theta_p \lambda_j^{2p} \right\} - 2d \overline{\log \lambda}.$$

The paper proceeds as follows. Section 2 briefly summarizes the statistical properties of long memory processes, and outlines the sieve bootstrap (both ‘raw’ and pre-filtered) in this context. The pre-filtered sieve bootstrap bias adjustment (PFSA(BA)) algorithm is also described in this section. In Section 3 we present the relevant approximations and exploit these to produce the error rates for the bootstrap technique. Section 4 outlines the iterated version of the bootstrap bias correction technique. Details of the simulation study are given in Section 5. Section 6 closes the paper.

2 Long memory processes, autoregressive approximation, and the sieve bootstrap

Let $y(t)$ for $t \in \mathbb{Z}$ denote a linearly regular, covariance-stationary process, with representation as in (1.2) where;

Assumption 1 *The transfer function in the representation (1.2) is given by $k(z) = \kappa(z)/(1-z)^d$ where $|d| < 0.5$ and $\kappa(z) \neq 0$, $|z| \leq 1$. The impulse response coefficients of $\kappa(z)$ satisfy $k(0) = 1$ and $\sum_{j \geq 0} j |\kappa(j)| < \infty$.*

Assumption 2 *The innovations process $\varepsilon(t)$ is an i.i.d. zero mean Gaussian white noise process with variance σ^2 .*

Assumption 1 serves to characterize the spectral features of quite a wide class of processes, including the ARFIMA family of models that are the focus of this paper. This assumption implies that the innovations in the unilateral representation in (1.2) are *fundamental*; meaning that $\varepsilon(t)$ lies in the space spanned by current and past values of $y(t)$, and $\varepsilon(t)$ and $y(s)$ are uncorrelated for all $s < t$. For a discussion of the role of fundamentality in the context of the autoregressive sieve bootstrap see Kreiss, Paparoditis and Politis (2011). Note that the regularity conditions employed in Kreiss et al. (2011) exclude fractional time series, but using the extension of Baxter’s inequality to long range dependent processes due to Inoue and Kasahara (2006) it is possible to generalize the results of Kreiss et al. (2011) to time series generated from a fractional transformation of a linear processes. In particular, since the statistics that we investigate are asymptotically pivotal the results in Kreiss et al. (2011, Section 3) can be extended to the statistics and class of processes under consideration here.

Assumption 2 implies that $y(t)$ is a Gaussian process. A basic property of such a process that underlies the sieve bootstrap methodology and the associated results is that $y(t)$ is linearly regular and the linear predictor

$$\bar{y}(t) = \sum_{j=1}^{\infty} \pi(j)y(t-j), \quad (2.1)$$

where $\sum_{j=0}^{\infty} \pi(j)z^j = (1-z)^d \kappa(z)^{-1}$, is the minimum mean squared error (MMSE) predictor (MMSEP) of $y(t)$ based upon its entire past. The need to invoke Gaussianity is unfortunate but is unavoidable here as we wish employ certain results from the existing literature where the assumption that $y(t)$ is a Gaussian process is adopted. It is likely that our results can be extended to more general linear processes, although the regularity conditions and prerequisites needed for such extensions are liable to be relatively involved.

The MMSEP of $y(t)$ based only on a finite number (h) of past observations (MMSEP(h)) is then

$$\bar{y}_h(t) = \sum_{j=1}^h \pi_h(j)y(t-j) \equiv - \sum_{j=1}^h \phi_h(j)y(t-j); \quad (2.2)$$

where the minor reparameterization from π_h to ϕ_h allows us, on also defining $\phi_h(0) = 1$, to conveniently write the corresponding prediction error as

$$\varepsilon_h(t) = \sum_{j=0}^h \phi_h(j)y(t-j). \quad (2.3)$$

The finite-order autoregressive coefficients $\phi_h(1), \dots, \phi_h(h)$ can be deduced from the Yule-Walker equations

$$\sum_{j=0}^h \phi_h(j)\gamma(j-k) = \delta_0(k)\sigma_h^2, \quad k = 0, 1, \dots, h, \quad (2.4)$$

in which $\gamma(\tau) = \gamma(-\tau) = E[y(t)y(t-\tau)]$, $\tau = 0, 1, \dots$, is the autocovariance function of the process $y(t)$, $\delta_0(k)$ is Kronecker's delta (i.e., $\delta_0(k) = 0 \forall k \neq 0$; $\delta_0(0) = 1$), and the MMSE is

$$\sigma_h^2 = E[\varepsilon_h(t)^2], \quad (2.5)$$

the prediction error variance associated with $\bar{y}_h(t)$.

The use of finite-order AR models to approximate an unknown (but suitably regular) process therefore requires that the optimal predictor $\bar{y}_h(t)$ determined from the autoregressive model of order h be a good approximation to the "infinite-order" predictor $\bar{y}(t)$ for sufficiently large h . The asymptotic validity, and properties, of finite-order autoregressive models when $h \rightarrow \infty$ with the sample size T under regularity conditions

that admit non-summable processes was proved in [Poskitt \(2007\)](#). Briefly, the order- h prediction error $\varepsilon_h(t)$ converges to $\varepsilon(t)$ in mean-square, the estimated sample-based covariances converge to their population counterparts (though at a slower rate than for a conventional $I(0)$ stationary process) and the ordinary least squares (Least Squares) and Yule-Walker estimators of the coefficients of the approximating autoregression are asymptotically equivalent and consistent. Furthermore, order selection by AIC, a commonly employed practice in the context of the sieve bootstrap ([Politis, 2003, §3](#)), is asymptotically efficient in the sense of minimizing Shibata's (1980) figure of merit. The sieve bootstrap, with order selected via an asymptotically efficient criterion, is accordingly a plausible “non-parametric” bootstrap technique for long memory processes.

2.1 The raw sieve bootstrap

Details of the raw sieve bootstrap (SB) for fractional processes are given in [Poskitt \(2008\)](#). For convenience we reproduce here the basic steps of the SB algorithm for generating a realization of a process $y(t)$, prior to presenting the PFSB(BA) algorithm adopted for bias-adjustment in this paper.

The raw sieve bootstrap (SB) algorithm:

SB1. Given data $y(t)$, $t = 1, \dots, T$, and using $y(1-j) = y(T-j+1)$, $j = 1, \dots, h$, as initial values, calculate parameter estimates of the $AR(h)$ approximation, denoted by $\bar{\phi}_h(1), \dots, \bar{\phi}_h(h)$, and evaluate the residuals

$$\bar{\varepsilon}_h(t) = \sum_{j=0}^h \bar{\phi}_h(j) y(t-j), \quad t = 1, \dots, T,$$

From $\bar{\varepsilon}_h(t)$, $t = 1, \dots, T$, construct the standardized residuals $\tilde{\varepsilon}_h(t) = (\bar{\varepsilon}_h(t) - \bar{\varepsilon}_h) / \bar{\sigma}_h$ where $\bar{\varepsilon}_h = T^{-1} \sum_{t=1}^T \bar{\varepsilon}_h(t)$ and $\bar{\sigma}_h^2 = T^{-1} \sum_{t=1}^T (\bar{\varepsilon}_h(t) - \bar{\varepsilon}_h)^2$.

SB2. Let $\varepsilon_h^+(t)$, $t = 1, \dots, T$, denote a simple random sample of *i.i.d.* values drawn from

$$U_{\tilde{\varepsilon}_h, T}(e) = T^{-1} \sum_{t=1}^T \mathbf{1}\{\tilde{\varepsilon}_h(t) \leq e\},$$

the probability distribution function that places a probability mass of $1/T$ at each of $\tilde{\varepsilon}_h(t)$, $t = 1, \dots, T$. Set $\varepsilon_h^*(t) = \bar{\sigma}_h \varepsilon_h^+(t)$, $t = 1, \dots, T$.

SB3. Construct the sieve bootstrap realization $y^*(1), \dots, y^*(T)$, where $y^*(t)$ is generated from the autoregressive process

$$\sum_{j=0}^h \bar{\phi}_h(j) y^*(t-j) = \varepsilon_h^*(t), \quad t = 1, \dots, T,$$

initiated at $y^*(1-j) = y(\tau - j + 1)$, $j = 1, \dots, h$, where τ is a discrete uniform random variable with support on the integers h, \dots, T .

Crucially, the rate of convergence of the coefficient estimates $\bar{\phi}_h(1), \dots, \bar{\phi}_h(h)$ evaluated in Step SB1 is dependent upon the value of the fractional index d , as formalized in the following theorem

Theorem 2.1 *Let $\sum_{j=0}^h \bar{\phi}_h(j)z^j$ denote the Burg, Least Squares or Yule-Walker estimator of $\sum_{j=0}^h \phi_h(j)z^j$. If $y(t)$ is a linearly regular, covariance-stationary process that satisfies Assumptions 1 and 2, then for all $h \leq H_T = a(\log T)^c$, $a > 0$, $c < \infty$,*

$$\sum_{j=1}^h |\bar{\phi}_h(j) - \phi_h(j)|^2 = O\left(h \left(\frac{\log T}{T}\right)^{1-2\max\{0,d\}}\right)$$

with probability one.

The proof of this theorem is placed in Appendix A, along with the proofs of other results presented in the paper.

2.2 The pre-filtered sieve bootstrap

Given the dependence of the convergence of $\sum_{j=0}^h \bar{\phi}_h(j)z^j$ to $\sum_{j=0}^h \phi_h(j)z^j$ on the value of d , the convergence of any bootstrap generated sampling distribution to the true unknown sampling distribution is also dependent on the value of d , see [Poskitt \(2008\)](#). In particular, in [Poskitt et al. \(2013\)](#) it is shown that under appropriate regularity the raw sieve bootstrap achieves a convergence rate of $O(T^{-(1-\max\{0,d\})+\beta})$ for all $\beta > 0$. Obviously, in the long memory case where $0 < d < 0.5$, the closer is d to zero the closer the convergence rate of $O(T^{-(1-d)+\beta})$ will be to the rate $O(T^{-1+\beta})$ achieved with short memory (and anti-persistent) processes. The empirical regularity of estimated values of d in the $0 < d < 0.5$ range thus provides motivation for the idea of pre-filtering the series prior to the application of the sieve. Specifically, we employ a modified sieve method wherein, for a given preliminary value of d , we pre-filter the data using this value, apply the AR approximation (and sieve bootstrap) to the pre-filtered data, before using the inverse filter to produce the final realization of $y(t)$. With this procedure, the raw sieve is applied (by construction) to filtered data with shorter memory; hence the achievement of an improved convergence rate.

For any $d > -1$ let $(1-z)^d = \sum_{j=0}^{\infty} \alpha_j^{(d)} z^j$ where $\alpha_j^{(d)}$, $j = 0, 1, 2, \dots$, denote the coefficients of the fractional difference operator when expressed in terms of its binomial expansion, as in the right hand side of [1.3](#). Given a preliminary value d^f of d , pre-filtered sieve bootstrap (PFSB) realizations of $y(t)$ are generated using the following algorithm:

PFBS1. Calculate the coefficients of the filter $(1 - z)^{d^f}$ and from the data generate the filtered values

$$w^f(t) = \sum_{j=0}^{t-1} \alpha_j^{(d^f)} y(t - j)$$

for $t = 1, \dots, T$.

PFBS2. Fit an AR approximation to $w^f(t)$ and generate a sieve bootstrap sample $w^{*f}(t)$, $t = 1, \dots, T$, of the filtered data as in Steps SB1–SB3 of the SB algorithm, with $w^f(t)$ and $w^{*f}(t)$ playing the role of $y(t)$ and $y^*(t)$ respectively therein.

PFBS3. Using the coefficients of the (inverse) filter $(1 - z)^{-d^f}$ construct a corresponding pre-filtered sieve bootstrap draw

$$y^{*f}(t) = \sum_{j=0}^{t-1} \alpha_j^{(-d^f)} w^{*f}(t - j)$$

of $y(t)$ for $t = 1, \dots, T$, where the superscript f is used to distinguish this bootstrap draw from the bootstrap draw produced by the raw sieve algorithm, in Step SB3 above.

In [Poskitt et al. \(2013\)](#) it is shown that given a judicious choice of d^f shorter memory will be induced by the preliminary filtering at Step PFBS1. The accuracy of the AR approximation and, thereby, the sieve bootstrap in Step PFBS2 will accordingly be increased, and this increase in accuracy will be passed on to the PFBS draws in Step PFBS3, resulting in a convergence rate equal to that obtained in the short memory case, namely $O(T^{-1+\beta})$. Using these results as motivation we proceed to work with the PFBS algorithm for the purpose of bias adjustment. More formal theoretical justification of the validity of the pre-filtered sieve when used for this particular purpose is provided in Section 3.

2.3 Bias correction via the pre-filtered sieve bootstrap

To bias adjust a chosen estimator, \widehat{d}_T , of d we proceed as follows:

BA1. Calculate \widehat{d}_T from the data $y(t)$, $t = 1, \dots, T$.

BA2. Use d^f as the preliminary value in Steps PFBS1–PFBS3 of the PFBS algorithm and produce B bootstrap realizations $y_b^{*f}(t)$, $t = 1, \dots, T$, $b = 1, \dots, B$, of the process $y(t)$. From these construct B bootstrap values of the estimator, $\widehat{d}_{T,b}^{*f}$, $b = 1, 2, \dots, B$, by evaluating the estimator \widehat{d}_T for each of the B independent bootstrap draws.

BA3. Estimate the bias of \widehat{d}_T by

$$\widehat{b}_{T,B}^{*f} = \left(\frac{1}{B} \sum_{b=1}^B \widehat{d}_{T,b}^{*f} \right) - d^f \quad (2.6)$$

and produce the bias-adjusted estimator

$$\widetilde{d}_T = \widehat{d}_T - \widehat{b}_{T,B}^{*f}. \quad (2.7)$$

We refer to this as the PFSB(BA) algorithm.

3 Some Theoretical Underpinnings

The use of the PFSB(BA) algorithm to correct the finite sample bias of an estimator is justified only if the method produces a bootstrap distribution that copies the true sampling distribution of the estimator to the appropriate order of magnitude. Not surprisingly, the rate of convergence of the bootstrap to the true sampling distribution is shown to be dependent on the proximity of the preliminary value employed in the PFSB, namely d^f , to the true value of d , as well as the order (h) of the autoregressive approximation used in the sieve component of the PFSB. Presuming that d^f is itself estimated from the data, $d^f = d_T^f$ say, the main content of these findings are presented in Theorems 3.1 and 3.2.

To begin, suppose that \widehat{d}_T (the estimator to be bias-corrected) is an asymptotically pivotal \sqrt{N} -CAN estimator of d and that the sampling distribution of $N^{\frac{1}{2}}(\widehat{d}_T - d)$ admits an Edgeworth expansion such that

$$\sup_x \left| \mathbb{P} \left\{ \frac{N^{\frac{1}{2}}(\widehat{d}_T - d)}{\nu} < x \right\} - G(x) \right| = o \left(\frac{N^{5/2}}{T^2} \right) \quad (3.1)$$

where $G(\cdot)$ denotes the standard normal distribution function. Let b_T denote the finite sample bias of \widehat{d}_T , that is,

$$b_T = E[\widehat{d}_T] - d. \quad (3.2)$$

Since $\lim_{T \rightarrow \infty} N^{\frac{1}{2}} E[\widehat{d}_T - d] = 0$ we have $b_T = o(N^{-\frac{1}{2}})$ (recall that $N \rightarrow \infty$ as $T \rightarrow \infty$ such that $N/T \rightarrow 0$). Here E denotes expectation taken with respect to the original probability space $(\Omega, \mathfrak{F}, P)$. Substituting (3.2) into (3.1) gives the approximation

$$\begin{aligned} \mathbb{P}\{N^{\frac{1}{2}}(\widehat{d}_T - E[\widehat{d}_T]) < x\} &= \mathbb{P}\{N^{\frac{1}{2}}(\widehat{d}_T - d) < x + b_T\} \\ &= G((x + N^{\frac{1}{2}}b_T)/\nu) + o(N^{-\frac{1}{2}}) \end{aligned} \quad (3.3)$$

for the distribution of the finite sample deviation $\widehat{d}_T - E[\widehat{d}_T]$.

Now let \widehat{d}_T^{*f} denote the value of \widehat{d}_T calculated from a bootstrap realization of the process, $y^{*f}(t)$, $t = 1, \dots, T$, constructed using the PFSB algorithm where; (i) the

pre-filtering value d_T^f satisfies the conditions stated above for \widehat{d}_T and, for the sake of argument; (ii) the innovations $\varepsilon_h^*(t)$, $t = 1, \dots, T$, used in Step PFBS2 are generated as i.i.d. $N(0, \bar{\sigma}_h^2)$. Since the process $\varepsilon_h^*(t)$ is now explicitly Gaussian, it follows that $y^{*f}(t)$ will be a fractionally integrated $AR(h)$ Gaussian process with parameters d_T^f and $\bar{\phi}_h(1), \dots, \bar{\phi}_h(h)$, and

$$\sup_x \left| \mathbb{P}^* \left\{ \frac{N^{\frac{1}{2}}(\widehat{d}_T^{*f} - d_T^f)}{\nu} < x \right\} - G(x) \right| = o\left(\frac{N^{5/2}}{T^2}\right) \quad (3.4)$$

where $(\Omega^*, \mathfrak{F}^*, P^*)$ denotes the probability space induced by the bootstrap process.² Denote the expectation associated with $(\Omega^*, \mathfrak{F}^*, P^*)$ by E^* . Proceeding as previously, replacing \widehat{d}_T by \widehat{d}_T^{*f} , d by d_T^f and $E[\widehat{d}_T]$ by $E^*[\widehat{d}_T^{*f}] = d_T^f + b_T^*$, with

$$b_T^* = E^*[\widehat{d}_T^{*f}] - d_T^f \quad (3.5)$$

by construction, and substiting (3.5) into (3.4) we obtain the approximation

$$\begin{aligned} \mathbb{P}^* \{ N^{\frac{1}{2}}(\widehat{d}_T^{*f} - E^*[\widehat{d}_T^{*f}]) < x \} &= \mathbb{P}^* \{ N^{\frac{1}{2}}(\widehat{d}_T^{*f} - d_T^f) < x + b_T^* \} \\ &= G((x + N^{\frac{1}{2}}b_T^*)/\nu) + o(N^{-\frac{1}{2}}) \end{aligned} \quad (3.6)$$

for the bootstrap deviation $\widehat{d}_T^{*f} - E^*[\widehat{d}_T^{*f}]$.

For a discussion of consistency and asymptotic normality of the LPR and SPLW estimators see, for example, [Hurvich, Deo and Brodsky \(1998\)](#) and [Giraitis and Robinson \(2003\)](#) respectively. [Giraitis and Robinson \(2003\)](#) also present Edgeworth expansions for the SPLW estimator. [Lieberman, Rousseau and Zucker \(2001\)](#) develop Edgeworth expansions for quadratic forms in Gaussian long memory series, and [Fay, Moulines and Soulier \(2004\)](#) provide a discussion of Edgeworth expansions in the context of linear statistics applied to long range dependent linear processes, with extensions to the LPR estimator presented in [Fay \(2010\)](#). From these references we can glean that the preceding \sqrt{N} -CAN and Edgeworth requisites require that the bandwidth parameter N be chosen such that $N \sim KT^\nu$ where $2/3 < \nu < 4/5$, $K \in (0, \infty)$. Asymptotic normality of the estimators considered here requires that $N = o(T^{4/5})$, hence the upper bound on ν . The lower bound on ν reflects that unless N increases sufficiently quickly with T terms due to bias of order $O(\log^3 N/N)$ (see (3.7) and (3.9) below) compete with more standard terms in the Edgeworth expansions.

² The innovations generated in Step PFBS2 are i.i.d. $(0, \bar{\sigma}_h^2)$ by construction (see Steps SB1–SB2 of the SB algorithm), and when $y(t)$ is Gaussian we can expect $\varepsilon_h^*(t)$, $t = 1, \dots, T$, based upon Steps SB1–SB2 to be approximately Gaussian. Replacing the innovations generated in Step PFBS2 by i.i.d. $N(0, \bar{\sigma}_h^2)$ innovations in the simulations (as would be strictly necessary to accord with the theoretical derivations) produced results that were virtually indistinguishable from those reported in Section 5 below.

Theorem 3.1 *Suppose that the process $y(t)$ satisfies Assumptions 1 and 2, and that the PFSB algorithm is applied to \widehat{d}_T using the preliminary value d_T^f and an $AR(h)$ approximation. Assume that d_T^f and \widehat{d}_T are \sqrt{N} -CAN estimators with bandwidth parameter chosen such that $N \sim KT^\nu$ where $2/3 < \nu < 4/5$, $K \in (0, \infty)$. Then for all $h \leq H_T = a(\log T)^c$, $a > 0$, $c < \infty$,*

$$\begin{aligned} & \sup_x \left| P\{N^{\frac{1}{2}}(\widehat{d}_T - E[\widehat{d}_T]) < x\} - P^*\{N^{\frac{1}{2}}(\widehat{d}_T^{*f} - E^*[\widehat{d}_T^{*f}]) < x\} \right| \\ &= O(N^{\frac{1}{2}} |b_T - b_T^*|) + o(N^{-1/2}), \end{aligned}$$

where b_T and b_T^* are as defined in (3.2) and (3.5) respectively.

Theorem 3.1 indicates that if the bandwidth of the estimators d_T^f and \widehat{d}_T is chosen appropriately then the PFSB distribution of $N^{\frac{1}{2}}(\widehat{d}_T^{*f} - E^*[\widehat{d}_T^{*f}])$ will closely approximate the true finite sampling distribution of $N^{\frac{1}{2}}(\widehat{d}_T - E[\widehat{d}_T])$ provided $N^{\frac{1}{2}}|b_T - \bar{b}_T|$ is sufficiently small. Given that $\widehat{b}_{T,B}^{*f}$ in (2.6) can be made arbitrarily close to the finite sample bias induced by the PFSB distribution by taking B sufficiently large, we can therefore anticipate that if $N^{\frac{1}{2}}|b_T - \bar{b}_T| \rightarrow 0$ sufficiently quickly then $N^{\frac{1}{2}}(\widehat{d}_T^{*f} - E^*[\widehat{d}_T^{*f}])$, with the sample mean of B bootstrap draws used to represent $E^*[\widehat{d}_T^{*f}]$, will closely approximate $N^{\frac{1}{2}}(\widehat{d}_T - E[\widehat{d}_T])$. This then provides a justification for using the PFSB(BA) algorithm to estimate the bias of \widehat{d}_T and, in turn, produce the bias-adjusted estimate.

To evaluate the magnitude of $|b_T - b_T^*|$ note that $|\kappa(e^{\lambda})|^2$ is a bounded, even function of λ , and we have the power series (McLaurin) expansion $|\kappa(e^{\lambda})|^2 = c_0 + \sum_{j \geq 1} c_j |\lambda|^{2j} = c_0 + c_1 |\lambda|^2 + o(|\lambda|^3)$ as $|\lambda| \rightarrow 0$. Then it can be shown that

$$b_T = -\beta \frac{2c_1}{9c_0} \left(\frac{N}{T}\right)^2 + o\left(\frac{N^2}{T^2}\right) + O\left(\frac{\log^3 N}{N}\right), \quad (3.7)$$

where $\beta = 1/(4\pi^2)$ for the SPLW estimator (Giraitis and Robinson, 2003) and $\beta = \pi^2$ for the LPR estimator (Hurvich et al., 1998). Similarly, set $\bar{\kappa}_h(z) = \sum_{j=0}^{\infty} \bar{\kappa}_h(j) z^j$ where the $\bar{\kappa}_h(j)$ and $\bar{\phi}_h(j)$ are related by the recursions

$$\bar{\phi}_h(0) = \bar{\kappa}_h(0) = 1, \quad \sum_{i=0}^j \bar{\kappa}_h(i) \bar{\phi}_h(j-i) = 0, \quad j = 1, 2, \dots \quad (3.8)$$

By construction $\bar{\kappa}_h(z) \bar{\phi}_h(z) = 1$ for all $|z| \leq 1$ and $\bar{\kappa}_h(z)$ yields the $AR(h)$ approximation to $\kappa(z)$ implicit in the PFSB. Then $|\bar{\kappa}_h(e^{\lambda})|^2 = |\sum_{j=0}^h \bar{\phi}_h(j) e^{\lambda j}|^{-2} = \bar{c}_0 + \bar{c}_1 |\lambda|^2 + o(|\lambda|^3)$ as $|\lambda| \rightarrow 0$ and

$$b_T^* = -\beta \frac{2\bar{c}_1}{9\bar{c}_0} \left(\frac{N}{T}\right)^2 + o\left(\frac{N^2}{T^2}\right) + O\left(\frac{\log^3 N}{N}\right). \quad (3.9)$$

Simple algebraic manipulation applied to 3.7 and 3.9 gives us the following bound

$$\begin{aligned} |b_T - b_T^*| &= \beta \frac{2}{9} \left| \frac{\bar{c}_1}{\bar{c}_0} - \frac{c_1}{c_0} \right| \left(\frac{N^2}{T^2} \right) + o \left(\frac{N^2}{T^2} \right) + O \left(\frac{\log^3 N}{N} \right) \\ &\leq \beta \frac{2}{9} \left(\left| \frac{c_1(\bar{c}_0 - c_0)}{c_0 \bar{c}_0} \right| + \left| \frac{(\bar{c}_1 - c_1)}{\bar{c}_0} \right| \right) \left(\frac{N^2}{T^2} \right) + o \left(\frac{N^2}{T^2} \right) + O \left(\frac{\log^3 N}{N} \right). \end{aligned}$$

The magnitude of $|b_T - b_T^*|$ obviously depends on the order of $(\bar{c}_0 - c_0)$ and $(\bar{c}_1 - c_1)$, and note that larger bandwidth entails larger bias and the need for more precise correction via the $AR(h)$ approximation to the short memory spectrum.

Let $\phi_h^f(z) = \sum_{j=0}^h \phi_h^f(j) z^j$ where $\phi_h^f(1), \dots, \phi_h^f(h)$ denote the coefficients in the MMSEP(h) of the process

$$w^f(t) = (1 - z)^{d^f} y(t) = \frac{\kappa(z)}{(1 - z)^{d - d^f}} \varepsilon(t),$$

and let σ_h^{f2} denote the MMSE. Set $\kappa^f(z) = \kappa(z)/(1 - z)^{d - d^f}$ and define $\kappa_h^f(z) = \{\phi_h^f(z)\}^{-1}$ by replacing the coefficients of $\bar{\phi}_h(z)$ by those of $\phi_h^f(z)$ in the recursions in equation (3.8). The magnitude of $(\bar{c}_0 - c_0)$ and $(\bar{c}_1 - c_1)$ can now be derived from the following lemma.

Lemma 3.1 *Suppose that the process $y(t)$ satisfies Assumptions 1 and 2, and that the PFSB algorithm is applied using; a preliminary value $d^f = d_T^f$ where d_T^f is such that $|d_T^f - d| < \delta_T$ where $\delta_T \log T \rightarrow 0$ almost surely (a.s.) as $T \rightarrow \infty$, and an $AR(h)$ approximation where $h \leq H_T = a(\log T)^c$, $a > 0$, $c < \infty$. Then*

$$\lim_{T \rightarrow \infty} \left| |\bar{\kappa}_h(e^{i\lambda})|^2 - |\kappa(e^{i\lambda})|^2 \right| \leq \nu_{1,T} + \nu_{2,T} + \nu_{3,T}$$

where for all $\lambda \in [2\pi/T, 2\pi N/T]$

$$\begin{aligned} \nu_{1,T} &= \left| |\bar{\kappa}_h(e^{i\lambda})|^2 - |\kappa_h^f(e^{i\lambda})|^2 \right| = O(h(\log T/T)^{\frac{1}{2} - \delta_T}) \\ \nu_{2,T} &= \left| |\kappa_h^f(e^{i\lambda})|^2 - |\kappa^f(e^{i\lambda})|^2 \right| = O(\delta_T h^{-|d|}) \quad \text{and} \\ \nu_{3,T} &= \left| |\kappa^f(e^{i\lambda})|^2 - |\kappa(e^{i\lambda})|^2 \right| = O(\delta_T \log T). \end{aligned}$$

with probability one.

Lemma 3.1 leads to the following result.

Theorem 3.2 *Suppose that the conditions in Theorem 3.1 hold. Assume also that $b_T = E[\hat{d}_T] - d$ and $\bar{b}_T = E[\hat{d}_T^{*f}] - d_T^f$ are expressed as in 3.7 and 3.9 respectively. If the PFSB algorithm is applied using; a preliminary value $d^f = d_T^f$ where d_T^f is such that $|d_T^f - d| < \delta_T$ where $\delta_T \log T \rightarrow 0$ a.s. as $T \rightarrow \infty$, and an $AR(h)$ approximation*

where $h \leq H_T = a(\log T)^c$, $a > 0$, $c < \infty$, then

$$|b_T - b_T^*| = O\left(\max\left\{h\left(\frac{\log T}{T}\right)^{\frac{1}{2}-\delta_T}, \delta_T h^{-|d|}, \delta_T \log T\right\}\right) + o\left(\frac{N^2}{T^2}\right) \quad a.s.$$

As preempted above, the convergence of \bar{b}_T to b_T depends on the order of the autoregressive approximation and the proximity of the preliminary value employed in the PFSB to the true d , that is, h and the value of δ_T implicit in the choice of d_T^f .

An optimal value of h can be achieved by selecting the order of the autoregression using AIC, or an equivalent criterion. Denote the said estimate by \hat{h}_{AIC} . Then $\hat{h}_{AIC} = \operatorname{argmin}_{h=0,1,\dots,H_T} \ln(\hat{\sigma}_h^2) + 2h/T$, where $\hat{\sigma}_h^2$ denotes the residual mean square obtained from a fitted $AR(h)$ model. Let $\bar{h}_T = \operatorname{argmin}_{h=0,1,\dots,H_T} L_T(h)$ where $L_T(h) = (\sigma_h^2 - \sigma^2) + h\sigma^2/T$ and σ^2 and σ_h^2 are as defined in Assumption (2) and equation (2.5) respectively. The function $L_T(h)$ was introduced by Shibata (1980) as a figure of merit and the $AR(\hat{h}_{AIC})$ model is asymptotically efficient in the sense that $L_T(\hat{h}_{AIC}) = L_T(\bar{h}_T)\{1 + o(1)\}$ *a.s.* as $T \rightarrow \infty$ (Poskitt, 2007, Theorem 9). It follows that $\hat{h}_{AIC}/\bar{h}_T \rightarrow 1$ *a.s.* as $T \rightarrow \infty$, so as T increases \hat{h}_{AIC} behaves like a deterministic sequence and yields an autoregressive order $h \sim K \log T$ *a.s.*

Appropriate selection of the pre-filtering value for d is less clear. From Theorem 3.2 we can see that we require d_T^f to be such that $|d_T^f - d| \log T = o(1)$ *a.s.*, but no other features of the result nor its derivation give us a guide as to suitable choices for d_T^f . If $N^{1/2}(d_T^f - d)$ were exactly $N(0, \nu)$ then it would follow from the tail area properties of the normal distribution that $\lim_{T \rightarrow \infty} P(|d_T^f - d| > \epsilon N^{-1/2+\delta}) \leq \exp(-\epsilon^2 N^{2\delta}/2\nu)$ for any δ , $0 < \delta < 0.5$ and $\epsilon > 0$. Since $\exp(-\epsilon N^\delta/2\nu) < |r|^{N^\delta}$ for all r such that $\exp(-\epsilon/2\nu) < |r| < 1$ we could then conclude from the Borel-Cantelli lemma that $N^{1/2-\delta}|d_T^f - d|$ converged to zero *a.s.* It would then follow that $|d_T^f - d| \log T = o(1)$ *a.s.* as required by Theorem 3.2 since $\log T/N^{1/2-\delta} \rightarrow 0$ for all $N \sim KT^\nu$ where $2/3 < \nu < 4/5$. Approximate Gaussianity associated with the pre-filtering value d_T^f being a \sqrt{N} -CAN estimator of d is not sufficient to establish the required result, however, because departures of $N^{1/2}(d_T^f - d)$ from zero that are inconsequential for weak convergence need not be immaterial for large-deviation probabilities. Nevertheless, the necessary large-deviation property can be derived on a case by case basis.

Proposition 3.1 *Let d_T^f denote any one of the estimators LPR, LPR-BA, SPLW or SPLW-BA. Then under the conditions of Theorem 3.2 $|d_T^f - d| \log T \rightarrow 0$ as $T \rightarrow \infty$ with probability one.*

Proposition 3.1 indicates that each of the estimators to be considered here can serve as a legitimate pre-filtering value, and in the simulation experiments we choose to set the (initial) pre-filtering value equal to the actual estimator to be bias-adjusted. Whilst the latter is perhaps not strictly necessary, it is an obvious choice to make and

a choice that is also consistent with the details of the proof provided in the paper for the convergence of the bootstrap bias in (3.5) to the actual finite sample bias in (3.2).³ Furthermore, in the context of the bootstrap algorithm, any bootstrap bias-adjusted version of an initial estimator can serve as a valid pre-filtering value in a subsequent application of the algorithm. This observation, in turn, prompts the following adaptation of the PFSB(BA) algorithm, in which successive bias-adjusted estimators play the role of the preliminary pre-filtering value within an iterative scheme.

4 A Recursive Bias Correction Procedure

Although the bias of the bias-adjusted estimator \tilde{d}_T in (2.7) will be smaller than that of \hat{d}_T , any bias remaining in $E[\tilde{d}_T] - d$ may still be large because the bias in any preliminary value d^f can be severe in finite samples, and $\hat{b}_{T,B}^{*f}$ will, as a consequence, be a biased estimate of its true counterpart b_T . To obtain a more accurate estimate of d we propose a further refinement to the proposed correction of \hat{d}_T through a recursive algorithm:

BA1'. Initialization: Set $k = 0$ and assign desirable tolerance levels $\tau_1 = \tau_1^{(0)}$ and $\tau_2 = \tau_2^{(0)}$. For the chosen estimator \hat{d}_T , set $\tilde{d}_T^{(0)} = \hat{d}_T$ (i.e. set $d^f = \hat{d}_T$). Now go to Step BA2'.

BA2'. Recursive Calculation: For the k th iteration set the preliminary value of d , namely d^f , to $\tilde{d}_T^{(k)}$ and repeat Steps BA2 and BA3 of the PFSB(BA) algorithm with \hat{d}_T therein replaced by $\tilde{d}_T^{(k)}$ to give, in an obvious notation, $\tilde{d}_T^{(k+1)} = \tilde{d}_T^{(k)} - \tilde{b}_{T,B}^{*f(k)}$. Proceed to Step BA3'.

BA3'. Stopping Rule: If $|\tilde{d}_T^{(k+1)} - \tilde{d}_T^{(k)}| > \tau_1$ and $|\tilde{d}_T^{(0)} - \tilde{d}_T^{(k)} - \tilde{b}_{T,B}^{*f(k)}| > \tau_2$ set $k = k + 1$, update the tolerance levels $\tau_1 = \tau_1^{(k)}$ and $\tau_2 = \tau_2^{(k)}$, and repeat Step BA2'. Otherwise set $\tilde{d}_T = \tilde{d}_T^{(k)}$ and stop.

The rationale behind the recursions is as follows: since the estimator $d^f = \hat{d}_T$ tends to be severely biased, $\hat{b}_{T,B}^{*f}$ will on average be a biased estimate of b_T , and the bias-adjusted estimate \tilde{d}_T will therefore still contain some bias. Replacing the initial values $\hat{d}_T = \tilde{d}_T^{(0)}$ and $\hat{b}_{T,B}^{*f} = \tilde{b}_{T,B}^{*f(0)}$ by $\tilde{d}_T^{(1)}$ and $\tilde{b}_{T,B}^{*f(1)}$, and (for general k) $\tilde{d}_T^{(k-1)}$ and $\tilde{b}_{T,B}^{*f(k-1)}$ by $\tilde{d}_T^{(k)}$ and $\tilde{b}_{T,B}^{*f(k)}$, and so on, produces more accurate estimates and bias assessments. Being based upon more accurate estimators, the updated estimate $\tilde{d}_T^{(k)}$ would be expected to be closer to the true value of d . The procedure is iterated until no meaningful gain in accuracy is achieved.

³ Specifically, the constant β that appears in the expressions (3.7) and (3.9) for b_T and b_T^* respectively is common to both expressions only if d_T^f is equivalent to the estimator being bias-adjusted. The presence of this common factor enables the result in Theorem 3.2 to be produced via convergence arguments concerning the quantities $|\bar{c}_0 - c_0|$ and $|\bar{c}_1 - c_1|$.

To determine if any meaningful gain in accuracy will be achieved by adding a further iteration, two criteria are used. The first, $|\tilde{d}_T^{(k+1)} - \tilde{d}_T^{(k)}| > \tau_1^{(k)}$, is based on Cauchy's convergence criterion. Given the stochastic nature of the bias correction mechanism we can think of this as a statistical decision rule in which $\tau_1^{(k)}$ governs the probability of moving from the k th to the $(k+1)$ th iteration. Now

$$\begin{aligned} \tilde{d}_T^{(k+1)} - \tilde{d}_T^{(k)} &= -\tilde{b}_{T,B}^{*f(k)} \\ &= \tilde{d}_T^{(k)} - \frac{1}{B} \sum_{b=1}^B \tilde{d}_{T,b}^{*f(k)} \\ &= -\frac{1}{B} \sum_{b=1}^B \left(\tilde{d}_{T,b}^{*f(k)} - \tilde{d}_T^{(k)} \right) \end{aligned}$$

and since \hat{d}_T is a \sqrt{N} -CAN estimator, given the data and the current and previous bootstrap iterations, $N^{\frac{1}{2}}(\tilde{d}_{T,b}^{*f(k)} - \tilde{d}_T^{(k)}) \xrightarrow{\mathcal{D}} N(0, v^2)$, where $\tilde{d}_{T,b}^{*f(k)}$ denotes the estimator produced from a bootstrap draw based on the PFSB(BA) algorithm, with $\tilde{d}_T^{(k)}$ used as the pre-filtering value. The conditional (asymptotic) variance of $B^{-1} \sum_{b=1}^B \left(\tilde{d}_{T,b}^{*f(k)} - \tilde{d}_T^{(k)} \right)$ is therefore v^2/NB , and using the rule that the overall variance equals the variance of the conditional mean (in this case $\text{Var}[\tilde{d}_T^{(k)}]$) plus the expectation of the conditional variance (in this case the constant v^2/NB) we can infer that the (asymptotic) variance of the difference between successive bias-adjusted estimators is given by

$$\text{Var}[\tilde{d}_T^{(k+1)} - \tilde{d}_T^{(k)}] = \text{Var}[\tilde{d}_T^{(k)}] + \frac{v^2}{NB}.$$

Furthermore, from the recurrence formula

$$\begin{aligned} \tilde{d}_T^{(k)} &= \tilde{d}_T^{(k-1)} - \tilde{b}_{T,B}^{*f(k-1)} \\ &= \tilde{d}_T^{(k-1)} - \frac{1}{B} \sum_{b=1}^B \left(\tilde{d}_{T,b}^{*f(k-1)} - \tilde{d}_T^{(k-1)} \right) \end{aligned}$$

it follows by a similar logic that

$$\text{Var}[\tilde{d}_T^{(k)}] = 2 \cdot \text{Var}[\tilde{d}_T^{(k-1)}] + \frac{v^2}{NB},$$

where $\text{Var}[\tilde{d}_T^{(1)}] = 2 \cdot \text{Var}[\tilde{d}_T^{(0)}] + v^2/NB = (2B+1)v^2/NB$. Moreover, at each iteration the bias-adjusted estimate is constructed as a linear combination of asymptotically normal random variables and is itself therefore asymptotically normal. This indicates that $\tau_1^{(k)}$ can be evaluated from percentile points of the normal approximation.

Similarly, the second convergence criterion, $|\tilde{d}_T^{(0)} - \tilde{d}_T^{(k)} - \tilde{b}_{T,B}^{*f(k)}| > \tau_2^{(k)}$, is perhaps best thought of as the decision rule that examines the difference between the current accumulated bias correction, $\tilde{d}_T^{(0)} - \tilde{d}_T^{(k)}$, and the current bootstrap estimate of the bias,

$\tilde{b}_{T,B}^{*f(k)}$. From the expression

$$\tilde{d}_T^{(0)} - \tilde{d}_T^{(k)} - \tilde{b}_{T,B}^{*f(k)} = \tilde{d}_T^{(0)} - \left(\frac{1}{B} \sum_{b=1}^B \tilde{d}_{T,b}^{*f(k)} \right),$$

it follows that the (asymptotic) variance,

$$\text{Var}[\tilde{d}_T^{(0)} - \tilde{d}_T^{(k)} - \tilde{b}_{T,B}^{*f(k)}] = \frac{v^2}{N} \left(1 + 2^{k-1} \left[1 + \frac{1}{B} \right] \right),$$

and the tolerance level $\tau_2^{(k)}$ can once again be set using percentile points from the asymptotic normal approximation.

The interpretation of the convergence criteria as statistical decision rules in which the tolerance levels govern the probability of going from the current to the next iteration suggests that $\tau_1^{(k)}$ and $\tau_2^{(k)}$ be set by reference to conventional critical values used in statistical hypothesis tests. When k is very small we might conjecture that $\tilde{d}_T^{(k)}$ still contains some bias and we may wish to iterate further unless there is strong evidence that so doing will produce very little change. On the other hand, when k is large the initial estimate $\tilde{d}_T^{(0)}$ has already undergone several adjustments to produce $\tilde{d}_T^{(k)}$ and we may prefer to terminate iteration unless there is strong evidence that further iteration will produce additional, substantial correction. We can therefore calibrate $\tau_1^{(k)}$ and $\tau_2^{(k)}$ using quantile points of the normal distribution $z_{(1-p_k/2)}$ (where $G(z_{(1-p)}) = 1-p$) and p_k , the probability of going from the k th to the $(k+1)$ th iteration, is assigned to be large when k is small and vice versa. In the experiments that follow we set $p_0 = 0.95$, $p_1 = 0.9$, and $p_k = (0.1)2^{(1-k)}$ for $k = 2, 3, \dots$ for uncorrected LPR and SPLW; and $p_0 = 0.9$, $p_k = (0.1)2^{-k}$ for $k = 1, 2, 3, \dots$ for LPR-BA and SPLW-BA with $P = 1$. We comment further on the stochastic stopping rules when discussing our experimental results below.

5 Simulation Exercise

5.1 Simulation Design

In this section we illustrate the performance of the bootstrap bias-corrected estimators via a small simulation experiment. Following [Andrews and Guggenberger \(2003\)](#) we simulate data from a Gaussian *ARFIMA*(1, d , 0) process,

$$(1-L)^d \Phi(z)y(t) = \varepsilon(t), \quad 0 < d < 0.5, \quad (5.1)$$

where $\Phi(z) = 1 - \phi z$ is the operator for a stationary AR(1) component and $\varepsilon(t)$ is zero-mean Gaussian white noise. The choice of this model is motivated, in part, by earlier work that highlights the distinct finite sample bias of the LPR estimator of d

in this setting, when the value of ϕ is positive and large (See [Agiakloglou et al., 1993](#)). Indeed, [Andrews and Guggenberger \(2003\)](#) document substantial remaining bias in the bias-corrected version of the LPR estimator in the presence of a large autoregressive parameter. That is, the impetus for applying bootstrap-based bias corrections to the various estimators is particularly strong in this setting.

The process in (5.1) is simulated $R = 1000$ times for $d = 0.0, 0.2, 0.3, 0.4$; $\phi = 0.3, 0.6, 0.9$, and sample sizes $T = 100, 200, 500$ via Levinson recursion applied to the autocovariance function (ACF) of the desired $ARFIMA(p, d, q)$ process and the generated pseudo-random $\varepsilon(t)$ (see, for instance, [Brockwell and Davis, 1991](#), §5.2). The ARFIMA ACF for given T , ϕ , θ , and d is calculated using Sowell's (1992) algorithm as modified by [Doornik and Ooms \(2001\)](#).

The estimators that we bias correct via the iterative PFSB(BA) algorithm are: LPR, LPR-BA, SPLW and SPLW-BA, implemented with a bandwidth $N = T^{0.7}$ and $B = 1000$. Values of $P = 1, 2$, are used for defining the two (analytically) bias-adjusted methods. For the log-periodogram regression estimators,

$$N^{\frac{1}{2}}(\widehat{d}_T - d) \xrightarrow{\mathcal{D}} N\left(0, \frac{\pi^2}{24}v_P^2\right), \quad (5.2)$$

where v_P^2 gives the variance inflation factor of the estimator. The inflation factor results from the modeling of $\log |\kappa(e^{-i\lambda})|^2$ by a polynomial of degree $2P$ that underlies the bias correction. For the local polynomial Whittle estimators,

$$N^{\frac{1}{2}}(\widehat{d}_T - d) \xrightarrow{\mathcal{D}} N\left(0, \frac{1}{4}v_P^2\right). \quad (5.3)$$

For both the LPR and the SPLW estimators the variance inflation factors are $v_0^2 = 1$, $v_1^2 = 2.25$ and $v_2^2 = 3.52$ where $v_0^2 = 1$ yields the baseline variance for the uncorrected estimator, see [Andrews and Guggenberger \(2003\)](#) and [Andrews and Sun \(2004\)](#). The estimators are known to be rate optimal when $N \sim KT^{4/5}$ in the uncorrected case ([Giraitis et al., 1997](#)) and $N \sim KT^{(4+4P)/(5+4P)}$ in the corrected case ([Andrews and Guggenberger, 2003](#); [Andrews and Sun, 2004](#)), but in practice optimal bandwidths seem not to be used much, the values $N = T^{2/5}, T^{1/2}, T^{3/5}$ and $T^{7/10}$ being popular choices. The order (h) of the autoregressive approximation underlying the sieve component of the bootstrap algorithm is chosen via *AIC*, and Burg's algorithm is used to estimate the autoregressive parameters.

Based on the R replications, for each estimator of d , we report the bias and mean square error (MSE). For comparative purposes, we also document the sampling performance of the unadjusted estimators (LPR, SPLW) and the estimators that are analytically adjusted (only) (LPR-BA and SPLW-BA; $P = 1, 2$). That is, we are interested, in particular, in documenting: 1) any improvement that can be had by using the bootstrap method *rather* than an analytical method to bias correct a given

estimator; and 2) any *additional* improvement associated with bias correcting (via the bootstrap) an estimator that has already been bias corrected via analytical means.

For each estimator considered (i.e. each of the two base estimators, LPR and SPLW, and all of the analytically and bootstrap bias-corrected versions thereof), we also document the empirical coverage (over the Monte Carlo replications) of the nominal 95% confidence intervals, plus the average length of the given intervals. The 95% confidence intervals (CIs) are constructed from each of the R bootstrap distributions (each, in turn, based on B bootstrap draws) as: $\{\tilde{d}_T(L), \tilde{d}_T(U)\}$, where $\tilde{d}_T(L)$ ($\tilde{d}_T(U)$) denotes the lower (upper) bound of a highest density interval, in which the narrowest interval with 95% coverage for the bootstrap distribution is selected. For any given estimator \tilde{d}_T , empirical coverage for each interval type is calculated as the proportion of times (in R replications) that each interval covers the true value of d . The average length of each interval (across the R replications) is also recorded. These coverage and length statistics for the bootstrap-based estimators are compared with the empirical coverage and (constant) length of 95% intervals constructed for the unadjusted (or analytically-adjusted) LPR and SPLW estimators, as based on the appropriate asymptotic distributions, in (5.2) and (5.3) respectively. Note that the value of B used here implies, from the Dvoretzky–Kiefer–Wolfowitz inequality, that $P(\sup_x |\bar{F}_{\tilde{d}_T, B}^*(x) - F_{\tilde{d}_T}^*(x)| > \delta) < 2 \exp(-\delta^2(1000))$, where $\bar{F}_{\tilde{d}_T, B}^*(x)$ is the empirical (bootstrap) distribution of \tilde{d}_T , based on B bootstrap draws, and $F_{\tilde{d}_T}^*(x)$ is the distribution of \tilde{d}_T under the probability law induced by the bootstrap.

We record results for the bootstrap-based estimators produced through formal application of the stopping rules described above. To the two stochastic stopping criteria we add a deterministic criterion, whereby the iterative scheme ceases if $\tilde{d}_T^{(k+1)} < -1$ or ≥ 1.5 and the estimator $\tilde{d}_T^{(k)}$ retained as the final choice. We also record results for the estimators based on only one and two iterations of the iterative method ($k = 1, 2$ in Steps BA2' and BA3'). In the following section we discuss all numerical results associated with the LPR estimator, and Section 5.3 all results for the SPLW estimator, with the relevant tables included in Appendix B. Note that most results for $T = 200$ and $d = 0.3$ are omitted for brevity. The coverage and length results for the three different values of ϕ are reported after averaging over all four values of d , including $d = 0.3$.

5.2 Simulation Results: LPR

Tables 1 and 2 record (for $T = 100$ and 500 respectively) the bias and MSE results for all estimators based on the LPR method. All results pertaining to the use of the bootstrapping to bias adjust are indicated by the subscript ‘sb’ appearing on the relevant acronym for the estimator (LPR or LPR-BA), both in the subsequent text

and the tables. In all tables the most favorable result for each parameter setting is highlighted in bold. The columns headed ‘SSR’ in the tables report the results based on the stochastic stopping rules discussed in Section 4 and modified (deterministically) as described at the end of Section 5.1.

The key message to be gleaned from the numerical results presented in Tables 1 and 2 is that the bootstrap technique *does* reduce bias, but with the most substantial gains to be had by using the bootstrap algorithm to bias-adjust an estimator that has already been bias adjusted analytically. For example, for $T = 100$, and for $\phi = 0.3, 0.6$, in all but one of the six cases, the smallest bias is produced by bias adjusting (via the bootstrap) the LPR-BA($P = 2$) estimator once, with no subsequent iteration: LPR-BA_{sb}($P = 2, k = 0$). For $T = 500$, this estimator is the least biased estimator for *all* three values of d and for $\phi = 0.3, 0.6$. Importantly, for these two values of ϕ (and for both sample sizes) if one compares the MSE of LPR-BA_{sb}($P = 2, k = 0$) with that of LPR-BA($P = 3$), the reduction in bias produced by the bootstrap technique is *not* obtained at the expense of MSE, with the two estimators having very similar MSE’s, and one not systematically dominating the other in terms of this performance measure. For $\phi = 0.9$, *all* versions of the LPR estimator, including the bootstrap bias-adjusted versions, are very biased. That said, for $T = 500$, the estimator with the *smallest* bias is the raw LPR ($P = 0$) estimator bootstrap bias-adjusted three times: LPR_{sb}($k = 2$).

A detailed examination of the simulation outcomes indicates that the stochastic stopping rules usually terminate the iterative procedure after zero, one or two iterations, with evidence for this provided by the nature of the bias and MSE results recorded in Tables 1 and 2. Looking first at the results in the middle panel of Table 1, we see that bias in the SSR column falls between the bias recorded for $k = 1$ and $k = 2$ respectively. The same observation can be made for the MSE. With the exception of the MSE results for $\phi = 0.9$, the same conclusion can be drawn for the results recorded in the middle panel of Table 2 for $T = 500$. For the cases where an analytical bias adjustment precedes the iterative bootstrap procedure (as recorded in the third panel of Tables 1 and 2) we find that the bias and MSE recorded in the SSR column almost always fall between the comparable results for $k = 0$ and $k = 1$. Hence, we can conclude that although a stochastic stopping rule tailors the number of iterations to the realization at hand, its use does not appear to *guarantee* an improvement in overall performance compared to using a fixed number of iterations.

Table 3 summarizes the empirical coverage performance of highest probability density (HPD) confidence intervals for the alternative estimators, for both sample sizes and based on a nominal coverage of 95%. The second panel of this table records the average length (across simulations) of the 95% HPD intervals for all cases. Coverage (and length) results for the nominal level of 90% are qualitatively similar, and hence are not reported.

In terms of coverage, a combination of analytical and bootstrap-based bias adjustments once again yields the best results overall, with either LPR-BA_{sb}($P = 1, k = 1$) (i.e. LPR-BA($P = 1$) bootstrap bias-adjusted twice) or LPR-BA_{sb}($P = 2, k = 0$) (i.e. LPR-BA($P = 2$) bootstrap bias-adjusted once) having the best empirical coverage – and very *accurate* empirical coverage – in all four cases recorded in Table 3 for $\phi = 0.3, 0.6$. Once again, all coverage results for $\phi = 0.9$ are poor, although, for what it is worth, for $T = 500$, the bootstrapped bias-adjusted LPR-BA ($P = 2$) produces the most accurate coverage interval (at 32%!).

In terms of the length of the 95% intervals, there are two key points to note. Firstly, it is the asymptotic intervals that are the most narrow, but this precision is at the expense of very inaccurate coverage. Secondly, the coverage accuracy yielded by the bootstrap is *not* at the expense of precision. That is, any bootstrap-based bias correction that improves coverage produces a negligible change in the length of the interval. This result provides an interesting contrast with the corresponding results for analytical bias-adjustment; i.e. any such analytical adjustment that improves coverage does so at the expense of a decrease in precision, with the 95% intervals widening as the value of P increases.

This raises the question of how the sieve bootstrap is able to bias correct the LPR or a LPR-BA estimator without incurring any loss of precision. The motivation underlying log-periodogram regression is that

$$\frac{I_T(\lambda)2\pi|1 - e^{-i\lambda}|^{2d}}{\sigma^2|\kappa(e^{i\lambda})|^2} \xrightarrow{\mathcal{D}} \text{Exp}(1), \quad (5.4)$$

and using the approximation $|1 - e^{-i\lambda}|^{2d} = |\lambda|^{2d}(1 + o(1))$ as $\lambda \rightarrow 0$ we have the linear regression model

$$\log(I_T(\lambda_j)) = \alpha_0 - 2d \log(\lambda_j) + \eta_j, \quad (5.5)$$

where $E[\eta_j] = 0$ and the intercept α_0 is presumed to capture the effects of the adjustments

$$a_j = \log |\kappa(1)|^2 + \log \left(\frac{|\kappa(e^{i\lambda_j})|^2}{|\kappa(1)|^2} \right) - d \log \left(\frac{|1 - e^{-i\lambda_j}|^2}{\lambda_j^2} \right) - C \quad (5.6)$$

$$= \log |\kappa(1)|^2 - C + O(N^2/T^2) \quad \text{for all } 1 \leq j \leq N, \quad (5.7)$$

where $C = 0.577216$ (Euler's constant).⁴ The presumption that α_0 absorbs the effects of the adjustment term assumes a_j approaches $\log |\kappa(1)|^2 - C$ sufficiently quickly that the deviations $a_j - \log |\kappa(1)|^2 + C$ can be ignored.

The analytical correction replaces the simple regression in (5.5) by the multiple

⁴ The expression in (5.7) follows as a consequence of the fact that $\log(|\kappa(e^{i\lambda})|^2/|\kappa(1)|^2) = \log(1 + (c_1/c_0)|\lambda|^2 + o(|\lambda|^3))$ and $\log(|1 - e^{-i\lambda}|^2/\lambda^2) = \log(1 - (1/12)|\lambda|^2 + o(|\lambda|^3))$ as $\lambda \rightarrow 0$.

regression

$$\log(I_T(\lambda_j)) = \sum_{p=0}^P \alpha_p \lambda_j^{2p} - 2d \log(\lambda_j) + \eta_j, \quad (5.8)$$

the rationale being that the term $\sum_{p=0}^P \alpha_p \lambda_j^{2p}$ provides a better approximation to the Maclaurin series expansion of the right hand side of (5.6) than supposing a_j is constant in a neighbourhood of zero. The introduction of λ_j^{2p} , $p = 1, \dots, P$, in (5.8) reduces the bias in the estimate of d , but it is also the presence of these additional regressors that causes the variance inflation seen in (5.2).

The PFSB, on the other hand, takes the specification of the regression in (5.5) or (5.8) as given and adjusts the estimator by mimicking the sampling behaviour of the regressand. Recall that $I_T(\lambda) = (2\pi)^{-1} \sum_{r=1-T}^{T-1} \hat{\gamma}(r) e^{i\lambda r}$. Hosking (1996) shows that when d is large the $\hat{\gamma}(r)$ have substantial negative bias relative to the true autocovariances, even for moderate to large samples. The PFSB reduces the memory in the “data” to which the sieve bootstrap is applied, via the pre-filtering procedure, so as to give a near optimal convergence rate when implicitly assessing the corresponding bias in $\log(I_T(\lambda))$. Whether it is applied to (5.5) or (5.8), the PFSB is thereby able to attack the problem of bias in the estimation of d without compromising the pivotal nature of the ratio in (5.4), the basic result that underlies the log-periodogram regressions and determines the estimators’ variance.

5.3 Simulation Results: SPLW

Tables 4 and 5 record (for $T = 100$ and 500 respectively) the bias and MSE results for all estimators based on the SPLW method (with the subscript ‘*sb*’ used as described above), whilst Table 6 records the 95% interval coverage and length statistics, for all cases. Once again, the most favorable result for each parameter setting is highlighted in bold in all tables. As with the LPR-based estimators, the bootstrap-based bias adjustment yields the largest bias reductions, but only when applied to an SPLW estimator that has already been analytically bias adjusted. In contrast with the LPR-based results, these bias gains are evident only for the larger of the two sample sizes ($T = 500$), with there being no gain (over full analytical adjustments) in the $T = 100$ case. The bias gains (for the $T = 500$ case) are for $\phi = 0.3, 0.6$ only, with the least biased estimator for $\phi = 0.9$ being the SPLW-BA ($P = 3$) estimator. The biases of all SPLW-based estimators are similar to the biases of the comparable LPR-based estimators, and as with the LPR-based estimators, the reduction in bias produced by the bootstrap technique (in certain cases) is not obtained at the expense of MSE. Once again, although the use of a stochastic stopping rule is appealing, as was the case for the LPR results it does not guarantee an improvement in performance over using a fixed number of iterations.

The coverage results for the SPLW-based estimators are qualitatively identical to those for the LPR case; in particular, the bootstrap bias adjustment of an already analytically adjusted estimator yields the best coverage for $\phi = 0.3, 0.6$, for both sample sizes - and very accurate coverage at that. Although the bootstrapped bias-adjustment of LPR-BA ($P = 2$) produces the most accurate coverage for the $\phi = 0.9$ case (for both sample sizes), the coverage results are poor for all estimators in this part of the parameter space. As for the LPR case, the bootstrap-based bias adjustment is not accompanied by an increase in interval length, in contrast with the analytical bias adjustment. As a consequence, the bootstrap method can be used to yield coverage that is close to the nominal level without sacrificing inferential precision.

6 Conclusion

This paper has developed a bootstrap method for bias correcting semi-parametric estimators of the long memory parameter in fractionally integrated processes. The method involves applying the sieve bootstrap to data pre-filtered by a preliminary semi-parametric estimate of the long memory parameter. In addition to providing theoretical (asymptotic) justification for using the bootstrap techniques, we document the results of simulation experiments, in which the finite sample performance of the (bias-adjusted) estimators is compared with that of both unadjusted estimators and estimators adjusted via analytical means. The numerical results are very encouraging, and suggest that the bootstrap bias correction *can* yield more accurate inferences about long memory dynamics in the types of samples that are encountered in practice.

Appendix A: Proofs

Proof of Theorem 2.1: For the Least Squares and Yule-Walker estimators see [Poskitt \(2007, Theorem 5 and Corollary 1\)](#) and the associated discussion. For the Burg estimator the result then follows from [Poskitt \(1994, Theorem 1\)](#). ■

Proof of Theorem 3.1: Subtracting (3.3) from (3.6) and using the triangular inequality we find that $|\mathbb{P}^*\{N^{\frac{1}{2}}(\widehat{d}_T^* - E^*[\widehat{d}_T^*]) < x\} - \mathbb{P}\{N^{\frac{1}{2}}(\widehat{d}_T - E[\widehat{d}_T]) < x\}|$ is less than or equal to

$$|G((x + N^{\frac{1}{2}}b_T)/v) - G((x + N^{\frac{1}{2}}b_T^*)/v)| + o(N^{-\frac{1}{2}}).$$

But

$$\sup_x |G((x + N^{\frac{1}{2}}b_T)/v) - G((x + N^{\frac{1}{2}}b_T^*)/v)| \leq \frac{N^{\frac{1}{2}}}{v\sqrt{2\pi}} |b_T - b_T^*|$$

by the first mean value theorem for integrals (Apostol, 1960, Theorem 7.30) and the theorem follows. \blacksquare

Proof of Lemma 3.1: Trivial addition and subtraction yields

$$\begin{aligned} |\bar{\kappa}_h(e^{i\lambda})|^2 - |\kappa(e^{i\lambda})|^2 &= (|\bar{\kappa}_h(e^{i\lambda})|^2 - |\kappa_h^f(e^{i\lambda})|^2) + (|\kappa_h^f(e^{i\lambda})|^2 - |\kappa^f(e^{i\lambda})|^2) \\ &\quad + (|\kappa^f(e^{i\lambda})|^2 - |\kappa(e^{i\lambda})|^2). \end{aligned} \quad (\text{A.1})$$

Consider the first term in (A.1), $|\bar{\kappa}_h(e^{i\lambda})|^2 - |\kappa_h^f(e^{i\lambda})|^2$. By definition

$$\bar{\kappa}_h(z) - \kappa_h^f(z) = \frac{\phi_h^f(z) - \bar{\phi}_h(z)}{\bar{\phi}_h(z)\phi_h^f(z)},$$

and since $\bar{\phi}_h(z) \neq 0$ and $\phi_h^f(z) \neq 0$, $|z| \leq 1$, there exists an $\epsilon > 0$ such that

$$\begin{aligned} |\bar{\kappa}_h(z) - \kappa_h^f(z)| &\leq \epsilon^{-2} |\phi_h^f(z) - \bar{\phi}_h(z)| \\ &\leq \epsilon^{-2} \sum_{j=0}^h |\phi_h^f(j) - \bar{\phi}_h(j)| \quad \text{for all } |z| \leq 1. \end{aligned}$$

But

$$\begin{aligned} \sum_{j=0}^h |\phi_h^f(j) - \bar{\phi}_h(j)| &\leq \left(h \sum_{j=0}^h |\phi_h^f(j) - \bar{\phi}_h(j)|^2 \right)^{\frac{1}{2}} \\ &= O \left(h \left(\frac{\log T}{T} \right)^{\frac{1}{2}(1-2\max\{0, d-d^f\})} \right) \\ &= O \left(h \left(\frac{\log T}{T} \right)^{\frac{1}{2}-\delta_T} \right) \quad a.s. \end{aligned}$$

by Theorem 2.1 and the fact that $|d^f - d| < \delta_T$ by assumption. It follows that $|\bar{\kappa}_h(e^{i\lambda}) - \kappa_h^f(e^{i\lambda})| = O(h(\log T/T)^{\frac{1}{2}-\delta_T})$ *a.s.* uniformly in λ , and hence that $\left| |\bar{\kappa}_h(e^{i\lambda})|^2 - |\kappa_h^f(e^{i\lambda})|^2 \right| = O(h(\log T/T)^{\frac{1}{2}-\delta_T})$ *a.s.* uniformly in λ . We can therefore interchange limit operations (Apostol, 1960, Theorem 13.3) to give

$$\lim_{T \rightarrow \infty} \lim_{\lambda \rightarrow 0} \left| |\bar{\kappa}_h(e^{i\lambda})|^2 - |\kappa_h^f(e^{i\lambda})|^2 \right| = \lim_{\lambda \rightarrow 0} \lim_{T \rightarrow \infty} \left| |\bar{\kappa}_h(e^{i\lambda})|^2 - |\kappa_h^f(e^{i\lambda})|^2 \right|,$$

which implies that $\nu_{1,T} = O(h(\log T/T)^{\frac{1}{2}-\delta_T})$ *a.s.* for all $\lambda \in [2\pi/T, 2\pi N/T]$.

For the second term in (A.1), $|\kappa_h^f(\rho)|^2 - |\kappa^f(e^{i\lambda})|^2$, we have

$$\kappa_h^f(z) - \kappa^f(z) = \frac{1 - \kappa^f(z)\phi_h^f(z)}{\phi_h^f(z)},$$

giving us the bound

$$|\kappa_h^f(z) - \kappa^f(z)| \leq \epsilon^{-1} |1 - \kappa^f(z)\phi_h^f(z)| \quad \text{for all } |z| \leq 1.$$

Let $\rho_h(z) = \sum_{j \geq 1} \rho_h(j) z^j = 1 - \kappa^f(z)\phi_h^f(z)$. Then from Parseval's relation

$$\sum_{j \geq 1} \rho_h(j)^2 = \int_{-\pi}^{\pi} |1 - \kappa^f(e^{i\lambda})\phi_h^f(e^{i\lambda})|^2 d\lambda = 2\pi\sigma^{-2}(\sigma_h^{f2} - \sigma^2)$$

and from the Levinson–Durbin recursions (Durbin, 1960; Levinson, 1947) we have $\sigma_h^{f2} = (1 - \phi_h^f(h)^2)\sigma_{h-1}^{f2}$. Substituting sequentially in the recurrence formula $\sigma_h^{f2} = \sigma_{h+1}^{f2} + \phi_h^f(h)^2\sigma_h^{f2}$ leads to the series expansion $\sigma_h^{f2} - \sigma^2 = \sum_{r=h}^{\infty} \phi_r^f(r)^2\sigma_r^{f2}$, from which we obtain the bound

$$\sum_{j \geq 1} \rho_h(j)^2 \leq 2\pi\sigma^{-2} E[w^f(t)^2] \sum_{r=h}^{\infty} \phi_r^f(r)^2.$$

But $\phi_h^f(h) \sim |d - d^f|/h$ as $h \rightarrow \infty$ (Inoue, 2002; Inoue and Kasahara, 2004) and therefore we can infer that

$$\sum_{j \geq 1} \rho_h(j)^2 \leq \text{const.} \frac{|d - d^f|^2}{h^{2|d|}} \zeta(2(1 - |d|)),$$

where $\zeta(\cdot)$ denotes the Riemann zeta function. It follows that $\lim_{h \rightarrow \infty} \rho_h(e^{i\lambda}) = 0$ and that $\lim_{T \rightarrow \infty} |\rho_h(e^{i\lambda})|^2 = O(\delta_T^2 h^{-2|d|})$ almost everywhere on $[-\pi, \pi]$. Hence we can conclude that $\lim_{T \rightarrow \infty} \lim_{\lambda \rightarrow 0} \left| |\kappa_h^f(e^{i\lambda})|^2 - |\kappa^f(e^{i\lambda})|^2 \right| = \lim_{\lambda \rightarrow 0} \lim_{T \rightarrow \infty} \left| |\kappa_h^f(e^{i\lambda})|^2 - |\kappa^f(e^{i\lambda})|^2 \right|$ and hence that $\nu_{2,T} = O(\delta_T^2 h^{-2|d|})$.

The third and final term in (A.1) is

$$|\kappa^f(e^{i\lambda})|^2 - |\kappa(e^{i\lambda})|^2 = |\kappa(e^{i\lambda})|^2 (|1 - e^{i\lambda}|^{2(d^f-d)} - 1). \quad (\text{A.2})$$

Substituting $|1 - e^{i\lambda}|^{2(d^f-d)} = \exp\{(d^f - d) \log |1 - e^{i\lambda}|^2\}$ into (A.2) and using the expansion $|1 - e^{i\lambda}|^2 = 2 \sum_{j=1}^{\infty} (-1)^{j-1} |\lambda|^{2j} / (2j)!$, which implies that $\log |1 - e^{i\lambda}|^2 = 2 \log |\lambda| + \log(1 + o(|\lambda|))$ as $\lambda \rightarrow 0$, we can deduce that

$$\left| |\kappa(e^{i\lambda})|^2 (|1 - e^{i\lambda}|^{2(d^f-d)} - 1) \right| \leq \left\{ \sup_{[-\pi, \pi]} |\kappa(e^{i\lambda})|^2 \right\} |\exp\{2(d^f - d) \log |\lambda| + o(|\lambda|)\} - 1|$$

as $\lambda \rightarrow 0$. Furthermore, by assumption $|d^f - d| \leq \delta_T$ where $\delta_T \log T \rightarrow 0$ as $T \rightarrow \infty$, and since $|\exp(x) - 1| = |x| \cdot |1 + \frac{1}{2}x + o(|x|)|$ for x in a neighbourhood of the origin, it follows that

$$\left| |\kappa(e^{i\lambda})|^2 (|1 - e^{i\lambda}|^{2(d^f-d)} - 1) \right| \leq 2 \left\{ \sup_{[-\pi, \pi]} |\kappa(e^{i\lambda})|^2 \right\} |d^f - d| (\log 2\pi N/T) + o(N/T)$$

for all $\lambda \in [2\pi/T, 2\pi N/T]$ as $T \rightarrow \infty$. We can therefore infer that (A.2) is $O(\delta_T \log T)$

or smaller, uniformly in λ for all $\lambda \in [2\pi/T, 2\pi N/T]$. The lemma now follows. \blacksquare

Proof of Theorem 3.2: It is sufficient to show that $|\bar{c}_0 - c_0|$ and $|\bar{c}_1 - c_1|$ are of order $O(T^2 M_T/N^2)$ or smaller where $M_T = \max\{h(\frac{\log T}{T})^{\frac{1}{2}-\delta_T}, \delta_T h^{-|d|}, \delta_T \log T\}$. Evaluating the expression

$$(\bar{c}_0 - c_0) + (\bar{c}_1 - c_1)|\lambda|^2 = |\bar{\kappa}_h(e^{i\lambda})|^2 - |\kappa(e^{i\lambda})|^2 + o(|\lambda|^3) \quad (\text{A.3})$$

at $\lambda = 2\pi/T$ and $2\pi N/T$, and solving for $\bar{c}_0 - c_0$ and $\bar{c}_1 - c_1$, it follows a consequence of Lemma 3.1 that $|\bar{c}_0 - c_0| = O(M_T) + o(T^{-3})$ and $|\bar{c}_1 - c_1| = O(T^2 M_T/N^2) + o(N/T)$. Extracting the dominant term gives the desired result. \blacksquare

Proof of Proposition 3.1: Let \hat{d}_T denote the LPR estimator. Then \hat{d}_T is the OLS coefficient of the regressor $-2 \log \lambda_j$ in the regression of $\log I_T(\lambda_j)$ on 1 and $-2 \log \lambda_j$. Substituting $a_j - 2d \log(\lambda_j) + \eta_j$ for $\log I_T(\lambda_j)$ in this regression leads to the expression

$$\begin{aligned} \hat{d}_T - d &= -\frac{\sum_{j=1}^N (\log \lambda_j - \overline{\log \lambda})(\eta_j + a_j)}{2 \sum_{j=1}^N (\log \lambda_j - \overline{\log \lambda})^2} \\ &= -\frac{1}{2} \sum_{j=1}^N r_j (\eta_j + a_j) \end{aligned} \quad (\text{A.4})$$

for the estimation error where η_j and a_j are defined in expressions (5.5) and (5.6), and $r_j = (\log \lambda_j - \overline{\log \lambda}) / \sum_{j=1}^N (\log \lambda_j - \overline{\log \lambda})^2$, $j = 1, \dots, N$. See the discussion associated with (5.5) and (5.6) for clarification.

By Theorem 2 of [Moulines and Soulier \(1999\)](#) there exists sequences e_j and f_j , $j = 1, \dots, N$, such that $\eta_j = e_j + f_j$, where the e_j , $j = 1, \dots, N$, are weakly dependent, centered Gumbel random variables with variance $\pi^2/6$ and covariance $\text{cov}\{e_k, e_j\} = O(\log^2(j)k^{-2|d|}j^{2(|d|-1)})$ for $1 \leq k < j \leq N$, and $|f_j| = O(\log(1+j)/j)$ with probability one. Since $\max_{1 \leq j \leq N} |\log \lambda_j - \overline{\log \lambda}| = O(\log N)$ and $\sum_{j=1}^N (\log \lambda_j - \overline{\log \lambda})^2 = O(N)$ it follows that $\sum_{j=1}^N r_j f_j = O(\log^3 N/N)$ *a.s.*. Given that $\sum_{j=1}^N r_j = 0$, it also follows from (5.7) that $\sum_{j=1}^N r_j a_j = O(N^2 \log N/T^2)$. We can therefore infer from (A.4) that

$$\hat{d}_T - d = -\frac{1}{2} \sum_{j=1}^N r_j e_j + R_N$$

where $|R_N| \log T = O(\nu^3 \log^4 T/T^\nu) + O(\nu \log^2 T/T^{2(1-\nu)}) = o(1)$ *a.s.*, $2/3 < \nu < 4/5$.

The desired result now follows because on application of a law of large numbers for triangular arrays of weakly dependent random variables we find that for all $\delta > 0$

$$\sum_{j=1}^N r_j e_j = o((\nu \log T)^{5/2} (\log(\nu \log T))^{(1+\delta)/2} T^{-\nu/2}) \quad \textit{a.s.} \quad .$$

More specifically, let $S_n = \sum_{j=1}^n r_j e_j$. Then by Doob's inequality $E[(\max_{n \leq 2^k} |S_n|)^2] \leq 4E[|S_{2^k}|^2]$, and using the bounds on the covariance of e_j we have

$$E[|S_n|^2] = \sum_{j=1}^n r_j^2 E[e_j^2] + 2 \sum_{1 \leq k < j \leq n} r_k r_j \text{cov}\{e_k, e_j\} = O(\log^4 n/n).$$

We can therefore conclude that for any $\delta > 0$

$$\sum_{k=1}^{\infty} \frac{2^k}{k^5 (\log k)^{1+\delta}} E[(\max_{n \leq 2^k} |S_n|)^2] \leq \sum_{k=1}^{\infty} \frac{2^k}{k^5 (\log k)^{1+\delta}} O\left(\frac{k^4}{2^k}\right) < \infty,$$

since $\sum_{k=1}^{\infty} 1/k(\log k)^{1+\delta} < \infty$, which by the Borel-Cantelli lemma implies $\max_{n \leq 2^k} |S_n| = o(k^{5/2}(\log k)^{(1+\delta)/2} 2^{-k/2})$ *a.s.*. Consequently $\sqrt{N}|S_N| = o((\log N)^{5/2}(\log \log N)^{(1+\delta)/2})$ *a.s.* since the function $(\log n)^{5/2}(\log \log n)^{(1+\delta)/2}$ is slowly varying at infinity.

Now let \widehat{d}_T denote the LPR-BA estimator. The analytically bias-adjusted LPR estimator is the OLS coefficient of the regressor $-2 \log \lambda_j$ in the regression of $\log I_T(\lambda_j)$ on 1 , $-2 \log \lambda_j$, and λ_j^{2p} , $p = 1, \dots, P$. Applying the Frisch-Waugh-Lovell theorem and projecting out the regressors λ_j^{2p} , $p = 1, \dots, P$, as well as unity we can express the estimation error $\widehat{d}_T - d$ exactly as in (A.4), save that the r_j are now defined in terms of $-2 \widetilde{\log \lambda_j}$, say, the component of $-2 \log \lambda_j$ orthogonal to 1 and λ_j^{2p} , $p = 1, \dots, P$. This projection does not alter the overall magnitudes, so for the orthogonalized regressor we have $\max_{1 \leq j \leq N} |\widetilde{\log \lambda_j}| = O(\log N)$ and $\sum_{j=1}^N (\widetilde{\log \lambda_j})^2 = O(N)$ (Andrews and Guggenberger, 2003, Lemma 2, parts (j) & (k)). The proof that $|\widehat{d}_T - d| \log T = o(1)$ *a.s.* now proceeds as previously with $r_j = \widetilde{\log \lambda_j} / \sum_{j=1}^N (\widetilde{\log \lambda_j})^2$, $j = 1, \dots, N$.

For the SPLW estimator the proposition follows directly from Giraitis and Robinson (2003, Lemma 5.8), which implies that the SPLW estimator satisfies $P(|\widehat{d}_T - d| \log T > \epsilon) = o(N^{-p})$, where $p > 1/\epsilon$ and N , the bandwidth, satisfies $T^\epsilon < N < T^{1-\epsilon}$ for some $\epsilon > 0$. For the SPLW-BA estimator the proposition can be established in a manner similar to that employed above for the LPR and LPR-BA estimators. Using Lemma 4 of Andrews and Sun (2004) we can express $\widehat{d}_T - d$, where \widehat{d}_T now denotes the SPLW-BA estimator, as a function of the standardized score and from Lemma 5 of Andrews and Sun (2004) we can conclude that the standardized score is of an order that implies that $|\widehat{d}_T - d| \log T = o(1)$ *a.s.*, cf. Andrews and Sun (2004, Theorem 4). ■

Appendix B: Tables

TABLE 1. Bias and mean square error (MSE) for all LPR-based estimators: $T = 100$. Unadjusted (LPR); analytically bias-adjusted (LPR-BA); bootstrap bias-adjusted (LPR-BA) for $k = 0, 1, 2$; bootstrap bias-adjusted *after* analytical adjustment (LPR-BA_{sb}). The lowest bias (in absolute value) and MSE for each parameter setting are highlighted in bold.

d	ϕ	LPR-BA			LPR _{sb}			LPR-BA _{sb}					
		$P = 1$	$P = 2$	$P = 3$	$k = 0$	$k = 1$	$k = 2$	SSR	$k = 0$	$k = 1$	$P = 2$ $k = 0$		
		Bias											
0	0.3	0.1445	0.0366	0.0138	0.0236	0.1255	0.0944	0.0368	0.0798	0.0165	-0.0063	0.0108	-0.0161
	0.6	0.3947	0.2000	0.1039	0.0725	0.3511	0.2799	0.1506	0.2655	0.1574	0.0919	0.1485	0.0564
	0.9	0.8230	0.7402	0.6540	0.5969	0.8000	0.7188	0.8053	0.7439	0.7031	0.6234	0.6915	0.6161
0.2	0.3	0.1400	0.0395	0.0161	0.0262	0.1207	0.0886	0.0252	0.0769	0.0220	-0.0116	0.0152	-0.0093
	0.6	0.3887	0.2017	0.1047	0.0746	0.3401	0.2609	0.1183	0.2459	0.1549	0.0805	0.1474	0.0601
	0.9	0.7968	0.7310	0.6558	0.5937	0.8180	0.8612	0.9425	0.8900	0.7301	0.6253	0.7162	0.6534
0.4	0.3	0.1374	0.0461	0.0194	0.0309	0.1110	0.0684	-0.0130	0.0590	0.0229	-0.0193	0.0127	-0.0047
	0.6	0.3780	0.2051	0.1063	0.0730	0.3319	0.2555	0.1178	0.2349	0.1546	0.0713	0.1368	0.0620
	0.9	0.7245	0.6910	0.6333	0.5706	0.8107	0.9839	1.2222	1.1407	0.7485	0.8018	0.7676	0.6859
		MSE											
0	0.3	0.0463	0.0753	0.1483	0.2369	0.0650	0.1396	0.3867	0.1869	0.1349	0.2549	0.1602	0.2525
	0.6	0.1810	0.1125	0.1543	0.2348	0.1711	0.2041	0.4095	0.2461	0.1515	0.2807	0.1841	0.2483
	0.9	0.7031	0.6189	0.5658	0.5861	0.6948	0.7188	0.8053	0.7552	0.6235	0.6697	0.6471	0.6341
0.2	0.3	0.0449	0.0737	0.1409	0.2247	0.0612	0.1276	0.3602	0.1664	0.1187	0.2478	0.1532	0.2220
	0.6	0.1765	0.1117	0.1493	0.2310	0.1640	0.1942	0.4002	0.2357	0.1422	0.2664	0.1620	0.2301
	0.9	0.6589	0.6026	0.5620	0.5752	0.7375	0.9677	1.6844	1.2435	0.6903	0.7542	0.7804	0.7180
0.4	0.3	0.0440	0.0747	0.1415	0.2372	0.0616	0.1201	0.3477	0.1507	0.1174	0.2565	0.1585	0.2253
	0.6	0.1676	0.1135	0.1498	0.2403	0.1635	0.2106	0.4723	0.2811	0.1486	0.2976	0.1944	0.2420
	0.9	0.5519	0.5458	0.5325	0.5532	0.7585	1.3521	2.8938	2.3084	0.7647	1.2723	1.0173	0.8160

TABLE 2. Bias and mean square error (MSE) for all LPR-based estimators: $T = 500$. Unadjusted (LPR); analytically bias-adjusted (LPR-BA); bootstrap bias-adjusted (LPR-BA) for $k = 0, 1, 2$; bootstrap bias-adjusted after analytical adjustment (LPR-BA_{sb}). The lowest bias (in absolute value) and MSE for each parameter setting are highlighted in bold.

d	ϕ	LPR-BA			LPR _{sb}			LPR-BA _{sb}					
		$P = 1$	$P = 2$	$P = 3$	$k = 0$	$k = 1$	$k = 2$	$k = 0$	$k = 1$	$k = 2$			
Bias													
0	0.3	0.0619	0.0097	0.0060	0.0026	0.0351	-0.0020	-0.0607	-0.0053	0.0025	-0.0089	0.0018	0.0001
	0.6	0.2221	0.0671	0.0244	0.0090	0.1603	0.0652	-0.1000	0.0446	0.0282	-0.0323	0.0168	-0.0016
	0.9	0.6736	0.4946	0.3707	0.2814	0.5927	0.4628	0.2351	0.4014	0.4114	0.2818	0.3802	0.2917
0.2	0.3	0.0601	0.0101	0.0066	0.0036	0.0330	-0.0044	-0.0642	-0.0063	0.0020	-0.0108	0.0019	-0.0014
	0.6	0.2205	0.0679	0.0253	0.0105	0.1561	0.0570	-0.1140	0.0344	0.0270	-0.0353	0.0166	-0.0027
	0.9	0.6691	0.4948	0.3713	0.2840	0.5972	0.4758	0.2610	0.4168	0.4045	0.2660	0.3765	0.2842
0.4	0.3	0.0613	0.0151	0.0126	0.0110	0.0320	-0.0079	-0.0720	-0.0087	0.0034	-0.0126	0.0049	0.0000
	0.6	0.2206	0.0725	0.0304	0.0174	0.1488	0.0392	-0.1489	0.0116	0.0262	-0.0418	0.0190	-0.0041
	0.9	0.6534	0.4908	0.3704	0.2856	0.6621	0.6785	0.6175	0.6529	0.4227	0.3126	0.3958	0.2876
MSE													
0	0.3	0.0103	0.0165	0.0293	0.0409	0.0131	0.0271	0.0783	0.0360	0.0236	0.0414	0.0284	0.0389
	0.6	0.0558	0.0210	0.0302	0.0413	0.0385	0.0400	0.1237	0.0817	0.0288	0.0624	0.0554	0.0463
	0.9	0.4603	0.2614	0.1675	0.1208	0.3636	0.2468	0.1611	0.3356	0.1999	0.1549	0.2361	0.1393
0.2	0.3	0.0102	0.0168	0.0303	0.0420	0.0130	0.0272	0.0782	0.0310	0.0235	0.0404	0.0307	0.0387
	0.6	0.0552	0.0213	0.0307	0.0416	0.0371	0.0382	0.1253	0.0798	0.0288	0.0620	0.0511	0.0456
	0.9	0.4542	0.2614	0.1675	0.1221	0.3715	0.2672	0.1980	0.3487	0.1941	0.1432	0.2369	0.1334
0.4	0.3	0.0103	0.0169	0.0303	0.0415	0.0127	0.0261	0.0748	0.0274	0.0219	0.0351	0.0265	0.0356
	0.6	0.0552	0.0219	0.0312	0.0420	0.0345	0.0349	0.1288	0.0953	0.0278	0.0587	0.0446	0.0430
	0.9	0.4342	0.2579	0.1678	0.1243	0.4631	0.5524	0.7172	0.5893	0.2180	0.2031	0.2479	0.1446

TABLE 3. Empirical coverage and length of (nominal 95%) HPD intervals for all LPR-based estimators: $T = 100, 500$. Unadjusted (LPR); analytically bias-adjusted (LPR-BA); bootstrap bias-adjusted (LPR- sb) for $k = 0, 1, 2$; bootstrap bias-adjusted *after* analytical adjustment (LPR-BA- sb). Figures are averaged over all values of d used in the experimental design for each value of ϕ . Coverages for the intervals based on the asymptotic distribution of the LPR and analytically bias-adjusted (LPR-BA) estimators are also reported for comparison. The empirical coverage closest to the nominal 95%, and the shortest length, are highlighted in bold.

ϕ	T	LPR-BA		LPR- sb			LPR-BA- sb			Asymptotic interval		
		LPR	$P = 1$ $P = 2$	$k = 0$ $k = 1$ $k = 2$	$k = 0$ $k = 1$ $k = 2$	$k = 0$ $k = 1$ $k = 2$	LPR	$P = 1$ $P = 2$	LPR-BA	$P = 1$ $P = 2$		
				Coverage								
				Interval length								
0.3	100	0.9015	0.9795 0.9730	0.9048	0.8880	0.8410	0.9773	0.9612	0.9635	0.7563	0.8408	0.8035
	500	0.8793	0.9748 0.9698	0.9120	0.9128	0.9075	0.9683	0.9555	0.9703	0.8343	0.9083	0.8873
0.6	100	0.2058	0.9160 0.9713	0.2475	0.3010	0.3328	0.9248	0.9092	0.9595	0.1918	0.7078	0.7860
	500	0.0698	0.9388 0.9710	0.0898	0.1590	0.2155	0.9435	0.9440	0.9738	0.1593	0.8565	0.8840
0.9	100	0.0000	0.1568 0.5945	0.0005	0.0013	0.0195	0.1898	0.2405	0.5880	0.0010	0.1020	0.3065
	500	0.0000	0.0030 0.2150	0.0000	0.0000	0.0005	0.0063	0.0140	0.3168	0.0000	0.0200	0.2670
ϕ	T			Interval length								
0.3	100	0.6413	1.1082 1.5664	0.6425	0.6452	0.6507	1.1070	1.0978	1.5542	0.5016	0.7523	0.9404
	500	0.3278	0.5267 0.6976	0.3275	0.3279	0.3303	0.5271	0.5275	0.6984	0.2856	0.4283	0.5354
0.6	100	0.6404	1.1046 1.5622	0.6409	0.6410	0.6392	1.1045	1.0931	1.5492	0.5016	0.7523	0.9404
	500	0.3308	0.5274 0.6983	0.3294	0.3290	0.3306	0.5269	0.5273	0.6989	0.2856	0.4283	0.5354
0.9	100	0.6114	1.0347 1.4638	0.6056	0.5716	0.5062	0.9663	0.9251	1.3499	0.5016	0.7523	0.9404
	500	0.3306	0.5224 0.6954	0.3325	0.3252	0.3150	0.5241	0.5244	0.6957	0.2856	0.4283	0.5354

TABLE 4. Bias and mean square error (MSE) for all SPLW-based estimators: $T = 100$. Unadjusted (SPLW); analytically bias-adjusted (SPLW-BA); bootstrap bias-adjusted (SPLW_{sb} for $k = 0, 1, 2$); bootstrap bias-adjusted after analytical adjustment (SPLW-BA_{sb}). The lowest bias (in absolute value) and MSE for each parameter setting are highlighted in bold.

d	ϕ	SPLW-BA			SPLW _{sb}			SPLW-BA _{sb}					
		$P = 1$	$P = 2$	$P = 3$	$k = 0$	$k = 1$	$k = 2$	SSR	$k = 0$	$k = 1$	$P = 2$ $k = 0$		
		Bias											
0	0.3	0.1327	-0.0064	-0.0393	-0.0715	0.1191	0.0997	0.0647	0.1003	0.0111	0.0315	0.0078	-0.0250
	0.6	0.3993	0.1629	0.0530	-0.0214	0.3697	0.3243	0.2456	0.3252	0.1530	0.1328	0.1492	0.0504
	0.9	0.8239	0.7139	0.6192	0.5165	0.8164	0.8035	0.7706	0.8043	0.7125	0.7044	0.7097	0.6209
0.2	0.3	0.1268	-0.0058	-0.0397	-0.0709	0.1127	0.0928	0.0572	0.0929	0.0119	0.0311	0.0084	-0.0216
	0.6	0.3922	0.1633	0.0538	-0.0214	0.3586	0.3062	0.2133	0.3072	0.1494	0.1228	0.1469	0.0535
	0.9	0.7997	0.7029	0.6154	0.5068	0.8296	0.8687	0.8438	0.8755	0.7227	0.7113	0.7207	0.6378
0.4	0.3	0.1246	0.0004	-0.0340	-0.0668	0.1081	0.0842	0.0395	0.0846	0.0129	0.0234	0.0109	-0.0141
	0.6	0.3831	0.1668	0.0586	-0.0193	0.3534	0.3035	0.2060	0.3039	0.1466	0.1124	0.1460	0.0565
	0.9	0.7363	0.6724	0.5942	0.4913	0.8291	0.8785	0.7524	0.9266	0.7419	0.6804	0.7288	0.6583
		MSE											
0	0.3	0.0352	0.0523	0.1128	0.1993	0.0393	0.0624	0.1518	0.0623	0.0830	0.1541	0.0896	0.1790
	0.6	0.1787	0.0789	0.1129	0.1921	0.1621	0.1556	0.1999	0.1562	0.1008	0.1720	0.1070	0.1777
	0.9	0.6969	0.5620	0.4913	0.4533	0.6973	0.7108	0.7566	0.7116	0.5869	0.6475	0.5835	0.5553
0.2	0.3	0.0339	0.0522	0.1104	0.1944	0.0379	0.0605	0.1479	0.0602	0.0766	0.1437	0.0832	0.1550
	0.6	0.1732	0.0789	0.1105	0.1885	0.1551	0.1488	0.1973	0.1493	0.0975	0.1627	0.0992	0.1628
	0.9	0.6575	0.5451	0.4853	0.4382	0.7311	0.8819	1.0841	0.8835	0.6199	0.7251	0.6557	0.6122
0.4	0.3	0.0334	0.0524	0.1088	0.1934	0.0368	0.0594	0.1454	0.0593	0.0739	0.1356	0.0728	0.1461
	0.6	0.1660	0.0803	0.1116	0.1892	0.1565	0.1635	0.2385	0.1663	0.1000	0.1714	0.1011	0.1639
	0.9	0.5619	0.5027	0.4597	0.4249	0.7508	1.1471	1.4360	1.0075	0.6790	0.8647	0.7115	0.6851

TABLE 5. Bias and mean square error (MSE) for all SPLW-based estimators: $T = 500$. Unadjusted (SPLW); analytically bias-adjusted (SPLW-BA); bootstrap bias-adjusted (SPLW- sb for $k = 0, 1, 2$); bootstrap bias-adjusted after analytical adjustment (SPLW-BA- sb). The lowest bias (in absolute value) and MSE for each parameter setting are highlighted in bold.

d	ϕ	SPLW-BA			SPLW- sb			SPLW-BA- sb					
		$P = 1$	$P = 2$	$P = 3$	$k = 0$	$k = 1$	$k = 2$	SSR	$k = 0$	$k = 1$	$P = 2$ $k = 0$		
		Bias											
0	0.3	0.0573	-0.0058	-0.0130	-0.0320	0.0323	-0.0013	-0.0517	-0.0009	0.0012	0.0076	-0.0014	0.0000
	0.6	0.2306	0.0550	0.0068	-0.0255	0.1755	0.0920	-0.0501	0.0922	0.0286	-0.0117	0.0293	0.0005
	0.9	0.7250	0.5273	0.3849	0.2659	0.6762	0.6045	0.4876	0.6050	0.4765	0.4002	0.4770	0.3340
0.2	0.3	0.0564	-0.0038	-0.0108	-0.0293	0.0316	-0.0018	-0.0513	-0.0012	0.0030	0.0091	0.0000	0.0009
	0.6	0.2292	0.0569	0.0090	-0.0228	0.1716	0.0847	-0.0630	0.0849	0.0292	-0.0122	0.0302	0.0017
	0.9	0.7195	0.5269	0.3854	0.2679	0.6846	0.6298	0.5374	0.6304	0.4696	0.3839	0.4685	0.3265
0.4	0.3	0.0582	0.0018	-0.0046	-0.0227	0.0316	-0.0040	-0.0567	-0.0034	0.0048	0.0070	0.0034	0.0024
	0.6	0.2296	0.0621	0.0146	-0.0163	0.1664	0.0719	-0.0889	0.0721	0.0292	-0.0180	0.0300	0.0017
	0.9	0.7020	0.5222	0.3839	0.2697	0.7464	0.8296	0.8771	0.8312	0.4852	0.4267	0.4826	0.3283
		MSE											
d	ϕ												
0	0.3	0.0075	0.0106	0.0194	0.0300	0.0076	0.0134	0.0356	0.0135	0.0137	0.0202	0.0133	0.0231
	0.6	0.0578	0.0137	0.0194	0.0295	0.0373	0.0241	0.0523	0.0241	0.0180	0.0348	0.0181	0.0281
	0.9	0.5312	0.2907	0.1694	0.1010	0.4652	0.3801	0.2747	0.3809	0.2461	0.1984	0.2477	0.1453
0.2	0.3	0.0074	0.0106	0.0196	0.0302	0.0075	0.0131	0.0345	0.0132	0.0134	0.0193	0.0129	0.0226
	0.6	0.0571	0.0139	0.0196	0.0298	0.0358	0.0223	0.0524	0.0224	0.0175	0.0332	0.0177	0.0276
	0.9	0.5232	0.2903	0.1700	0.1024	0.4792	0.4201	0.3502	0.4209	0.2405	0.1878	0.2469	0.1403
0.4	0.3	0.0077	0.0108	0.0201	0.0305	0.0075	0.0130	0.0344	0.0131	0.0131	0.0181	0.0128	0.0221
	0.6	0.0573	0.0147	0.0204	0.0302	0.0341	0.0205	0.0570	0.0205	0.0173	0.0322	0.0175	0.0273
	0.9	0.4986	0.2854	0.1692	0.1037	0.5749	0.7557	1.1358	0.7496	0.2618	0.2452	0.2659	0.1461

TABLE 6. Empirical coverage and length of (nominal 95%) HPD intervals for all SPLW-based estimators: $T = 100, 500$. Unadjusted (SPLW); analytically bias-adjusted (SPLW-BA); bootstrap bias-adjusted (SPLW_{sb}) for $k = 0, 1, 2$; bootstrap bias-adjusted *after* analytical adjustment (SPLW-BA_{sb}). Figures are averaged over all values of d used in the experimental design for each value of ϕ . Coverages for the intervals based on the asymptotic distribution of the SPLW and analytically bias-adjusted (SPLW-BA) estimators are also reported for comparison. The empirical coverage closest to the nominal 95%, and the shortest length, are highlighted in bold.

ϕ	T	SPLW-BA		SPLW _{sb}				SPLW-BA _{sb}				Asymptotic interval	
		$P = 1$	$P = 2$	$k = 0$	$k = 1$	$k = 2$	$k = 0$	$k = 1$	$k = 2$	$P = 1$	$P = 2$	$SPLW$	$SPLW-BA$
		Coverage											
		Interval length											
0.3	100	0.8563	0.9715	0.8673	0.8773	0.8580	0.9718	0.9630	0.9663	0.9663	0.6765	0.7940	0.7565
	500	0.7883	0.9685	0.8645	0.8938	0.9010	0.9590	0.9448	0.9648	0.9648	0.7890	0.8978	0.8575
0.6	100	0.1400	0.9268	0.1503	0.1768	0.2000	0.9205	0.8955	0.9600	0.9663	0.0713	0.6768	0.7480
	500	0.0438	0.9300	0.0480	0.0613	0.0725	0.9375	0.9385	0.9663	0.9663	0.0468	0.8505	0.8640
0.9	100	0.0000	0.1268	0.0000	0.0000	0.0018	0.1370	0.1549	0.5608	0.5608	0.0000	0.0400	0.2205
	500	0.0000	0.0020	0.0000	0.0000	0.0000	0.0023	0.0045	0.1540	0.0000	0.0000	0.0013	0.1100
0.3	100	0.5400	0.9555	0.5413	0.5445	0.5529	0.9569	0.9587	1.3789	1.3789	0.3911	0.5866	0.7332
	500	0.2630	0.4289	0.2634	0.2645	0.2674	0.4291	0.4297	0.5775	0.5775	0.2226	0.3340	0.4175
0.6	100	0.5434	0.9562	0.5445	0.5480	0.5538	0.9586	0.9592	1.3756	1.3756	0.3911	0.5866	0.7332
	500	0.2676	0.4305	0.2680	0.2712	0.2809	0.4313	0.4340	0.5777	0.5777	0.2226	0.3340	0.4175
0.9	100	0.5034	0.8847	0.4742	0.4248	0.4073	0.8236	0.7900	1.1904	1.1904	0.3911	0.5866	0.7332
	500	0.2638	0.4235	0.2690	0.2632	0.2503	0.4270	0.4319	0.5785	0.5785	0.2226	0.3340	0.4175

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