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Specification Testing for Nonlinear Multivariate Cointegrating Regressions*

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Abstract

This paper considers a general model specification test for nonlinear multivariate cointegrating regressions where the regressor consists of a univariate integrated time series and a vector of stationary time series. The regressors and the errors are generated from the same innovations, so that the model accommodates endogeniety. A new and simple test is proposed and the resulting asymptotic theory is established. The test statistic is constructed based on a natural distance function between a nonparametric estimate and a smoothed parametric counterpart. The asymptotic distribution of the test statistic under the parametric specification is proportional to that of a local–time random variable with a known distribution. In addition, the finite sample performance of the proposed test is evaluated through using both simulated and real data examples.

Key words: Cointegration, endogeneity, nonparametric kernel estimation, parametric model specification, time series.

JEL Classification: C12, C14, C22.

Abbreviated Title: Model Specification in Nonstationary Cointegration.

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1 Introduction

In recent years, there has been an increasing interest in discussing model estimation and specification testing problems involving nonparametric regression models associated with integrated time series. Recent literature includes Phillips and Park (1998), Karlsen and Tjøstheim (2001), Karlsen et al. (2007), Wang and Phillips (2009a,b), Wang and Phillips (2011), and Wang (2014) in the area of nonparametric estimation. Such existing studies are all limited to the case where the integrated time series regressor is univariate without additional regressors involved. A main reason for this is that the null recurrent structure of integrated time series typically reduces the amount of time that such time series spend in the vicinity of any one point, thereby exacerbating the sparse data problem or the "curse of dimensionality" in nonparametric modelling of multivariate integrated time series. As discussed by Gao and Phillips (2013, p. 62), meanwhile, nonparametric kernel estimation may not be working in the multivariate integrated time series case. Therefore, some semiparametric regression models are being proposed to deal with modelling multivariate integrated time series. Existing studies include Chen et al. (2012), and Gao and Phillips (2013) in the field of semiparametric regression modelling of multivariate integrated time series. Meanwhile, Cai et al. (2009) and Xiao (2009) consider using varying–coefficient models as an alternative. A recent study by Dong et al. (2015) proposes using a single–index model to deal with a vector of integrated regressors.

In these latter studies, the nonparametric part of the model again contains only one univariate nonstationary regressor and additional stationary regressors cannot be involved in the same nonparametric part. An exception is the recent paper by Dong et al. (2015) for nonparametrically modelling a vector of integrated regressors. In the parametric integrating case, however, both stationary and nonstationary time series regressors can be involved in the same regression model (see, for example, Chang et al. 2001), and there are in fact good reasons for studying such models in addressing empirical problems. Examples include modelling the relationship between the consumption time series and the income time series, in which a short–term interest rate variable can be naturally involved as a stationary time series regressor, while both the consumption and income time series regressors are known to be nonstationary. Section 5 below examines the suitability of linear models and applicability of nonlinear models. Additionally, endogeneity is naturally inherited in many economic variables, such as, supply and demand, and disposable income and expenditure consumption. In the estimation case, to the best of our knowledge, Wang and Phillips (2009b) were among the first investigating a nonparametric cointegrating regression for the case where the errors are defined by a functional form of innovations which also are building blocks of the regressors. More recently, Wang and Phillips (2016) extend a kernel based test originally proposed in an earlier version of this paper by Gao et al. (2012) to the case where there is a type of endogeneity, although their paper mainly focuses on nonparmetric kernel estimation. For specification testing purposes, this paper shall establish some large and small sample properties for a newly proposed test statistic under a very general type of endogeneity.

A main objective of using a parametric model specification is to find a best available parametric function to approximate an unknown nonparametric function. As shown in the literature (such as, Karlsen and Tjøstheim 2001; Wang and Phillips 2009a), nonparametric kernel estimation for the integrated time series case often results in a rate of convergence at the order of $\sqrt{\sqrt{n}\,h}$, slower than the rate of $\sqrt{n\,h}$ for the stationary time series case, where h is a bandwidth parameter. By contrast, parametric estimation in the integrated time series case can achieve the conventional rate–$\sqrt{n}$ or even faster than it. As a consequence, one would prefer a parametric cointegrating model to a nonparametric cointegrating model when possible. This thus means that using parametric specification in the integrated time series case may be more relevant and necessary than that for the stationary time series case.

In this paper, we are interested in a multivariate time series model of the form

$$y_t = m(x_t, z_t) + e_t, \tag{1.1}$$

where x_t is a univariate nonstationary time series, $z_t = (z_{t1}, \cdots, z_{td})^\tau$ is a d–dimensional vector of stationary time series regressors, x_t and z_t can be either mutually independent of each other or highly correlated, $\{e_t\}$ is a linear process and can be correlated with x_t, and $m(\cdot, \cdot)$ is an unknown function over $\mathbb{R}^{d+1}$.

Our primary interest is in testing the null hypothesis:

$$H_0: \; P\left(m(x_t, z_t) = g(x_t, z_t; \theta_0)\right) = 1, \tag{1.2}$$

versus a sequence of local alternatives of the form:

$$H_1: \; P\left(m(x_t, z_t) = g(x_t, z_t; \theta_0) + \Delta_n(x_t, z_t)\right) = 1, \tag{1.3}$$

where $g(\cdot, \cdot; \theta)$ is a known parametric function indexed by θ, a vector of unknown parameters, $\theta_0 \in \Theta_0$ with Θ_0 being a compact subset of $\mathbb{R}^k$ for $k \geq 1$, and $\Delta_n(x, z)$ is a sequence of nonparametrically unknown departure functions.

Recent studies in the field of nonparametric model specification of integrated time series models include Marmer (2007), Kasparis (2008), Gao et al. (2009b,a), Hong and Phillips (2010), Kasparis (2010), Kasparis and Phillips (2012), Kasparis et al. (2012), and Wang and Phillips (2012). The proposed tests by Gao et al. (2009b) and Wang and Phillips (2012) use exactly the same type of tests as those originally developed for the stationary time series case (see, for example, Chapter 3 of Gao 2007). In other words, the full nature of nonstationarity of $\{x_t\}$ has not been taken into account in the construction of the proposed tests. This paper takes the full feature of the integrated structure of $\{x_t\}$ into account and proposes a simple and powerful test for testing H_0 versus H_1. Both theoretical and empirical comparisons given in Sections 2–4, moreover, show that the proposed test is preferred to the test proposed in Gao et al. (2009b) and Wang and Phillips (2012).

The contributions and organisation of this paper are given as follows. Section 2 constructs our new test and then establishes a general asymptotic theory for the case where the probabilistic structure of (x_t, z_t, e_t) is quite general such that x_t, z_t and e_t can be highly correlated, indicating the existence of endogeneity in our system. Section 3 discusses some power properties of the proposed test and then compares such properties with those of an existing test. A set of simulated examples are given in Section 4. Section 5 considers an empirical application. The paper concludes in Section 6. Some crucial and preliminary lemmas are given in Appendix A. Appendix B then gives the proof of the main results of the paper. The proof of the lemmas listed in Appendix A and some secondary results are given in Appendices C and D, respectively.

Throughout the paper the following notation is used. Any integral "$\int$" stands for an integration on $\mathbb{R}$ or $\mathbb{R}^d$ which would be clear in the context; "$\|\cdot\|$" denotes Euclidean norm for a vector; "$\equiv$" means equal by definition; C, C_1 and C_2 are absolutely constants which may be different at each appearance.

2 Nonparametric specification test

Before we construct our test, we have a look at how to estimate θ_0 and $m(\cdot,\cdot)$, respectively. It follows from model (1.1) that

$$y_t = m(x_t, z_t) + e_t = g(x_t, z_t; \theta_0) + e_t \quad \text{under } H_0. \tag{2.1}$$

Hence, θ_0 in model (2.1) is estimated by $\widehat{\theta}$ that minimises

$$\frac{1}{n}\sum_{t=1}^{n} [y_t - g(x_t, z_t; \theta)]^2 \quad \text{over all possible } \theta. \tag{2.2}$$

Meanwhile, model (2.1) suggests estimating $m(\cdot,\cdot)$ by

$$\widehat{m}(x,z) = \frac{\sum_{t=1}^{n} K\left(\frac{x_t - x}{h_1}, \frac{z_t - z}{h_2}\right) y_t}{\sum_{t=1}^{n} K\left(\frac{x_t - x}{h_1}, \frac{z_t - z}{h_2}\right)}, \tag{2.3}$$

where $K(\cdot,\cdot) = K_1(\cdot)K_2(\cdot)$, and $K_1(\cdot)$ is a univariate probability kernel function, K_2 is a d–dimensional probability kernel function, and h_1 and h_2 are bandwidth parameters.

Note that the conventional estimation method used in (2.3) may not be extendable to the case where x_t is a vector integrated time series. The failure of the estimation in this case is discussed in Gao and Phillips (2013, p. 62). A recent paper by Mykelbust et al. (2012) discusses a similar issue.

In order to construct our test, we introduce a smoothed version of $g(\cdot,\cdot;\theta_0)$ of the form

$$\widehat{g}(x,z;\theta_0) = \frac{\sum_{t=1}^{n} K\left(\frac{x_t - x}{h_1}, \frac{z_t - z}{h_2}\right) g(x_t, z_t; \theta_0)}{\sum_{t=1}^{n} K\left(\frac{x_t - x}{h_1}, \frac{z_t - z}{h_2}\right)}. \tag{2.4}$$

To test H_0, a statistical measuring the distance function between $\widehat{m}(x,z)$ and $\widehat{g}(x,z;\widehat{\theta})$ is needed. In order to circumvent introducing some random denominator problem, however, we propose using a distance function by comparing the following modified quantities:

$$\begin{aligned}\widetilde{m}(x,z) =& \sqrt{\frac{d_n}{nh_1h_2^d}}\sum_{t=1}^{n} K\left(\frac{x_t-x}{h_1}, \frac{z_t-z}{h_2}\right) y_t \\ \widetilde{g}(x,z;\theta_0) =& \sqrt{\frac{d_n}{nh_1h_2^d}}\sum_{t=1}^{n} K\left(\frac{x_t-x}{h_1}, \frac{z_t-z}{h_2}\right) g(x_t,z_t;\theta_0),\end{aligned} \tag{2.5}$$

where $d_n(\sim \sqrt{n})$ will be explicitly defined after Assumption 2.1. This paper now proposes using a test statistic of the form

$$\widehat{L}_n \equiv \widehat{L}_n(h_1,h_2) = \iint \left(\widetilde{m}(x,z) - \widetilde{g}(x,z;\widehat{\theta})\right)^2 \pi_1(x)\pi_2(z)dzdx \tag{2.6}$$

where $\widehat{\theta}$ is as defined in (2.2), $\pi_i(\cdot)$, $i = 1, 2$, are both known probability weight functions satisfying Assumption 2.3 below. Since $\widehat{L}_n = \iint \left(\widehat{m}(x,z) - \widehat{g}(x,z;\widehat{\theta})\right)^2 \widehat{p}^2(x,z)\ \pi_1(x)\pi_2(z)dzdx$ with $\widehat{p}(x,z) = \sum_{t=1}^{n} K\left(\frac{x_t-x}{h_1}, \frac{z_t-z}{h_2}\right)$, it is similar to the original proposal by Härdle and Mammen (1993) for the independent sample case.

Before we impose certain conditions to establish an asymptotic distribution for $\widehat{L}_n(h_1,h_2)$, we have a look at a closed–form approximation to $\widehat{L}_n$ under H_0. Similarly to the proof of Theorem 2.1 given in Appendix B below, we have under H_0

$$\widehat{L}_n \equiv \widetilde{L}_n + o_P(1), \qquad \widetilde{L}_n \equiv \widetilde{S}_{1n} + \widetilde{S}_{2n} \tag{2.7}$$

where $\widetilde{S}_{1n} = C_2(K)\frac{d_n}{n}\sum_{t=1}^{n} \widehat{e}_t^2\ \pi_1(x_t)\pi_2(z_t)$ and

$$\widetilde{S}_{2n} = \frac{2d_n}{n}\sum_{t=2}^{n}\sum_{s=1}^{t-1} \widehat{e}_t\ \widehat{e}_s\ \pi_1(x_s)\pi_2(z_s)\ L_1\left(\frac{x_t-x_s}{h_1}\right) L_2\left(\frac{z_t-z_s}{h_2}\right), \tag{2.8}$$

in which $\widehat{e}_t = y_t - g(x_t,z_t;\widehat{\theta})$, $L_i(v) = \int K_i(u)K_i(u+v)du$ for $i = 1, 2$, $C_2(K) = \iint K^2(u,v)dvdu$. Note that $\widetilde{L}_n$ is a closed form approximation to $\widehat{L}_n$ and our experience in Sections 4 and 5 below shows that it is computationally easier to use $\widetilde{L}_n$ than $\widehat{L}_n$, which involves an integral in $\mathbb{R}^{1+d}$.

Mainly because of the fact that $\widetilde{S}_{1n}$ already converges in distribution to a random variable, there is no need to standardise $\widetilde{L}_n$ by a random denominator of a quadratic form as has been done in Gao et al. (2009b). Moreover, existing literature (Gao et al. 2009b; Wang and Phillips 2012) shows that it is much harder to show that a standardised version of $\widetilde{S}_{2n}$ converges in distribution to a standard normal random variable than to prove that $\widetilde{S}_{2n}$ converges in probability to zero as will be done in this paper. In the stationary case where $\{x_t\}$ is also stationary, however, researchers will need to use a standardised version of the form

$$\overline{M}_n = \frac{\widetilde{L}_n - \widetilde{S}_{1n}}{\widetilde{\sigma}_n} \tag{2.9}$$

as a test statistic (see, for example, Härdle and Mammen 1993; Fan and Yao 2003; Gao 2007; Li and Racine 2007), where $\widetilde{\sigma}_n$ is a random denominator of a quadratic form. This is mainly due to the fact that $\frac{1}{\sqrt{n}}\widetilde{S}_{1n} \to_P C$, where C is a non–random constant. In other words, $\widetilde{S}_{1n}$ itself cannot be normalised to be a test statistic.

In order to precisely establish and show an asymptotic distribution for $\widehat{L}_n(h_1, h_2)$, we need to introduce the following assumptions; their justifications are given below.

Assumption 2.1. *Let $\{\epsilon_j, j \in \mathbb{Z}\}$ be a scalar sequence of independent and identically distributed (i.i.d.) with $E[\epsilon_1] = 0$, $E[\epsilon_1^2] = 1$ and $E[\epsilon_1^4] < \infty$. Suppose further that ϵ_1 has a density function $f_\epsilon(\cdot)$ and its characteristic function $\varphi(\cdot)$ satisfies $\int |\lambda|\varphi(\lambda)d\lambda < \infty$.*

(i) Consider $x_t = x_{t-1} + u_t$ with $x_0 = O_P(1)$. Here, $u_t = \sum_{i=0}^{\infty} \phi_i \epsilon_{t-i}$, where $\{\phi_i : i \geq 0\}$ is a sequence of real numbers such that $\phi := \sum_{j=0}^{\infty} \phi_j \neq 0$ and $\sum_{j=0}^{\infty} j^\beta |\phi_j| < \infty$ with $\beta > 1/2$.

(ii) Let $z_t = \psi(\epsilon_t, \cdots, \epsilon_{t-m_0+1})$ for some fixed $m_0 \geq 1$ and $\psi(v) = (\psi_1(v), \cdots, \psi_d(v)) : \mathbb{R}^{m_0} \mapsto \mathbb{R}^d$ a d-vector measurable function for $v = (v_1, \cdots, v_{m_0})$ such that $E\|z_t\|^2 < \infty$. Suppose that the joint density of $(z_t, \epsilon_t, \cdots, \epsilon_{t-m_0+1})$ exists and has a bounded derivative with respect to the first argument. Moreover, let $p(z)$ be the density of z_t. Suppose that $E[e_t \pi_2(z_t) p(z_t)] = 0$.

(iii) e_t is generated by $e_t = \sum_{j=0}^{\infty} \rho_j \epsilon_{t-j}$ where $\{\rho_i, i \geq 0\}$ is a real sequence such that $\rho \equiv \sum_{j=0}^{\infty} \rho_j \neq 0$ and $\sum_{j=0}^{\infty} j|\rho_j| < \infty$.

In Assumption 2.1 the innovation sequence $\{\epsilon_j, j \in \mathbb{Z}\}$ is the building blocks for a stationary structure of $\{e_t, u_t, z_t\}$. Hence, it is evident that the system possesses endogeneity. It covers a variety possibilities of the relations among u_t, z_t and e_t noting by enormous choices of the coefficients involved.

Note that x_t is integrated by u_t, and the linear process u_t admits Beveridge-Nelson decomposition (Phillips and Solo, 1992, p. 972), namely, $u_t = \phi\epsilon_t + \tilde{u}_{t-1} - \tilde{u}_t$ where $\tilde{u}_t = \sum_{j=0}^{\infty} \tilde{\phi}_j \epsilon_{t-j}$ with $\tilde{\phi}_j = \sum_{k=j+1}^{\infty} \phi_k$. Due to the condition on ϕ_j, $\sum_{j=0}^{\infty} |\tilde{\phi}_j|^2 < \infty$, implying that $\tilde{u}_t$ is a stationary process.

Hence, $x_t = \phi(\epsilon_1 + \cdots + \epsilon_t) + x_0 + \tilde{u}_0 - \tilde{u}_t$ and $d_t^2 \equiv E[x_t^2] = \phi^2 \cdot t(1 + o(1))$. The invariant principle is applied,

$$d_n^{-1} x_{[nr]} \to_D B(r), \qquad r \in [0, 1], \tag{2.10}$$

where $B(r)$ is a standard Brownian motion.

Note that z_t is defined to be a vector of functions of $(\epsilon_t, \cdots, \epsilon_{t-m_0+1})$ and therefore z_t and e_t are usually dependent on each other. When e_t and z_t are independent of each other, we have $E[e_t \pi_2(z_t) p(z_t)] = 0$. Additionally, it is also satisfied when $E[\epsilon_j \pi_2(z_t) p(z_t)] = 0$ for $t - m_0 + 1 \leq j \leq t$. Considering the case of $m_0 = 1$, we will have $E[\epsilon_t \pi_2(\epsilon_t) p(\epsilon_t)] = 0$ if the density function, $f_\epsilon(\cdot)$, of ϵ_1 is symmetric and we choose $z_t = \epsilon_t$, $\pi_2(\cdot) = p(\cdot) = f_\epsilon(\cdot)$.

Meanwhile, it is easily seen that x_t is correlated with e_t, since in general $E[x_t e_t] \neq 0$.

Furthermore, the following development relies on the local process of the Brownian motion $B(r)$ defined by

$$L_B(r,a) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^r I\left[|B(s) - a| < \delta\right] ds,$$

which measures the sojourning time of the Brownian motion around point a during time period $[0,r]$. Notice that $L_B(1,0)$ has cumulative distribution function given by

$$P\left(L_B(1,0) \le x\right) = \begin{cases} 2\Phi(x) - 1, & x \ge 0, \\ 0, & x < 0, \end{cases} \tag{2.11}$$

in which $\Phi(x)$ is the cdf of $N(0,1)$.

Assumption 2.2. *(i) The bandwidths h_1 and h_2 satisfy that $h_1 + h_2 \to 0$, $nh_1^2h_2^{2d} \to \infty$, $\sqrt{n}h_1^2h_2^{2d+1} \to 0$, $h_1^2h_2^{2d}\ln(n) \to 0$ as $n \to \infty$.*

(ii) Let $\widehat{\theta}$ be a consistent estimator of θ_0 such that for any small $\varepsilon > 0$ and as $n \to \infty$, we have $P\left(\sqrt{n}h_1h_2^d\|\widehat{\theta} - \theta_0\|^2 > \varepsilon\right) \to 0$.

As our test statistic is proposed based on kernel estimation, the bandwidths h_1 and h_2 in Assumption 2.2(i) satisfy the usual requirement in the literature for the kernel estimation method. All of them are reasonable and compatible. Particularly, the condition that $\sqrt{n}h_1^2h_2^{2d+1} \to 0$ is a quite common requirement for the bandwidths. To see this, letting $h_1 = h_2 = h$ and $d = 1$, it reduces to requiring $\sqrt{n}h^5 \to 0$, which is satisfied trivially.

Since weak convergence with a rate is needed to establish the asymptotic distribution in Theorem 2.1 below, Assumption 2.2(ii) imposes a condition on the rate of convergence directly rather than imposing certain conditions to imply the asymptotic consistency. However, this may be easily satisfied when $\widehat{\theta} - \theta_0$ achieves either a slow rate of $n^{-1/4}$ or a fast rate of $n^{-1/2}$ as has been established in the literature (see, for example, Chang et al. 2001; Park and Phillips 2001) where e_t is independent of x_t or $(e_t, \mathcal{F}_t)$ is a martingale difference such that x_t is adapted with respect to $\mathcal{F}_{t-1}$. Hereafter, $\mathcal{F}_t = \sigma(\epsilon_s, s \le t)$ the sigma field generated by the innovation up to t.

Assumption 2.3. *(i) Let $K_i(\cdot)$ be symmetric probability kernel functions with*

$$\int \|u\|^j K_i^2(u)du < \infty \quad \textit{and} \quad \int \left(\int \|v\|^j K_i(u+v)K_i(v)dv\right)^2 du < \infty$$

for $i = 1,2$ and $j = 0,1,2$.

(ii) Let $\pi_1(\cdot)$ and $\pi_2(\cdot)$ be known probability weight functions such that $0 < \int \pi_1^i(u)du < \infty$ and $0 < \int \pi_2^i(z)dz < \infty$ for $i = 1,2$. Moreover, there are functions $D_1(x):\ \mathbb{R} \mapsto \mathbb{R}$ and $D_2(z):\ \mathbb{R}^d \mapsto \mathbb{R}$, satisfying $\int(|D_1(x)| + |D_1(x)|^2)dx < \infty$ and $\int |D_2(z)|^2p(z)dz < \infty$ where $p(z)$ is the density of the stationary process z_t, such that

$$|\pi_1(y) - \pi_1(x)| \le D_1(x) \cdot |y - x|, \quad (x,y) \in \Omega_1(\epsilon) = \{(x,y) : |y-x| \le \epsilon, x, y \in \mathbb{R}\},$$
$$|\pi_2(y) - \pi_2(x)| \le D_2(x) \cdot \|y - x\|, \quad (x,y) \in \Omega_2(\epsilon) = \{(x,y) : \|y-x\| \le \epsilon, x, y \in \mathbb{R}^d\},$$

where $\epsilon > 0$ is some small constant.

(iii) $g(x, z; \theta)$ is differentiable with respect to θ and there are some function $G_i(x, z; \theta_0)$ for $i = 1, 2$ and small $\varepsilon > 0$ such that

$$|g(x, z; \theta) - g(x, z; \theta_0) - (G_1(x, z; \theta_0))^{\tau} (\theta - \theta_0)| \leq G_2(x, z; \theta_0) \|\theta - \theta_0\|^2$$

and $\iint \|G_i(x, z; \theta)\|^2 \pi_1^j(x) \pi_2^j(z) p^j(z) dz dx < \infty$ for all $\theta \in \Theta(\varepsilon) = \{\theta : \|\theta - \theta_0\| \leq \varepsilon\}$ and for $i, j = 1, 2$.

The condition on the kernels commonly used in the kernel estimation context is postulated in Assumption 2.3(i), while smoothness condition on the weight functions $\pi_1(x)$ and $\pi_2(z)$ is imposed by Assumption 2.3 (ii). Note that all such conditions may not be the weakest possible, but are all quite mild and verifiable.

Assumption 2.3(iii) imposes some mild conditions to ensure the integrability of the first partial derivative of $g(x, z; \theta)$ with respect to θ. Due to the involvement of $\pi_1(x)$ in particular, various functional forms of $g(x, z; \theta)$, including the conventional integrable functions and non–integrable polynomial functions, can be covered in Assumption 2.3(iii) when $\pi_1(x)$ is suitably chosen. Specifically, one may choose $\pi_1(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ when the partial derivatives of $g(x, z; \theta)$ with respect to θ are of polynomial forms. As a consequence, there may be no need to individually consider the case where $g(x, z; \theta)$ is either integrable with respect to (x, z) or asymptotically homogeneous with respect to (x, z) as has been done in the literature (see, for example, Park and Phillips 2001). In summary, the differentiability condition on $g(x, z; \theta)$ with respect to θ, along with the integrability of $G_i(x, z; \theta)$, is quite flexible and easily verifiable.

In addition, Assumption 2.3(iii) involves G_1 and G_2 with the rate of convergence of $\widehat{\theta} - \theta_0$. This is not unreasonable, since similar assumptions have been used in the literature (see, for example, Theorems 5.1-5.2 of Park and Phillips 2001; Assumption 3.1 of Gao et al. 2009a and Corollary 5.4 of Park and Phillips 2001). One particular example is $g(x_t, z_t; \theta_0) = \theta_0 x_t^2$, which implies $G_1(x_t, z_t; \theta_0) = x_t^2$ and $G_2(x_t, z_t; \theta_0) = 0$. In this case, Assumption 2.3(iii) can easily be satisfied.

We now state the main theorem of this paper; its proof is given in Appendix B.

THEOREM 2.1. *Let Assumptions 2.1–2.3 hold. Then under H_0*

$$\widehat{L}_n \quad \to_D \quad C_1(K; \pi_2, \sigma_e) \cdot L_B(1, 0) \quad \text{as } n \to \infty, \tag{2.12}$$

where $C_1(K; \pi_2, \sigma_e) = \iint K^2(u, v) dv du \cdot E[e_1^2 \; \pi_2(z_1)]$.

When $E[e_1^2 \; \pi_2(z_1)]$ is unknown, it is estimated by $\widehat{\Xi}_n = \frac{1}{n} \sum_{t=1}^n \left(y_t - g(x_t, z_t; \widehat{\theta}) \right)^2 \pi_2(z_t)$ under H_0. Under some mild condition, it is also possible to estimate ϕ in $d_n = |\phi| \sqrt{n} (1 + o(1))$ by $\{x_t, t = 1, \cdots, n\}$ directly to give $\widehat{d}_n = |\widehat{\phi}| \sqrt{n}$, and as shown in Lemma 3.1 of Dong and Gao (2014), $\widehat{d}_n / d_n \to_P 1$. The readers are referred to the paper and the references therein. Since the estimation of ϕ is a secondary issue, we do not repeat a set of additional conditions, but, in order to state Corollary 2.1 below, we need to refer the conditions in Lemma 3.1 of Dong and Gao (2014) to as **Assumption 2.4** below. We then have the following corollary; its proof is given in the supplementary document of the paper.

Assumption 2.4. *Let u_t satisfy Assumption A.2 of Dong and Gao (2014).*

COROLLARY 2.1. *Consider model (1.1). Let Assumptions 2.1–2.4 hold. Then under H_0, $\widehat{\Xi}_n \to_P E[e_1^2\ \pi_2(z_1)]$, and consequently*

$$(\widehat{\Xi}_n)^{-1}\widehat{L}_n^* \to_D C_2(K) \cdot L_B(1,0), \tag{2.13}$$

where $\widehat{L}_n^*$ is the $\widehat{L}_n$ with d_n replaced by $\widehat{d}_n$, $C_2(K) = \iint K^2(u,v)dvdu$.

Note that Corollary 2.1 shows that the asymptotic distribution is proportional to $L_B(1,0)$ that has a known distribution function given in (2.11). Note also that it is quite common in the parametric case to have a functional of Brownian motion as a limiting distribution for a unit–root test statistic. It is therefore natural to have the local–time process as the limiting distribution of the proposed test statistic of this paper.

Meanwhile, Section 3 below discusses asymptotic power properties of the proposed test and its natural competitor, and shows that the proposed test can be more powerful than a natural competitor under a sequence of local alternatives. The finite–sample study in Section 4 further confirms this.

3 Asymptotic power properties

Since the methodologies and techniques required for us to rigorously study the power function of the proposed test are not readily available, this section briefly discusses some theoretical properties of the proposed test and the natural competitor under a sequence of asymptotically localised alternatives.

We now consider an extended form of the test statistic proposed in Gao et al. (2009b) and then used in Wang and Phillips (2012) as follows:

$$\widehat{M}_n = \frac{\sum_{t=1}^n \sum_{s=1,\neq t}^n K\left(\frac{x_t-x_s}{h_1}, \frac{z_t-z_s}{h_2}\right)\ \widehat{e}_s\ \widehat{e}_t}{\sqrt{2\sum_{t=1}^n \sum_{s=1}^n K^2\left(\frac{x_t-x_s}{h_1}, \frac{z_t-z_s}{h_2}\right)\ \widehat{e}_s^2\ \widehat{e}_t^2}}. \tag{3.1}$$

Let $\widehat{M}_{2n}^2 = 2\sum_{t=1}^n \sum_{s=1}^n K^2\left(\frac{x_t-x_s}{h_1}, \frac{z_t-z_s}{h_2}\right)\ \widehat{e}_s^2\ \widehat{e}_t^2$. Similarly to the derivations of equation (3.5) below, we have under H_0: $\widehat{M}_{2n}^2 = \widetilde{M}_{2n}^2 + o_P(1)$, where

$$\begin{aligned}\widetilde{M}_{2n}^2 \equiv& 2\sum_{t=2}^n \sum_{s=1}^n K^2\left(\frac{x_t-x_s}{h_1}, \frac{z_t-z_s}{h_2}\right)\ e_s^2\ e_t^2, \quad \text{and} \\ \sigma_{2n}^2 \equiv& E\left[\widetilde{M}_{2n}^2\right] = C(1+o(1)) \cdot \sqrt{n}^3 h_1 h_2^d.\end{aligned} \tag{3.2}$$

We then show that $\widehat{L}_n(h_1, h_2)$ is asymptotically more powerful than $\widehat{M}_n(h_1, h_2)$ under a sequence of local alternatives of (1.3) satisfying the following assumption.

Assumption 3.1. *Let $\Delta_n(x, z) = \delta_n \cdot \Delta(x, z)$ where $\delta_n \to 0$ and $\delta_n^2 \sqrt{n} h_1 h_2^d \to \infty$ as $n \to \infty$, and $\Delta(x, z)$ is a nonnegative function such that*

$$0 < \iint \Delta^2(x, z) p(z) dz dx < \infty. \tag{3.3}$$

where $p(z)$ is the density of the stationary process z_t.

This assumption imposes a set of conditions on the bandwidths and the local departure function such that our test statistic is able to differentiate the null hypothesis from the alternative. Similar conditions also are imposed in the related literature. See, for example, Assumption C of Dong and Gao (2014).

In order to compare our test with $\widehat{M}_n$, we relax (3.3) to be

$$\iint \Delta^2(x, z) p^2(z) \pi_1^j(x) \pi_2^j(z) dz dx < \infty, \tag{3.4}$$

for $j = 1, 2$. It is clear that $\Delta(x, z)$ satisfying (3.3) is covered by the condition of (3.4).

Note that $\widehat{e}_t = y_t - g(x_t, z_t; \widehat{\theta}) = e_t + \Delta_n(x_t, z_t) + g(x_t, z_t; \theta_0) - g(x_t, z_t; \widehat{\theta})$ under H_1. As shown in Lemma A.2 in Appendix A below, under H_1 we have as $n \to \infty$

$$\begin{aligned}
\widehat{L}_n =& \frac{d_n}{n h_1 h_2^d} \sum_{t=1}^n \sum_{s=1}^n \widehat{e}_s\, \widehat{e}_t \iint K\left(\frac{x_t - x}{h_1}, \frac{z_t - z}{h_2}\right) K\left(\frac{x_s - x}{h_1}, \frac{z_s - z}{h_2}\right) \pi_1(x) \pi_2(z) dz dx \\
\equiv& L_{1n} + R_{1n} + o_P(1),
\end{aligned}$$

where

$$\begin{aligned}
L_{1n} =& \frac{d_n}{n h_1 h_2^d} \sum_{t=1}^n \sum_{s=1}^n e_s\, e_t \iint K\left(\frac{x_t - x}{h_1}, \frac{z_t - z}{h_2}\right) K\left(\frac{x_s - x}{h_1}, \frac{z_s - z}{h_2}\right) \pi_1(x) \pi_2(z) dz dx, \\
R_{1n} =& \frac{\delta_n^2 d_n}{n h_1 h_2^d} \sum_{t=1}^n \sum_{s=1}^n \Delta(x_s, z_s) \Delta(x_t, z_t) \iint K\left(\frac{x_t - x}{h_1}, \frac{z_t - z}{h_2}\right) \\
& \times K\left(\frac{x_s - x}{h_1}, \frac{z_s - z}{h_2}\right) \pi_1(x) \pi_2(z) dz dx.
\end{aligned}$$

Let $\widehat{M}_{1n} = \frac{1}{\sigma_{2n}} \sum_{t=1}^n \sum_{s=1}^n K\left(\frac{x_t - x_s}{h_1}, \frac{z_t - z_s}{h_2}\right) \widehat{e}_s\, \widehat{e}_t$. As can be deduced from the proof of Lemma A.2 in the supplementary document of the paper, we have under H_1,

$$\begin{aligned}
\widehat{M}_{1n} &= \frac{1}{\sigma_{2n}} \sum_{t=1}^n \sum_{s=1}^n K\left(\frac{x_t - x_s}{h_1}, \frac{z_t - z_s}{h_2}\right) \widehat{e}_s\, \widehat{e}_t \\
&= \frac{1}{\sigma_{2n}} \sum_{t=1}^n \sum_{s=1}^n K\left(\frac{x_t - x_s}{h_1}, \frac{z_t - z_s}{h_2}\right) e_s\, e_t \\
&\quad + \frac{1}{\sigma_{2n}} \sum_{t=1}^n \sum_{s=1}^n K\left(\frac{x_t - x_s}{h_1}, \frac{z_t - z_s}{h_2}\right) \Delta_n(x_s, z_s)\, \Delta_n(x_t, z_t) + o_P(1).
\end{aligned}$$

$$\equiv \quad M_{2n} + R_{2n} + o_P(1). \tag{3.5}$$

It may then be shown that under Assumptions 2.2 and 3.1, as $n \to \infty$,

$$\begin{aligned} E[R_{1n}] &= C_1(1+o(1)) \cdot \delta_n^2 \cdot \sqrt{n} h_1 h_2^d, \\ E[R_{2n}] &= C_2(1+o(1)) \cdot \delta_n^2 \cdot \sqrt{\sqrt{n} h_1 h_2^d}. \end{aligned} \tag{3.6}$$

The derivation of (3.6) is given in Appendix B.

In view of Assumption 2.2(i) that $\sqrt{n} h_1 h_2^d \to \infty$, equation (3.6) therefore implies that there is some $C_0 > 0$ such that

$$\frac{E\left[R_{1n}\right]}{E\left[R_{2n}\right]} = C_0 \sqrt{\sqrt{n} h_1 h_2^d} \to \infty. \tag{3.7}$$

Equation (3.7) indicates that $\widehat{L}_n(h_1, h_2)$ may be more powerful than $\widehat{M}_n(h_1, h_2)$ under a sequence of local departure functions that satisfy (3.3). As a matter of fact, $\widehat{L}_n(h_1, h_2)$ is indeed more powerful than $\widehat{M}_n(h_1, h_2)$. Indeed, let c_α be the α–level critical value of the limiting distribution of $\widehat{L}_n(h_1, h_2)$ and d_α be the α–level critical value of the limiting distribution of $\widehat{M}_n(h_1, h_2)$. Let $F_{n,i}(x)$ be the distributional function of $\widehat{L}_n(h_1, h_2)$ and $G_{n,i}(x)$ be the distributional function of $\widehat{M}_n(h_1, h_2)$ under H_i for $i = 0, 1$. Then, by Theorem 2.1 of this paper and the proof of Theorem 2.1 of Gao et al. (2009a), we may have as $n \to \infty$

$$\begin{aligned} F_{n,1}(x) &= P\left(\widehat{L}_n \le x\right) = (1+o(1)) P\left(L_{1n} \le x - r_{1n}\right) \\ &= (1+o(1)) F_{n,0}(x - r_{1n}), \\ G_{n,1}(x) &= P\left(\widehat{M}_n \le x\right) = (1+o(1)) P\left(M_{2n} \le x - r_{2n}\right) \\ &= (1+o(1)) G_{n,0}(x - r_{2n}), \end{aligned} \tag{3.8}$$

where $r_{1n} = E\left[R_{1n}\right]$ and $r_{2n} = E\left[R_{2n}\right]$.

Then, the power functions of $\widehat{L}_n$ and $\widehat{M}_n$ can be respectively represented by

$$\begin{aligned} \beta_{l,n}(h) &= P\left(\widehat{L}_n > c_\alpha | H_1\right) = (1+o(1))\left(1 - F_{n,0}(c_\alpha - r_{1n})\right) \\ &= (1+o(1))\left(1 - F_l(c_\alpha - r_{1n})\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \beta_{m,n}(h) &= P\left(\widehat{M}_n > d_\alpha | H_1\right) = (1+o(1))\left(1 - G_{n,0}(d_\alpha - r_{1n})\right) \\ &= (1+o(1))\left(1 - \Phi(d_\alpha - r_{2n})\right), \end{aligned} \tag{3.10}$$

where $F_l(\cdot)$ is the limit distribution of $\widehat{L}_n(h_1, h_2)$ as stated in Theorem 2.1 above, and $\Phi(\cdot)$ is the standard normal distribution as the limiting distribution of $\widehat{M}_n(h_1, h_2)$ being established in Theorem 2.1 of Gao *et al* (2009b). Further study of the power functions could be done when Edgeworth expansions of the power functions for the stationary stationary time series case become available. Proposition 3.1 of Chen et al. (2011) has only established an Edgeworth expansion for the case where e_t is independent and identically distributed.

In view of the fact that $F_l(x) = 0$ for $x < 0$ and $r_{1n} \to \infty$, we have

$$\beta_{l,n}(h) - \beta_{m,n}(h) = (1+o(1))\left(\Phi(d_\alpha - r_{2n}) - F_l(c_\alpha - r_{1n})\right)$$

$$=(1+o(1)) \cdot \Phi(d_\alpha - r_{2n}) \geq 0. \tag{3.11}$$

Our simulation study in Section 4 below supports the fact that $\widehat{L}_n(h_1, h_2)$ is indeed more powerful than $\widehat{M}_n(h_1, h_2)$ in the case where $\Delta_n(x, z) = \delta_n\ \Delta(x, z)$.

We then summarise the above discussion into the following proposition; its proof follows immediately from equation (3.9).

PROPOSITION 3.1. *Suppose that Assumptions 2.1–2.3 and 3.1 hold. Then under H_1 we have*

$$\lim_{n\to\infty} P\left(\widehat{L}_n > C\right) = 1, \tag{3.12}$$

for any positive constant $C > \ell_\alpha$, where ℓ_α is the α–level critical value of the limiting distribution of $\widehat{L}_n$.

It should be pointed out that there is a kind of trade–off between ensuring that $\widehat{L}_n$ is more powerful than $\widehat{M}_n$ and involving the weight functions $\pi_1(\cdot)$ and $\pi_2(\cdot)$, in addition to requiring Assumption 2.3 (ii). This is mainly because $\widehat{M}_n$ can be more powerful than $\widehat{L}_n$ when (3.4) is satisfied but (3.3) is not necessarily satisfied. Examples include the case where $\Delta(x, z) = \alpha x^2 + \beta z^2$. In this case, $\Delta(x, z)$ is not integrable with respect to x, but it is asymptotically homogeneous with respect to x (see, for example, Definition 2.2 of Chen *et al* 2011). However, this paper is not interested in such a case for power comparison. The main reason is that the departure function $\Delta_n(x, z)$ can be asymptotically 'large' even though $\delta_n \to 0$ with a rate. Let us just consider the univariate case where $g(x, \theta) = \alpha + \beta x$ and $\Delta_n(x_t) = \delta_n\ \Delta(x_t)$ with

$$\delta_n = \frac{1}{2} n^{-\frac{1}{8}} \log(n) \quad \text{and} \quad \Delta(x_t) = x_t^2, \tag{3.13}$$

where $x_t = x_{t-1} + u_t$ with $x_0 = o_P(\sqrt{n})$ and $u_t \sim N(0, 1)$ (an example of this type has been considered in the simulation section of Wang and Phillips 2012).

Since $E[x_n^2] = n(1 + o(1))$, we have

$$E\left[\Delta_n(x_n)\right] = \delta_n\ E[x_n^2] = \frac{1}{2} n^{\frac{7}{8}} \log(n) \to \infty \tag{3.14}$$

even though $\delta_n \to 0$.

This shows that the choice of a polynomial form for the departure function in the integrated time series case may not be so interesting because of the explosive nature of polynomial functions of such integrated time series. We are therefore only interested in the case where $\Delta(x, z)$ is a 'small' integrable function with respect to x as required in equation (3.3). As a consequence, the departure function $\Delta_n(x, z)$ can be asymptotically negligible because $\delta_n \to 0$ as $n \to \infty$. As shown in Section 4 below, the proposed test $\widehat{L}_n$ has power to pick up such 'small' departure and is more powerful than $\widehat{M}_n$ when $\Delta_n(x, z)$ is asymptotically negligible. In summary, the theoretical discussion in Sections 2 and 3, along with the finite–sample study in Section 4 below, shows that $\widehat{L}_n$ is a more powerful test than $\widehat{M}_n$ under a sequence of local alternatives.

4 Simulation evaluation

This section uses several simulated examples to show how to implement the proposed test in practice and then examine whether the proposed test works numerically. Example 4.1 considers the case where the model under the null hypothesis is a simple linear model. Some nonlinear models are used in Example 4.2. In both Examples 4.1 and 4.2, the dimensionality of z_t is $d = 1$.

Recall that we are interested in the following hypotheses:

$$\begin{aligned} H_0 &: \ P\left(m(x_t, z_t) = g(x_t, z_t; \theta_0)\right) = 1 \quad \text{versus} \\ H_1 &: \ P\left(m(x_t, z_t) = g(x_t, z_t; \theta_0) + \Delta_n(x_t, z_t)\right) = 1. \end{aligned}$$

Define the following test statistic:

$$\widehat{L}_{1n} \equiv \widehat{M}_n = \frac{\sum_{t=1}^{n} \sum_{s=1,\neq t}^{n} \widehat{e}_s K_1\left(\frac{x_s - x_t}{h_1}\right) K_2\left(\frac{z_s - z_t}{h_2}\right) \widehat{e}_t}{\widehat{\sigma}_{1n}}, \tag{4.1}$$

where $\widehat{\sigma}_{1n}^2 = 2 \sum_{t=1}^{n} \sum_{s=1,\neq t}^{n} \widehat{e}_s^2 K_1^2\left(\frac{x_s - x_t}{h_1}\right) K_2^2\left(\frac{z_s - z_t}{h_2}\right) \widehat{e}_t^2$ with $\widehat{e}_t = \left(y_t - g(x_t, z_t; \widehat{\theta})\right)$, in which $\widehat{\theta}$ is the nonlinear least squares estimator of θ defined by minimising

$$\frac{1}{n} \sum_{t=1}^{n} \left(y_t - g(x_t, z_t; \theta)\right)^2 \quad \text{over } \theta.$$

In view of equations (2.7) and (2.8), define another test statistic as an approximation to $\widehat{L}_n$:

$$\begin{aligned} \widehat{L}_{2n} \ \equiv \ \widetilde{L}_n &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \widehat{e}_t^2 \ \pi_1(x_t)\pi_2(z_t) \ \iint K_1^2(u) K_2^2(v) dv du \\ &+ \frac{2}{\sqrt{n}} \sum_{t=2}^{n} \sum_{s=1}^{t-1} \widehat{e}_t \ \widehat{e}_s \ \pi_1(x_s)\pi_2(z_s) \ L_1\left(\frac{x_t - x_s}{h_1}\right) L_2\left(\frac{z_t - z_s}{h_2}\right), \end{aligned} \tag{4.2}$$

where $\pi_1(x)$ and $\pi_2(z)$ are probability weight functions.

Our experience shows that the choice of $\pi_1(x)$ and $\pi_2(z)$ has little impact on both the size and power properties of the proposed test. In the simulated and real data examples below, we choose $\pi_1(x) = \frac{1}{\pi \ (1+x^2)}$ and $\pi_2(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. In addition, we choose $K_i(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ for $i = 1, 2$. In this case, we then have $\int_{-\infty}^{\infty} K_i^2(u) du = \frac{1}{2\sqrt{\pi}}$ and $L_i(u) = \int_{-\infty}^{\infty} K_i(v) K_i(u + v) dv = \frac{1}{2\sqrt{\pi}} e^{-\frac{u^2}{4}}$.

Mainly due to the fact that Edgeworth expansions for both $\widehat{L}_{1n}(h_1, h_2)$ and $\widehat{L}_{2n}(h_1, h_2)$ are not readily available, we are unable to adopt the power–function approach for the choice of optimal bandwidths (as discussed in Chapter 3 of Gao (2007)). Instead, we propose using estimation–based optimal bandwidths of the form:

$$\left(\widehat{h}_{1cv}, \widehat{h}_{2cv}\right) = \arg \min_{(h_1,h_2)\in H_{cv}} \frac{1}{n} \sum_{t=1}^{n} \left(y_t - \widehat{m}_{-t}(x_t, z_t; h_1, h_2)\right)^2, \tag{4.3}$$

where $\widehat{m}_{-t}(x_t, z_t; h_1, h_2) = \frac{\sum_{s=1,\neq t}^{n} K_1\left(\frac{x_t - x_s}{h_1}\right) K_2\left(\frac{z_t - z_s}{h_2}\right) y_s}{\sum_{u=1,\neq t}^{n} K_1\left(\frac{x_t - x_u}{h_1}\right) K_2\left(\frac{z_t - z_s}{h_2}\right)}$ and

$$H_{\text{cv}} = \left[c_1 \ n^{-\frac{1}{12} - c_0}, c_2 \ n^{-\frac{1}{12} + c_0}\right] \times \left[d_1 \ n^{-\frac{1}{6} - d_0}, d_2 \ n^{-\frac{1}{6} + d_0}\right]$$

for some $0 < c_1 < c_2 < \infty$, $0 < c_0 < \frac{1}{48}$, $0 < d_1 < d_2 < \infty$ and $0 < d_0 < \frac{1}{24}$. The choice of (c_1, c_2) and (d_0, d_1, d_2) is based on existing studies (see, for example, Chapter 3 of Gao (2007)). Before selecting H_{cv}, we actually calculated equation (4.3) over all possible intervals. Our preliminary simulation studies indicate that H_{cv} is the shortest possible interval on which the CV function attains its smallest value.

Let $Q_n(h_1, h_2)$ denote either $\widehat{L}_{1n}$ or $\widehat{L}_{2n}$. Our experience with Examples 4.1 and 4.2 shows that $\widehat{L}_{2n}$ already has some stable sizes and good power values under the choice of $\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$. This may be because this pair of bandwidths may be either exactly identical or very close to such bandwidth values that maximise the power function while controlling the size function. In the stationary time series case, the theory developed in Chapter 3 of Gao (2007) shows that such estimation–based optimal bandwidth values may also be optimal for testing purposes.

Let $\widehat{m}(x, z; \widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}) = \frac{\sum_{s=1}^{n} K_1\left(\frac{x - x_s}{\widehat{h}_{1\text{cv}}}\right) K_2\left(\frac{z - z_s}{\widehat{h}_{2\text{cv}}}\right) y_s}{\sum_{s=1}^{n} K_1\left(\frac{x - x_s}{\widehat{h}_{1\text{cv}}}\right) K_2\left(\frac{z - z_s}{\widehat{h}_{2\text{cv}}}\right)}$. The rest of this section proposes using a bootstrap simulation procedure to approximate the critical values of the test statistic. The main reason for using the bootstrap method is to avoid the possible size distortion which often occurs when the critical values of a test are chosen by using an asymptotic distribution theory. Let q_r be the asymptotic critical value of $Q_n\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$ at the significance level r. We then propose using the following bootstrap method to find a simulated critical value, q_r^*, to approximate q_r.

Step 1: Let $\widehat{e}_t = \left(y_t - \widehat{m}(x_t, z_t; \widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}})\right)$. Generate the bootstrap residuals $\{e_t^*\}$ by $e_t^* = \widehat{e}_t \eta_t^*$, where $\{\eta_t^*, 1 \leq t \leq n\}$ is a sequence of i.i.d. random variables drawn from

$$P\left(\eta_1^* = -\frac{\sqrt{5} - 1}{2}\right) = \frac{\sqrt{5} + 1}{2\sqrt{5}} \quad \text{and} \quad P\left(\eta_1^* = \frac{\sqrt{5} + 1}{2}\right) = \frac{\sqrt{5} - 1}{2\sqrt{5}}. \tag{4.4}$$

Step 2: Obtain $y_t^* = \widehat{m}(x_t, z_t; \widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}) + e_t^*$. The resulting sample $\{(y_t^*, x_t, z_t), 1 \leq t \leq n\}$ is called a bootstrap sample.

Step 3: Use the data set $\{(y_t^*, x_t, z_t), 1 \leq t \leq n\}$ to re–estimate $m(x, z)$ and denote its estimate by $\widehat{m}^*(x, z; \widehat{h}_{1\text{cv}}^*, \widehat{h}_{2\text{cv}}^*)$, in which $\widehat{h}_{1\text{cv}}^*$ and $\widehat{h}_{2\text{cv}}^*$ are calculated based on the data $\{(y_t^*, x_t, z_t), 1 \leq t \leq n\}$. Then calculate the test statistic $\widehat{Q}_n^*\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$, which is the corresponding version of $\widehat{Q}_n\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$ by replacing $\{(y_t, x_t, z_t)\}$ and $\widehat{m}(x, z; \widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}})$ with $\{(y_t^*, x_t, z_t)\}$ and $\widehat{m}^*(x, z; \widehat{h}_{1\text{cv}}^*, \widehat{h}_{2\text{cv}}^*)$, respectively.

Step 4: Repeat Steps 1–3 $M = 250$ times and produce $M = 250$ versions of $\widehat{Q}_n^*\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$. Denote the M versions of $\widehat{Q}_n^*\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$ by $\widehat{Q}_{n,m}^*(h_1, h_2)$, $m = 1, 2, \cdots, M$. Then, we construct the empirical distributions of $\widehat{Q}_{n,m}^*\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$. That is,

$$\mathsf{P}^*\left(\widehat{Q}_n^*\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right) \leq x\right) = \mathsf{P}\left(\widehat{Q}_n^*\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right) \leq x | \mathcal{W}_n\right),$$

where $\mathcal{W}_n = \{(y_t, x_t, z_t), 1 \le t \le n\}$.

For each pair $\left(\widehat{h}^*_{1\text{cv}}, \widehat{h}^*_{2\text{cv}}\right)$, choose q^*_r such that $P^*\left(\widehat{Q}^*_n\left(\widehat{h}^*_{1\text{cv}}, \widehat{h}^*_{2\text{cv}}\right) > q^*_r\right) = r$ and estimate q_r by q^*_r. For $\widehat{L}_{1n}$ or $\widehat{L}_{2n}$, we approximate the asymptotic critical value z_r or l_r by z^*_r or l^*_r, respectively. For $M = 250$, let also $f^*_{j\text{cv}}$ denote the frequency of $\widehat{L}_{1n}(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}; j) > z^*_r$ for $j = 0, 1$ under H_0 or H_1, and $g^*_{j\text{cv}}$ denote the frequency of $\widehat{L}_{2n}(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}; j) > l^*_r$ for $j = 0, 1$ under H_0 or H_1.

In the case where (e_t, u_t, z_t) is a vector of stationary time series, one should use a block bootstrap method. Since we only consider the case where (e_t, u_t, z_t) is a vector of i.i.d. random variables in Examples 4.1 and 4.2 below, the regression bootstrap with the choice of $\{\eta^*_t\}$ works well. In both Examples 4.1 and 4.2, we use the chosen bandwidths and then the simulated critical values z^*_r for $\widehat{L}_{1n}$ and l^*_r for $\widehat{L}_{2n}$. The corresponding simulation results are reported in Tables 4.1–4.2 below.

Example 4.1. Consider a linear time series model of the form:

$$H_0: \ y_t = \alpha + \beta x_t + \gamma z_t + e_t, \quad t = 1, 2, \cdots, n, \tag{4.5}$$

versus

$$H_1: \ y_t = \alpha + \beta x_t + \gamma z_t + \Delta_n(x_t, z_t) + e_t, \quad t = 1, 2, \cdots, n, \tag{4.6}$$

where $x_t = x_{t-1} + u_t$ with $x_0 = 0$, $\alpha = 0$, $\beta = \gamma = 1$, and $\{(e_t, z_t, u_t) : 1 \le t \le n\}$ are independent and identically distributed as

$$(e_t, z_t, u_t) \sim N(0, \Sigma_0), \quad \text{with } \Sigma_0 = \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix}, \tag{4.7}$$

with $\rho_i = 0$ for $i = 1, 2, 3$ or $\rho_1 = 0$ and $\rho_i = 0.9$ for $i = 2, 3$, and

$$\Delta_n(x, z) = \frac{\delta_n \ z^2}{\sqrt{1 + x^2}} \quad \text{with } \delta_n = \frac{\log(n)}{2n^{\frac{1}{8}}}. \tag{4.8}$$

Table 4.1: Bootstrap with $M_b = 250$ and $M = 1000$ for Example 4.1

	$\rho_1 = \rho_2 = \rho_3 = 0$				$\rho_1 = 0$ and $\rho_2 = \rho_3 = 0.9$			
H_0	$\widehat{L}_{1n}$		$\widehat{L}_{2n}$		$\widehat{L}_{1n}$		$\widehat{L}_{2n}$	
n	1%	5%	1%	5%	1%	5%	1%	5%
100	0.009	0.022	0.018	0.055	0.009	0.026	0.011	0.051
300	0.007	0.028	0.015	0.063	0.003	0.022	0.009	0.064
500	0.014	0.032	0.014	0.050	0.014	0.030	0.006	0.046
H_1								
100	0.491	0.578	0.594	0.706	0.827	0.882	0.871	0.912
300	0.692	0.754	0.853	0.901	0.938	0.962	0.941	0.967
500	0.789	0.838	0.854	0.908	0.945	0.962	0.967	0.980

Note that there is an endogeneity between e_t and (u_t, z_t) when $\rho_i \neq 0$, such as $\rho_2 = E[e_t u_t] = 0.9$ as chosen in Table 4.1. Note also that the choice of δ_n in theory is to ensure that $\delta_n \to 0$ and $\delta_n^2 \sqrt{n} h_1 h_2 \to \infty$ required in equation (3.6). Since the leading orders of h_1 and h_2 are chosen as $n^{-\frac{1}{12}}$ and $n^{-\frac{1}{6}}$, respectively in the cross–validation method in (4.3), the choice of δ_n in (4.8) satisfies the theoretical requirements. Table 4.1 gives the simulated sizes and power values at the level of $r = 1\%$ and 5%.

Example 4.2. Consider one nonlinear time series model of the form for **Case A**:

$$H_0: \; y_t = \alpha \; e^{-\beta \; x_t^2} + \gamma z_t + e_t, \quad t = 1, 2, \cdots, n, \tag{4.9}$$

versus

$$H_1: \; y_t = \alpha \; e^{-\beta \; x_t^2} + \gamma z_t + \Delta_n(x_t, z_t) + e_t, \quad t = 1, 2, \cdots, n, \tag{4.10}$$

and another nonlinear time series model of the form for **Case B**:

$$H_0: \; y_t = \alpha \; (1 + x_t^2)^{\beta} + \gamma z_t + e_t, \quad t = 1, 2, \cdots, n, \tag{4.11}$$

versus

$$H_1: \; y_t = \alpha \; (1 + x_t^2)^{\beta} + \gamma z_t + \Delta_n(x_t, z_t) + e_t, \quad t = 1, 2, \cdots, n, \tag{4.12}$$

where $x_t = x_{t-1} + u_t$ with $x_0 = 0$, $\alpha = \beta = \gamma = \frac{1}{2}$, and $\{(e_t, z_t, u_t) : 1 \leq t \leq n\}$ are independent and identically distributed as

$$(e_t, z_t, u_t) \sim N(0, \Sigma_0), \tag{4.13}$$

where Σ_0 is the same as defined in (4.7) with $\rho_i = 0$ for $i = 1, 2, 3$ or $\rho_1 = 0$ and $\rho_i = 0.9$ for $i = 2, 3$, and $\Delta_n(x, z) = \delta_n \; z^2/\sqrt{1 + x^2}$ with $\delta_n = \log(n)/(2n^{1/8})$.

The choice of δ_n is the same as in (4.8). Tables 4.2 below gives the simulated sizes and power values at the level of $r = 1\%$ and 5% for both **Cases A** and **B**.

Remark 4.1 (i) Tables 4.1 and 4.2 show that both $\widehat{L}_{1n}$ and $\widehat{L}_{2n}$ work well numerically even though the sample size is as small as $n = 100$. Meanwhile, Tables 4.1–4.2 show that the proposed test $\widehat{L}_{2n}$ is more powerful than $\widehat{L}_{1n}$ when the alternative form is chosen as in (4.8).

(ii) In both examples, we also used an asymptotic critical value and the fixed bandwidths $h_1 = n^{-\frac{1}{12}}$ and $h_2 = n^{-\frac{1}{6}}$ in each case. For $\widehat{L}_{1n}$, we used $z_{0.01} = 2.33$ at the 1% level and $z_{0.05} = 1.645$ at the 5% level. For $\widehat{L}_{2n}$, we used the critical value, l_r, of $\sigma^2(K) \; \widehat{\sigma}_e^2 \; L_B(1, 0)$ at the 1% level and at the 5% level. Our simulation results show that both tests have relatively stable sizes and good power values, but both tests are slightly under sized and the power values are uniformly smaller than the corresponding values reported in Tables 4.1 and 4.2.

(iii) Tables 4.1 and 4.2 also show that both $\widehat{L}_{1n}$ and $\widehat{L}_{2n}$ work well when there is endogeneity between (x_t, z_t) and e_t. Similar observations from their simulation study are given in Wang and Phillips (2012) for the univariate version of $\widehat{L}_{1n}$. Tables 4.1 and 4.2 clearly support that the asymptotic theory remains true for the case where there is a type of endogeneity between (x_t, z_t) and e_t as imposed in Assumption 2.1.

Table 4.2a: Bootstrap with $M_b = 250$ and $M = 1000$ for Case A in Example 4.2

	$\rho_1 = \rho_2 = \rho_3 = 0$				$\rho_1 = 0$ and $\rho_2 = \rho_3 = 0.9$			
H_0	$\widehat{L}_{1n}$		$\widehat{L}_{2n}$		$\widehat{L}_{1n}$		$\widehat{L}_{2n}$	
n	1%	5%	1%	5%	1%	5%	1%	5%
100	0.003	0.017	0.020	0.080	0.009	0.023	0.020	0.054
300	0.004	0.023	0.016	0.061	0.011	0.039	0.015	0.052
500	0.018	0.048	0.014	0.052	0.011	0.034	0.012	0.048
H_1	$\widehat{L}_{1n}$		$\widehat{L}_{2n}$		$\widehat{L}_{1n}$		$\widehat{L}_{2n}$	
100	0.142	0.245	0.500	0.614	0.325	0.467	0.884	0.921
300	0.262	0.373	0.655	0.756	0.590	0.701	0.946	0.958
500	0.368	0.484	0.717	0.819	0.672	0.755	0.957	0.974

Table 4.2b: Bootstrap with $M_b = 250$ and $M = 1000$ for Case B in Example 4.2

	$\rho_1 = \rho_2 = \rho_3 = 0$				$\rho_1 = 0$ and $\rho_2 = \rho_3 = 0.9$			
H_0	$\widehat{L}_{1n}$		$\widehat{L}_{2n}$		$\widehat{L}_{1n}$		$\widehat{L}_{2n}$	
n	1%	5%	1%	5%	1%	5%	1%	5%
100	0.004	0.008	0.021	0.070	0.004	0.017	0.024	0.071
300	0.010	0.026	0.013	0.059	0.005	0.017	0.015	0.065
500	0.007	0.018	0.015	0.049	0.003	0.016	0.012	0.054
H_1	$\widehat{L}_{1n}$		$\widehat{L}_{2n}$		$\widehat{L}_{1n}$		$\widehat{L}_{2n}$	
100	0.565	0.638	0.619	0.727	0.895	0.930	0.895	0.950
300	0.696	0.767	0.723	0.849	0.931	0.955	0.940	0.974
500	0.782	0.824	0.809	0.908	0.961	0.967	0.972	0.985

5 An empirical application

Example 5.1. This example considers the data set from the Bureau of Economic Analysis (USA Economic Accounts) at the website: http://www.bea.gov/. Let c_t = log(consumption expenditure), I_t = log(disposable income), z_t =(nominal interest rate) or w_t =(real interest rate). Note that the data sets used were quarterly data of 199 observations. The period considered here is from the first quarter of 1960 to the last quarter of 2009. Note also that the real interest rate was calculated by deducting the inflation rate over the following quarter from the nominal interest rate. Figures 1 and 2 below give the plots of the relevant data sets.

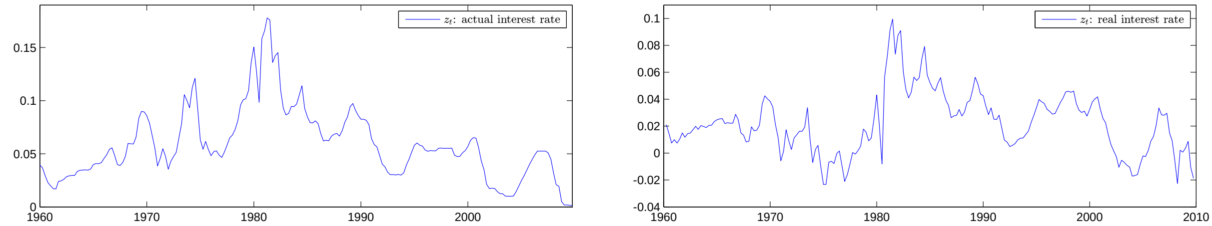
Let $y_t = c_t$, $x_t = I_t$, and $\{w_t\}$ is the real interest rate. Consider using a simple linear model of the form

$$y_t = \alpha + \beta x_t + \gamma w_t + e_t, \tag{5.1}$$

where (α, β, γ) is a vector of unknown parameters.

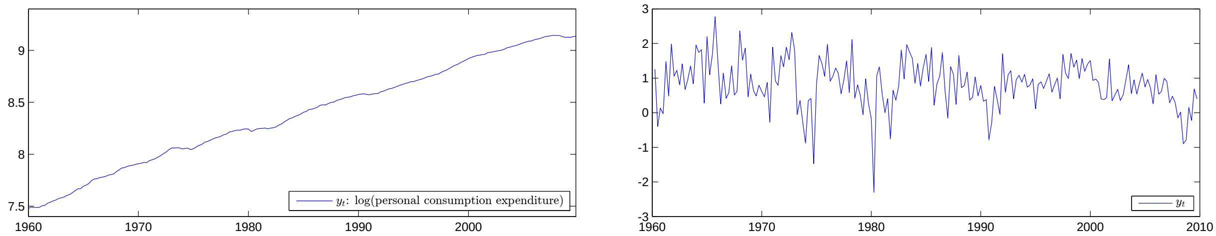
Meanwhile, there is a growing literature (see, for example, Gylfason 1981; Faff and Brooks 1998; Hahm and Steigerwald 1999; Cai et al. 2009; Xiao 2009; Gao and Phillips 2014) to support

Figure 1: The interest datasets



(a) Nominal interest rate (b) Real interest rate

Figure 2: Datasets and their difference



(a) Plot of c_t (b) Plot of $c_t - c_{t-1}$

(c) Plot of I_t (d) Plot of $I_t - I_{t-1}$

that $\beta = \beta(\cdot)$ should be treated as a function of w_t. With regard to the issue of which particular form should be chosen for $\beta(\cdot)$, polynomial functions have been commonly used (see, for example, Faff and Brooks 1998).

This section then proposes using a varying–coefficient model of the form

$$y_t = \alpha + \beta(w_t)x_t + \gamma w_t + e_t, \tag{5.2}$$

where $\beta(\cdot)$ is an unknown function, and γ is still an unknown parameter.

Existing estimation methods (see, for example, Chapter 2 of Gao (2007)) then produce a semiparametric estimate of the form $\widehat{\beta}(w)$. Its plot is given in Figure 3 below. Meanwhile, a second–order polynomial approximate form, $\widetilde{\beta}(w)$, of $\widehat{\beta}(w)$ is also given in Figure 3.

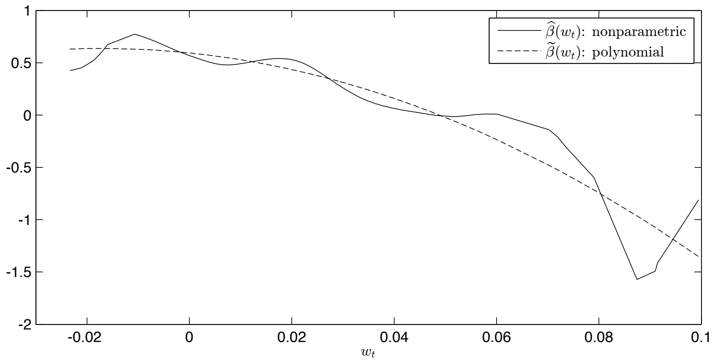
Figure 3, along with model (5.2), motivates us to rigorously support this parametric specification by testing

$$H_0: \; y_t = \theta_0 + \theta_1 x_t + \theta_2 x_t w_t + \theta_3 x_t w_t^2 + \theta_4 w_t + e_t \quad \text{versus} \tag{5.3}$$

$$H_1: \; y_t = \theta_0 + \theta_1 x_t + \theta_2 x_t w_t + \theta_3 x_t w_t^2 + \theta_4 w_t + \Delta_n(x_t, w_t) + e_t, \tag{5.4}$$

where $\Delta_n(x, z)$ is unknown and can be estimated under H_1.

Figure 3: Plots of $\widehat{\beta}(w)$ and $\widetilde{\beta}(w)$



Let $\widehat{Q}_n = \widehat{L}_{1n}\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$ or $\widehat{L}_{2n}\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$. To check whether model (5.3) is appropriate, we propose using the following simulation procedure.

- **Step 1**: Let $g(x, w; \theta) = \theta_0 + \theta_1 x_t + \theta_2 x_t w_t + \theta_3 x_t w_t^2 + \theta_4 w_t$ and $\widehat{\theta}$ be the least squares estimator. Generate the bootstrap residuals $\{e_t^*\}$ by $e_t^* = \widehat{e}_t \eta_t^*$, where $\widehat{e}_t = \left(y_t - g(x_t, w_t; \widehat{\theta})\right)$, and $\{\eta_t^*, 1 \le t \le n\}$ is a sequence of i.i.d. random variables drawn from

$$P\left(\eta_t^* = -\frac{\sqrt{5}-1}{2}\right) = \frac{\sqrt{5}+1}{2\sqrt{5}} \quad \text{and} \quad P\left(\eta_t^* = \frac{\sqrt{5}+1}{2}\right) = \frac{\sqrt{5}-1}{2\sqrt{5}}. \tag{5.5}$$

- **Step 2**: Obtain $y_t^* = g(x_t, w_t; \widehat{\theta}) + e_t^*$. Use the data set $\{(y_t^*, x_t), 1 \le t \le n\}$ to re–estimate θ and denote their estimators by $\widehat{\theta}^*$. Then calculate the test statistic $\widehat{Q}_n^*$, which is the corresponding version of $\widehat{Q}_n$ by replacing $\{(y_t, x_t, w_t)\}$ and $\widehat{\theta}$ with $\{(y_t^*, x_t, w_t)\}$ and $\widehat{\theta}^*$, respectively.

- **Step 3**: Repeat Steps 1–2 $M = 250$ times, find the bootstrap distribution of $\widehat{Q}_n^*$ and then compute the the proportion of $\widehat{Q}_n < \widehat{Q}_n^*$ for model (5.3). This proportion is an approximate P–value of $\widehat{Q}_n$ in each case.

An application of the proposed tests $\widehat{L}_{1n}\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$ and $\widehat{L}_{2n}\left(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}}\right)$, along with the proposed simulation procedure, shows that the simulated P–values are 0.1024 and 0.1436, respectively. This indicates that there is some evidence to suggest accepting a second–order polynomial form to approximate $\beta(w)$.

Models (5.2) and (5.3) show that the slope parameter $\beta(w)$ should be treated as a second–order polynomial function of w_t rather than as a constant parameter. In other words, a simple linear model of the form:

$$y_t - y_{t-1} = \alpha_0 + \alpha_1(x_t - x_{t-1}) + \alpha_2 w_t + \varepsilon_t \tag{5.6}$$

commonly used in the literature (see, for example, Campbell and Mankiw 1989; Campbell et al. 1997; Jawadi and Leoni 2012), may not be justifiable and suitable for such data set.

Let $\widetilde{y}_t = y_t - y_{t-1}$ and $\widetilde{x}_t = x_t - x_{t-1}$. In view of models (5.1)–(5.6), we propose using a first–order polynomial function of w_t to replace α_1 in model (5.6) and then compare the following model

$$\widetilde{y}_t = \beta_0 + (\beta_1 + \beta_2 w_t)\widetilde{x}_t + \beta_3 w_t + \eta_t \tag{5.7}$$

with a commonly used model of the form

$$\widetilde{y}_t = \gamma_0 + \gamma_1 \widetilde{x}_t + \gamma_2 w_t + \zeta_t. \tag{5.8}$$

The estimated versions of models (5.7) and (5.8) become respectively

$$\widetilde{y}_t = 0.554 + (0.345 - 0.366 w_t)\,\widetilde{x}_t + 0.116 w_t \quad \text{and} \tag{5.9}$$

$$\widetilde{y}_t = 0.558 + 0.341 \widetilde{x}_t - 0.199 w_t, \tag{5.10}$$

with the estimated standard deviations of the parameter estimates being between 0.0241 and 0.0632. An application of $\widehat{L}_{1n}(\widehat{h}_{1\text{cv}}, \widehat{h}_{2\text{cv}})$ to check whether either model (5.7) or model (5.8) is appropriate as a parametric model gives the simulated P–values of 0.1209 and 0.04387, respectively. This indicates that there is some evidence to support using model (5.7) in practice for such data sets.

In summary, our findings show that the β parameter involved in model (5.1) should be treated as a varying–coefficient function of w and an appropriately chosen polynomial form may be appropriate for this kind of empirical analysis.

6 Conclusions and discussions

We have proposed a new testing method for a general model specification in a nonlinear time series model with multivariate regressors. A new asymptotic theory has been established for the proposed test. Simulated examples have been used to evaluate the finite–sample performance of the proposed test as well as comparison with a natural competitor. Meanwhile, the proposed test has also been applied to test the suitability of a simple linear model commonly used in the consumption–income literature. Further studies are needed to investigate whether the new approach proposed in this paper is applicable to deal with the autoregressive case where $x_t = y_{t-1}$ in model (1.1). Such an issue is left for future research.

A Lemmas

This section consists of two parts, the first containing crucial lemmas to support the proof of the main result of the paper while the second providing some preliminary lemmas. The proof of all lemmas is relegated to Appendix C in the supplementary document of the paper.

A.1 Crucial Lemmas

Lemma A.1. *Let the conditions of Theorem 2.1 hold. Denote*

$$\begin{aligned}
S_{1n} =& \frac{d_n}{nh_1h_2^d}\sum_{t=1}^n e_t^2 \iint K^2\left(\frac{x_t-x}{h_1},\frac{z_t-z}{h_2}\right)\pi_1(x)\pi_2(z)dzdx,\\
\widehat{S}_{1n} =& \frac{d_n}{nh_1h_2^d}\sum_{t=1}^n \widehat{e}_t^2 \iint K^2\left(\frac{x_t-x}{h_1},\frac{z_t-z}{h_2}\right)\pi_1(x)\pi_2(z)dzdx,\\
S_{2n} =& \frac{d_n}{nh_1h_2^d}\sum_{t\neq s=1}^n e_se_t \iint K\left(\frac{x_t-x}{h_1},\frac{z_t-z}{h_2}\right)K\left(\frac{x_s-x}{h_1},\frac{z_s-z}{h_2}\right)\pi_1(x)\pi_2(z)dzdx,\\
\widehat{S}_{2n} =& \frac{d_n}{nh_1h_2^d}\sum_{t\neq s=1}^n \widehat{e}_s\widehat{e}_t \iint K\left(\frac{x_t-x}{h_1},\frac{z_t-z}{h_2}\right)K\left(\frac{x_s-x}{h_1},\frac{z_s-z}{h_2}\right)\pi_1(x)\pi_2(z)dzdx.
\end{aligned}$$

Under H_0, we then have as $n\to\infty$

$$\widehat{S}_{1n} = S_{1n} + o_P(1), \qquad \widehat{S}_{2n} = S_{2n} + o_P(1). \tag{A.1}$$

Lemma A.2. *Let the conditions of Theorem 2.1 hold. Let $\widehat{S}_{1n}$ and $\widehat{S}_{2n}$ be the same as in the preceding lemma. Under H_1 and Assumption 3.1, we have as $n\to\infty$*

$$\widehat{S}_{1n} = \frac{d_n}{nh_1h_2^d}\sum_{t=1}^n \Delta_n^2(x_t,z_t)\iint K^2\left(\frac{x_t-x}{h_1},\frac{z_t-z}{h_2}\right)\pi_1(x)\pi_2(z)dzdx(1+o_P(1)), \tag{A.2}$$

$$\begin{aligned}
\widehat{S}_{2n} =& \frac{d_n}{nh_1h_2^d}\sum_{t\neq s=1}^n \Delta_n(x_s,z_s)\Delta_n(x_t,z_t)\\
&\times\iint K\left(\frac{x_t-x}{h_1},\frac{z_t-z}{h_2}\right)K\left(\frac{x_s-x}{h_1},\frac{z_s-z}{h_2}\right)\pi_1(x)\pi_2(z)dzdx(1+o_P(1)).
\end{aligned}\tag{A.3}$$

Lemma A.3. *Let Assumptions 2.1–2.3 hold. Then as $n\to\infty$*

$$\frac{d_n}{n}\sum_{t=1}^n e_t^2\varpi_1(x_t)\varpi_2(z_t) \to_D L_B(1,0)\cdot E\left[e_1^2\varpi_2(z_1)\right]\cdot\int\varpi_1(x)dx, \tag{A.4}$$

provided that $E\left[e_1^2\varpi_2(z_1)\right]\neq 0$, where $\varpi_i(\cdot)=\pi_i(\cdot)$ or $D_i(\cdot)$ for $i=1,2$ are all defined in Assumption 2.3.

Note that Lemma A.3 and its proof may be of general interest. Before the proof of the main result, three preliminary lemmas are presented next.

A.2 Preliminary Lemmas

We need to study some densities of variables consisting of regressors. Recall that $\mathcal{F}_t=\sigma\{\epsilon_s, s\le t\}$ a sigma filtration generated by the innovation sequence up to t. Without loss of generality, let $x_t=0$ almost surely. In view of the definition of u_i in Assumption 2.1,

$$x_t = \sum_{i=1}^t u_i = \sum_{\ell=-\infty}^t b_{t,\ell}\epsilon_\ell, \quad \text{with } b_{t,\ell} = \sum_{i=\max(1,\ell)}^t \phi_{i-\ell}. \tag{A.5}$$

For $s < t$, we have

$$x_t = \sum_{i=1}^{t} u_i = x_s + \sum_{i=s+1}^{t} u_i = x_{ts}^* + x_{ts}, \tag{A.6}$$

$$\text{where } \ x_{ts} \equiv \sum_{\ell=s+1}^{t} b_{t,\ell}\epsilon_\ell, \quad x_{ts}^* = x_s + \bar{x}_{ts}, \quad \text{with } \ \bar{x}_{ts} = \sum_{\ell=-\infty}^{s} \left(\sum_{i=s+1}^{t} \phi_{i-\ell} \right) \epsilon_\ell.$$

Notice that x_{ts}^* adapted with $\mathcal{F}_s$ and x_{ts} are mutually independent. It follows that

$$d_{ts}^2 := E[x_{ts}^2] = \sigma_0^2 \sum_{\ell=s+1}^{t} b_{t,\ell}^2 = O(t-s),$$

using $b_{t,\ell} = \sum_{i=0}^{t-\ell} \phi_i = \phi(1+o(1))$ when $t-\ell$ and, moreover, in view of Assumption 2.1,

$$E[\bar{x}_{ts}^2] = \sigma_0^2 \sum_{\ell=-\infty}^{s} \left(\sum_{i=s+1}^{t} \phi_{i-\ell} \right)^2 = O(1) \sum_{\ell=-\infty}^{s} (s+1-\ell)^{-2\beta} = O(1),$$

since $\beta > 1/2$ and $\sum_{i=0}^{\infty} i^\beta |\phi_i| < \infty$.

In addition, denote

$$\lambda_{t,m_0} \equiv \lambda(\epsilon_t, \cdots, \epsilon_{t-m_0+1}) = \sum_{\ell=t-m_0+1}^{t} b_{t,\ell}\epsilon_\ell, \tag{A.7}$$

$$x_{t(m_0)} \equiv \sum_{\ell=-\infty}^{t-m_0} b_{t,\ell}\epsilon_\ell, \quad \text{and} \quad x_{ts(m_0)} \equiv \sum_{\ell=s+1}^{t-m_0} b_{t,\ell}\epsilon_\ell. \tag{A.8}$$

Thus, we have different version decompositions of x_t, (1) $x_t = x_{ts} + x_{ts}^*$ for $t > s$; (2) $x_t = \lambda_{t,m_0} + x_{t(m_0)}$ for $t > m_0$ and (3) $x_t = \lambda_{t,m_0} + x_{ts(m_0)} + x_{ts}^*$ for $t - s > m_0$. Such versions would facilitate the calculation of expectations in the proof of the results.

Lemma A.4. *Let Assumption 2.1(i) hold.*

(1) For large t, $d_t^{-1} x_t$ have densities $f_t(x)$ which are uniformly bounded over $x \in \mathbb{R}$ and t. Meanwhile, the derivatives of $f_t(x)$ are uniformly bounded as well. Consequently, $f_t(x)$ satisfy a uniform Lipschitz condition

$$\sup_x |f_t(x + \triangle x) - f_t(x)| \leq C|\triangle x| \tag{A.9}$$

for some $C > 0$ and any $\triangle x \in \mathbb{R}$.

(2) For large $t-s$, $d_{ts}^{-1} x_{ts}$ have uniformly bounded densities $f_{ts}(x)$ over all $x \in \mathbb{R}$ and t, s; additionally, $f_{ts}(x)$ have bounded derivatives and satisfy Lipshitz condition similar to (A.9) *as well.*

For fixed j, $j \leq t$, define $x_{t/j}$ by

$$x_{t/j} = \sum_{\ell=-\infty, \neq j}^{t} b_{t,\ell}\epsilon_\ell,$$

and hence, $x_t = b_{t,j}\epsilon_j + x_{t/j}$ and $x_{t/j}$ and ϵ_j are mutually independent.

In addition, if $s < j \leq t$, define $x_{ts/j}$ by

$$x_{ts/j} = \sum_{\ell=s+1, \neq j}^{t} b_{t,\ell}\epsilon_\ell.$$

and then $x_{ts} = b_{t,j}\epsilon_j + x_{ts/j}$. Clearly, $x_{ts/j}$ is independent of ϵ_j.

Lemma A.5. *Let Assumption 2.1(i) hold.*

(1) *For $j \le t$ and large t, $d_t^{-1} x_{t/j}$ have densities $f_{t/j}(x)$ which are uniformly bounded over $x \in \mathbb{R}$, t and j; in addition, $f_{t/j}(x)$ satisfy Lipshitz condition similar to* (A.9). *This is also true for $d_t^{-1} x_{t(m_0)}$, where their density is denoted by $f_{t(m_0)}(x)$.*

(2) *For $s < j \le t$ and large $t - s$, $d_{ts}^{-1} x_{ts/j}$ have densities $f_{ts/j}(x)$ which are uniformly bounded over $x \in \mathbb{R}$ and $t, j, s \ge 1$; moreover, $f_{ts/j}(x)$ satisfy Lipshitz condition similar to* (A.9). *This is also true for $d_{ts}^{-1} x_{ts(m_0)}$, where their density is denoted by $f_{ts(m_0)}(x)$.*

Lemma A.6. *Let Assumptions 2.1 hold.*

(1). *(A) Let $p_t(x, z, u)$ be the joint density function of $(d_t^{-1} x_t, z_t, \epsilon_j)$ for $t - m_0 + 1 \le j \le t$, $q(z, u)$ be the joint density of (z_t, ϵ_j). Then, for large t*

$$|p_t(x, z, u) - f_{t(m_0)}(x) q(z, u)| \le C d_t^{-1} \ \nu(z, u), \tag{A.10}$$

where $\nu(z, u)$ is given by

$$\nu(z, u) = |u| q(z, u) + \sum_{\ell = t - m_0 + 1, \neq j}^{t} \int |w_\ell| q(z, u, w_\ell) dw_\ell \tag{A.11}$$

in which $q(z, u, w_\ell)$ is the joint density of $(z_t, \epsilon_j, \epsilon_\ell)$, $j \neq \ell$, and $\nu(z, u)$ has the following properties: (i) $\nu(z, u)$ is nonnegative and integrable, $\iint \nu(z, u) dz du < \infty$; (ii) the moments $\iint |u|^i \nu(z, u) dz du < \infty$ exist up to the third order $i = 3$.

(B) Let $p_{ts}(x, z, u)$ be the joint density function of $(d_{ts}^{-1} x_{ts}, z_t, \epsilon_j)$ for $t - m_0 + 1 \le j \le t$. Then for large $t - s$

$$|p_{ts}(x, z, u) - f_{ts(m_0)}(x) q(z, u)| \le C d_{ts}^{-1} \ \nu(z, u) \tag{A.12}$$

where $\nu(z, u)$ is defined by (A.11).

(C) Let $p_{ts}(x, z, u, v)$ be the joint density function of $(d_{ts}^{-1} x_{ts}, z_t, \epsilon_{j_1}, \epsilon_{j_2})$ for $t - m_0 + 1 \le j_1 \neq j_2 \le t$. Then

$$|p_{ts}(x, z, u, v) - f_{ts(m_0)}(x) q(z, u, v)| \le C d_{ts}^{-1} \nu(z, u, v),$$

where $q(z, u, v)$ is the joint density of $(z_t, \epsilon_{j_1}, \epsilon_{j_2})$ and $\nu(z, u, v)$ is given by

$$\nu(z, u, v) = (|u| + |v|) q(z, u, v) + \sum_{\ell = t - m_0 + 1, \neq j_1, j_2}^{t} \int |w_\ell| q(z, u, v, w_\ell) dw_\ell, \tag{A.13}$$

in which $q(z, u, v, w_\ell)$ is the joint density of $(z_t, \epsilon_{j_1}, \epsilon_{j_2}, \epsilon_\ell)$.

(2). *(A) Let $q_t(x, z)$ be the joint density function of $(d_t^{-1} x_t, z_t)$ and $p(z)$ be the density of z_t. Then, for large t*

$$|q_t(x, z) - f_{t(m_0)}(x) p(z)| \le C d_t^{-1} \mu(z), \tag{A.14}$$

where $\mu(z)$ is given by

$$\mu(z) = \sum_{\ell=t-m_0+1}^{t} \int |w_\ell| \cdot q(z, w_\ell) dw_\ell \tag{A.15}$$

where $q(z, w_\ell)$ is the joint density of (z_t, ϵ_ℓ); $\mu(z)$ is a nonnegative and integrable function.

(B) Let $q_{ts}(x, z)$ be the joint density function of $(d_{ts}^{-1} x_{ts}, z_t)$. Then, for large $t - s$,

$$|q_{ts}(x, z) - f_{ts(m_0)}(x)p(z)| \leq C d_{ts}^{-1} \mu(z).$$

(C) Let $q_{ts/j}(x, z)$ be the joint density function of $(d_{ts}^{-1} x_{ts/j}, z_t)$ for $s + 1 \leq j \leq t - m_0$. Then, for large $t - s$,

$$|q_{ts/j}(x, z) - f_{ts(m_0)/j}(x)p(z)| \leq C d_{ts}^{-1} \mu(z),$$

where $f_{ts(m_0)/j}(x)$ is the density of $d_{ts}^{-1} x_{ts(m_0)/j}$. Here, $x_{ts(m_0)/j}$ is the one excluding ϵ_j from $x_{ts(m_0)}$ for $s + 1 \leq j \leq t - m_0$.

Note: In the lemma, p_{ts} is used to denote both the density of $(d_{ts}^{-1} x_{ts}, z_t, \epsilon_j)$ and the density of $(d_{ts}^{-1} x_{ts}, z_t, \epsilon_{j_1}, \epsilon_{j_2})$; meanwhile, q is used to stand for the density of (z_t, ϵ_j), the density of $(z_t, \epsilon_{j_1}, \epsilon_{j_2})$ and the density of $(z_t, \epsilon_{j_1}, \epsilon_{j_2}, \epsilon_\ell)$. Their difference is reflected by their arguments only.

B Proof of the Main Results

This section gives an outline proof of Theorem 2.1, followed by the derivations of (3.6) and (3.7). The detailed proof of Theorem 2.1 is relegated to the supplement of the paper.

Outline of The Proof of Theorem 2.1. Note by Lemma A.1 that $\widehat{L}_n = \widehat{S}_{1n} + \widehat{S}_{2n} = S_{1n} + S_{2n} + o_P(1)$. In order to prove Theorem 2.1, it suffices to show that as $n \to \infty$

$$S_{1n} \to_D L_B(1, 0) \cdot E\left[e_1^2 \pi_2(z_1)\right] \iint K^2(u, v) dv du, \tag{B.1}$$

$$S_{2n} = o_P(1). \tag{B.2}$$

We start with the proof of (B.1). Under Assumptions 2.1 and 2.3, using Lemma A.3, we have as $n \to \infty$

$$\begin{aligned}
S_{1n} =& \frac{d_n}{n} \sum_{t=1}^{n} e_t^2 \iint K^2(u, v) \pi_1(x_t - uh_1) \pi_2(z_t - vh_2) du dv \\
=& \frac{d_n}{n} \sum_{t=1}^{n} e_t^2 \iint K^2(u, v) \pi_1(x_t) \pi_2(z_t) du dv \\
&+ \frac{d_n}{n} \sum_{t=1}^{n} e_t^2 \iint K^2(u, v) \left(\pi_1(x_t - uh_1) - \pi_1(x_t)\right) \left(\pi_2(z_t - vh_2) - \pi_2(z_t)\right) du dv \\
&+ \frac{d_n}{n} \sum_{t=1}^{n} e_t^2 \iint K^2(u, v) \left(\pi_1(x_t - uh_1) - \pi_1(x_t)\right) \pi_2(z_t) dv du \\
&+ \frac{d_n}{n} \sum_{t=1}^{n} e_t^2 \iint K^2(u, v) \pi_1(x_t) \left(\pi_2(z_t - vh_2) - \pi_2(z_t)\right) du dv
\end{aligned}$$

$$
\begin{aligned}
=&\frac{d_n}{n}\sum_{t=1}^{n} e_t^2\pi_1(x_t)\pi_2(z_t)\iint K^2(u,v)dudv\\
&+O(h_1h_2^d)\cdot\frac{d_n}{n}\sum_{t=1}^{n} e_t^2 D_1(x_t)D_2(z_t)\iint |u||v|K^2(u,v)dvdu\\
&+O(h_1)\cdot\frac{d_n}{n}\sum_{t=1}^{n} e_t^2 D_1(x_t)\pi_2(z_t)\iint |u|K^2(u,v)dvdu\\
&+\ O(h_2^d)\cdot\frac{d_n}{n}\sum_{t=1}^{n} e_t^2\pi_1(x_t)D_2(z_t)\iint |v|K^2(u,v)dvdu\\
=&\frac{d_n}{n}\sum_{t=1}^{n} e_t^2\pi_1(x_t)\pi_2(z_t)\iint K^2(u,v)dudv+o_P(1)\\
\to_D& L_B(1,0)\cdot E\left[e_1^2\pi_2(z_1)\right]\cdot\iint K^2(u,v)dudv,
\end{aligned}
$$

which completes the proof of (B.1).

We then prove (B.2). Denote $S_{2n}=\frac{2d_n}{nh_1h_2^d}\sum_{t=2}^{n}\sum_{s=1}^{t-1}B(t,s)e_te_s$, where, recalling that $K(u,v)=K_1(u)K_2(v)$,

$$
B(t,s)\equiv\iint K_1\left(\frac{x_t-x}{h_1}\right)K_2\left(\frac{z_t-z}{h_2}\right)K_1\left(\frac{x_s-x}{h_1}\right)K_2\left(\frac{z_s-z}{h_2}\right)\pi_1(x)\pi_2(z)dzdx. \tag{B.3}
$$

Similarly to the proof of (B.1), we have

$$
\begin{aligned}
B(t,s)=&h_1h_2^d\int K_1\left(\frac{x_t-x_s}{h_1}+u\right)K_1(u)\,\pi_1(x_s-uh_1)du\\
&\times\int K_2\left(\frac{z_t-z_s}{h_2}+v\right)K_2(v)\pi_2(z_s-vh_2)dv\\
=&h_1h_2^d\pi_1(x_s)\ \pi_2(z_s)\cdot L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\\
&+h_1h_2^d\cdot\int K_1\left(\frac{x_t-x_s}{h_1}+u\right)K_1(u)\left(\pi_1(x_s-uh_1)-\pi_1(x_s)\right)du\\
&\qquad\times\int K_2\left(\frac{z_t-z_s}{h_2}+v\right)K_2(v)\left(\pi_2(z_s-vh_2)-\pi_2(z_s)\right)dv\\
&+h_1h_2^d\pi_2(z_s)L_2\left(\frac{z_t-z_s}{h_2}\right)\int K_1\left(\frac{x_t-x_s}{h_1}+u\right)K_1(u)\left(\pi_1(x_s-uh_1)-\pi_1(x_s)\right)du\\
&+h_1h_2^d\pi_1(x_s)L_1\left(\frac{x_t-x_s}{h_1}\right)\int K_2\left(\frac{z_t-z_s}{h_2}+v\right)K_2(v)\left(\pi_2(z_s-vh_2)-\pi_2(z_s)\right)dv\\
\equiv&\ h_1h_2^d\left(B_1(t,s)+B_2(t,s)+B_3(t,s)+B_4(t,s)\right),
\end{aligned}
$$

where $L_i(v)=\int K_i(u+v)K_i(u)du$ for $i=1,2$, and

$$
\begin{aligned}
B_1(t,s)=&\pi_1(x_s)\ \pi_2(z_s)\cdot L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\\
B_2(t,s)=&\int K_1\left(\frac{x_t-x_s}{h_1}+u\right)K_1(u)\left(\pi_1(x_s-uh_1)-\pi_1(x_s)\right)du\\
&\times\int K_2\left(\frac{z_t-z_s}{h_2}+v\right)K_2(v)\left(\pi_2(z_s-vh_2)-\pi_2(z_s)\right)dv \qquad\qquad (B.4)\\
B_3(t,s)=&\pi_2(z_s)L_2\left(\frac{z_t-z_s}{h_2}\right)\int K_1\left(\frac{x_t-x_s}{h_1}+u\right)K_1(u)\left(\pi_1(x_s-uh_1)-\pi_1(x_s)\right)du
\end{aligned}
$$

$$B_4(t,s) = \pi_1(x_s) L_1\left(\frac{x_t - x_s}{h_1}\right) \int K_2\left(\frac{z_t - z_s}{h_2} + v\right) K_2(v) \left(\pi_2(z_s - vh_2) - \pi_2(z_s)\right) dv.$$

Then, we have

$$\begin{aligned} S_{2n} =& \frac{2d_n}{n}\sum_{t=2}^{n}\sum_{s=1}^{t-1} B_1(t,s)e_t e_s + \frac{2d_n}{n}\sum_{t=2}^{n}\sum_{s=1}^{t-1} B_2(t,s)e_t e_s \\ &+ \frac{2d_n}{n}\sum_{t=2}^{n}\sum_{s=1}^{t-1} B_3(t,s)e_t e_s + \frac{2d_n}{n}\sum_{t=2}^{n}\sum_{s=1}^{t-1} B_4(t,s)e_t e_s \\ :=& 2(B_{n1} + \cdots + B_{n4}), \quad \text{say.} \end{aligned} \tag{B.5}$$

We shall show that $B_{ni} = o_P(1)$ for $i = 1, \cdots, 4$ in (B.5). Since the proof is much technical and tedious, it is provided in the supplement of the paper.

Derivations of (3.6) and (3.7). In view of the definitions of R_{2n} in (3.5) and $\Delta_n(\cdot,\cdot)$, we consider $R_{2n} \equiv \frac{1}{\sigma_{2n}}\delta_n^2 \Pi_n$, where

$$\begin{aligned} \Pi_n =& \sum_{t=1}^{n}\sum_{s=1}^{n} K_1\left(\frac{x_t - x_s}{h_1}\right) K_2\left(\frac{z_t - z_s}{h_2}\right) \Delta(x_s, z_s)\, \Delta(x_t, z_t) \\ =& K_1(0)K_2(0)\sum_{t=1}^{n}\Delta^2(x_t, z_t) + 2\sum_{t=2}^{n}\sum_{s=1}^{t-1} K_1\left(\frac{x_t - x_s}{h_1}\right) K_2\left(\frac{z_t - z_s}{h_2}\right) \Delta(x_s, z_s)\Delta(x_t, z_t) \\ \equiv& \Pi_{1n} + \Pi_{2n}. \end{aligned}$$

Ignoring unimportant constants and using Lemma A.6 for $t \geq m_n$ where $m_n \to \infty$ and $m_n^2/n \to o$ with $n \to \infty$,

$$\begin{aligned} E[\Pi_{1n}] =& o(1) + \sum_{t=m_n}^{n} E\Delta^2(x_t, z_t) = \sum_{t=m_n}^{n} \iint \Delta^2(d_t x, z) q_t(x,z)dxdz \\ =& \sum_{t=m_n}^{n} d_t^{-1} \iint \Delta^2(x,z) q_t(d_t^{-1}x, z)dxdz + o(1) \\ =& \sum_{t=m_n}^{n} d_t^{-1} \iint \Delta^2(x,z) f_{t(m_0)}(d_t^{-1}x)p(z)dxdz(1 + o(1)) \\ =& O(1)n^{1/2} \iint \Delta^2(x,z)p(z)dxdz + o(1) = O\left(n^{1/2}\right), \end{aligned}$$

where $q_t(d_t^{-1}x, z)$ is replaced by $f_{t(m_0)}(d_t^{-1}x)p(z)$ because their difference is a higher infinitesimal in view of Lemma A.6, and additionally the fact that $f_{t(m_0)}(\cdot)$ is approximated by the standard normal density $\phi(\cdot)$ is used. See the proof of Corollary 2.2 of Wang and Phillips (2009a).

To proceed further, let $m_n \to \infty$ and $m_n^4/n \to 0$ as $n \to \infty$. We have

$$\begin{aligned} \Pi_{2n} =& \sum_{t=2}^{n}\sum_{s=1}^{t-1} K_1\left(\frac{x_t - x_s}{h_1}\right) K_2\left(\frac{z_t - z_s}{h_2}\right) \Delta(x_s, z_s)\Delta(x_t, z_t) \\ =& \sum_{t=2}^{m_n}\sum_{s=1}^{t-1} K_1\left(\frac{x_t - x_s}{h_1}\right) K_2\left(\frac{z_t - z_s}{h_2}\right) \Delta(x_s, z_s)\Delta(x_t, z_t) \\ &+ \sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n} K_1\left(\frac{x_t - x_s}{h_1}\right) K_2\left(\frac{z_t - z_s}{h_2}\right) \Delta(x_s, z_s)\Delta(x_t, z_t) \end{aligned}$$

$$+\sum_{t=m_n+1}^{n}\sum_{s=1}^{m_n} K_1\left(\frac{x_t-x_s}{h_1}\right)K_2\left(\frac{z_t-z_s}{h_2}\right)\Delta(x_s,z_s)\Delta(x_t,z_t)$$
$$+\sum_{t=m_n+1}^{n}\sum_{s=t-m_n+1}^{t} K_1\left(\frac{x_t-x_s}{h_1}\right)K_2\left(\frac{z_t-z_s}{h_2}\right)\Delta(x_s,z_s)\Delta(x_t,z_t).$$

It is known that the second term is the leading term of Π_{2n}. We thus consider its expectation by noting the nonnegativity of $\Delta(\cdot,\cdot)$. By virtue of Lemma A.6,

$$\begin{aligned}
&E\left[K_1\left(\frac{x_t-x_s}{h_1}\right)K_2\left(\frac{z_t-z_s}{h_2}\right)\Delta(x_s,z_s)\Delta(x_t,z_t)|\mathcal{F}_s\right]\\
=&E\left[K_1\left(\frac{x_{ts}+x_{ts}^*-x_s}{h_1}\right)K_2\left(\frac{z_t-z_s}{h_2}\right)\Delta(x_s,z_s)\Delta(x_{ts}+x_{ts}^*,z_t)|\mathcal{F}_s\right]\\
=&\iint K_1\left(\frac{d_{ts}x+x_{ts}^*-x_s}{h_1}\right)K_2\left(\frac{z-z_s}{h_2}\right)\Delta(x_s,z_s)\Delta(d_{ts}x+x_{ts}^*,z)q_{ts}(x,z)dxdz\\
=&h_1h_2^d d_{ts}^{-1}\iint K_1(x)K_2(z)\Delta(x_s,z_s)\Delta(h_1x+x_s,h_2z+z_s)\\
&\times q_{ts}\left(\frac{h_1x-x_{ts}^*+x_s}{d_{ts}},h_2z+z_s\right)dxdz\\
=&h_1h_2^d d_{ts}^{-1}\iint K_1(x)K_2(z)\Delta(x_s,z_s)\Delta(h_1x+x_s,h_2z+z_s)\\
&\times f_{ts(m_0)}\left(\frac{h_1x-x_{ts}^*+x_s}{d_{ts}}\right)p(h_2z+z_s)dxdz(1+o_P(1))\\
=&h_1h_2^d d_{ts}^{-1}\iint K_1(x)K_2(z)\Delta(x_s,z_s)\Delta(h_1x+x_s,h_2z+z_s)\\
&\times f_{ts(m_0)}\left(\frac{h_1x}{d_{ts}}\right)p(h_2z+z_s)dxdz(1+o_P(1)),
\end{aligned}$$

where $q_t(\cdot,\cdot)$ is substituted by the product of $f_{ts(m_0)}(\cdot)$ and $p(\cdot)$, and due to $x_{ts}^*-x_s=\bar{x}_{ts}=O_p(1)$, $f_{ts(m_0)}\left(\frac{h_1x-x_{ts}^*+x_s}{d_{ts}}\right)=f_{ts(m_0)}\left(\frac{h_1x}{d_{ts}}\right)(1+o_P(1))$ by the Lipschitz condition. We continue to calculate

$$\begin{aligned}
&E\iint K_1(x)K_2(z)\Delta(x_s,z_s)\Delta(h_1x+x_s,h_2z+z_s)f_{ts(m_0)}\left(\frac{h_1x}{d_{ts}}\right)p(h_2z+z_s)dxdz\\
=&\int\cdots\int K_1(x)K_2(z)\Delta(d_sy,v)\Delta(h_1x+d_sy,h_2z+v)f_{ts(m_0)}\left(\frac{h_1x}{d_{ts}}\right)\\
&\times p(h_2z+v)q_s(y,v)dydvdxdz\\
=&d_s^{-1}\int\cdots\int K_1(x)K_2(z)\Delta(y,v)\Delta(h_1x+y,h_2z+v)f_{ts(m_0)}\left(\frac{h_1x}{d_{ts}}\right)p(h_2z+v)\\
&\times f_{s(m_0)}\left(\frac{y}{d_s}\right)p(v)dydvdxdz\\
=&d_s^{-1}f_{ts(m_0)}(0)f_{s(m_0)}(0)\iint K_1(x)K_2(z)dxdz\iint\Delta^2(y,v)p^2(v)dydv=O(1)d_s^{-1},
\end{aligned}$$

where the approximation of $\sup_x|f_t(x)-\phi(x)|\le O(1)d_t^{-1}$ for $f_{ts(m_0)}(\cdot)$ and $f_{s(m_0)}(\cdot)$ is used where $\phi(x)$ is the density of a standard normal variable (see again the proof of Corollary 2.2 of Wang and Phillips 2009a) as well as Taylor expansion to derive the leading term. We finally have

$$\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n} EK_1\left(\frac{x_t-x_s}{h_1}\right)K_2\left(\frac{z_t-z_s}{h_2}\right)\Delta(x_s,z_s)\Delta(x_t,z_t)$$

$$=O(1)h_1h_2^d\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}d_{ts}^{-1}d_s^{-1}=C_1h_1h_2^dn,$$

for some constant $C_2>0$, giving that $E[\Pi_n]=C_1h_1h_2^dn$.

In addition, straightforward but tedious calculation yields as $n\to\infty$,

$$\sigma_{2n}^2\equiv E\left[\sum_{t=1}^{n}\sum_{s=1}^{n}K_1^2\left(\frac{x_t-x_s}{h_1}\right)K_2^2\left(\frac{z_t-z_s}{h_2}\right)e_s^2e_t^2\right]=C_2(1+o(1))\ n^{\frac{3}{2}}h_1h_2^d.$$

Similarly, for R_{1n} we may show that as $n\to\infty$,

$$\begin{aligned}
&\sum_{t=2}^{n}\sum_{s=1}^{t-1}E\left[\iint K_1\left(\frac{x_t-x}{h_1}\right)K_2\left(\frac{z_t-z}{h_2}\right)K_1\left(\frac{x_s-x}{h_1}\right)K_2\left(\frac{z_s-z}{h_2}\right)\right.\\
&\qquad\left.\times\ \Delta(x_s,z_s)\Delta(x_t,z_t)\right]\pi_1(x)\pi_2(z)dzdx\\
=&h_1h_2^d\sum_{t=2}^{n}\sum_{s=1}^{t-1}E\left[\iint K_1\left(\frac{x_t-x_s}{h_1}+x\right)K_2\left(\frac{z_t-z_s}{h_2}+z\right)K_1(x)K_2(z)\right.\\
&\qquad\left.\times\ \Delta(x_s,z_s)\Delta(x_t,z_t)\pi_1(x_s-h_1x)\pi_2(z_s-h_2z)\right]dzdx\\
=&(1+o(1))h_1h_2^d\sum_{t=2}^{n}\sum_{s=1}^{t-1}E\left[\Delta(x_s,z_s)\Delta(x_t,z_t)L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\pi_1(x_s)\pi_2(z_s)\right]\\
=&C_2(1+o(1))\ nh_1^2h_2^{2d},
\end{aligned}$$

for some constant $C_2>0$, where $L_i(u)=\int K_i(u+v)K_i(v)dv$, $i=1,2$ and Taylor expansion is used for $\pi_1(x_s-h_1x)$ and $\pi_2(z_s-h_2z)$ at regions $|x|<\varepsilon h_1^{-1}$ and $\|z\|<\varepsilon h_2^{-1}$, respectively, for some fixed $\varepsilon>0$, and beyond the regions the property of K_1 and K_2, i.e. $\int|x|K_1(x)dx<\infty$ and $\int\|z\|K_2(z)dz<\infty$, is used to make sure the reminder terms are of smaller order.

In view of the expression of R_{1n} and R_{2n}, we therefore have as $n\to\infty$

$$\begin{aligned}
E[R_{2n}]=&O(1)\frac{nh_1h_2^d\delta_n^2}{n^{\frac{3}{4}}\sqrt{h_1h_2^d}}=O(1)\ \delta_n^2\ \sqrt{\sqrt{n}h_1h_2^d},\\
E[R_{1n}]=&O(1)\ \delta_n^2\ \sqrt{n}h_1h_2^d,
\end{aligned}\tag{B.6}$$

which complete the derivations of (3.6) and (3.7).

Abstract

This supplementary material provides the proofs of Lemmas A.1–A.6 in Appendix C, Theorem 2.1 and Corollary 2.1 in Appendix D.

Appendix C Proofs of Lemmas

This appendix gives the full proofs of Lemmas A.1–A.6 listed in Appendix A of the paper. However, it is reasonable to first prove Lemmas A.4–A.6, since they provide some preliminary results for the establishment of all other results.

Proof of Lemma A.4 (1) Under Assumption 2.1, $\int |\lambda\varphi(\lambda)|d\lambda < \infty$ where $\varphi(\cdot)$ is the characteristic function of ϵ_1. Let $\Psi_t(\alpha)$ be the characteristic function of $d_t^{-1}x_t$ for $\alpha \in \mathbb{R}$. Recall that $x_t = \sum_{j=-\infty}^{t} b_{t,j}\epsilon_j$ in Appendix A. Denote $x_t = x_t^+ + x_t^-$ where x_t^+ includes all ϵ_j with $j > 0$ in x_t, while x_t^- includes all ϵ_j with $j \leq 0$ in x_t. It follows that

$$\begin{aligned}
&\int |\alpha||\Psi_t(\alpha)|d\alpha = \int |\alpha||E\exp(i\alpha d_t^{-1}x_t)|d\alpha \leq \int |\alpha||E\exp(i\alpha_1 d_t^{-1}x_t^+)|d\alpha \\
=&\int |\alpha| \left| E\exp\left[i\left(\alpha d_t^{-1}\sum_{j=1}^{t} b_{t,j}\epsilon_j \right)\right]\right| d\alpha = \int |\alpha| \left| \prod_{j=1}^{t} E\exp(i\alpha d_t^{-1}b_{t,j}\epsilon_j)\right| d\alpha \\
=&\int |\alpha| \prod_{j=1}^{t} \left|\varphi(\alpha d_t^{-1}b_{t,j})\right| d\alpha.
\end{aligned}$$

It is clear that there exists a $\delta_0 > 0$ such that $|\varphi(\lambda)| < e^{-|\lambda|^2/4}$ whenever $|\lambda| \leq \delta_0$ and $|\varphi(\lambda)| < \eta$ if $|\lambda| > \delta_0$ for some $0 < \eta < 1$. (Wang and Phillips, 2009a, p. 730).

Note also that $b_{t,j} = \phi_0 + \cdots + \phi_{t-j}$. If $t-j$ is large, $b_{t,j} = \phi(1+o(1))$ where $\phi = \sum_j \phi_j \neq 0$. Let $\nu = \nu_t$ be a function of t such that $\nu \to \infty$ and $\nu/t \to 0$ as $t \to \infty$. Thus, for $1 \leq j \leq t-\nu$, there exist constants c_1, c_2 such that $0 < c_1 < c_2 < \infty$ and $c_1 < |b_{t,j}| < c_2$. Indeed, we may take $c_1 = |\phi|/2$ and $c_2 = 3|\phi|/2$. Therefore, letting $\delta = \delta_0/c_2$,

$$\begin{aligned}
&\int |\alpha| \prod_{j=1}^{t} \left|\varphi(\alpha d_t^{-1}b_{t,j})\right| d\alpha \leq \int |\alpha| \prod_{j=1}^{t-\nu} \left|\varphi(\alpha d_t^{-1}b_{t,j})\right| d\alpha \\
=&\left(\int_{|\alpha|\leq d_t\delta} + \int_{|\alpha|>d_t\delta} \right) |\alpha| \prod_{j=1}^{t-\nu} \left|\varphi(\alpha d_t^{-1}b_{t,j})\right| d\alpha \\
\leq&\int_{|\alpha|\leq d_t\delta} |\alpha| e^{-\alpha^2 d_t^{-2}\sum_{j=1}^{t-\nu} b_{t,j}^2/4} d\alpha + \eta^{t-\nu-1}\int_{|\alpha|>d_t\delta} |\alpha| \left|\varphi\left(\alpha d_t^{-1}b_{t,1}\right)\right| d\alpha \\
\leq&\int_{|\alpha|\leq d_t\delta} |\alpha| e^{-\alpha^2 c_1(1-\nu/t)/4} d\alpha + b_{t,1}^{-2} d_t^2 \eta^{t-\nu-1}\int_{|\alpha|>\delta} |\alpha| \left|\varphi(\alpha)\right| d\alpha \\
\leq&\int |\alpha| e^{-\alpha^2 c_1/4} d\alpha + b_{t,1}^{-2} d_t^2 \eta^{t-\nu-1}\int |\alpha| \left|\varphi(\alpha)\right| d\alpha < \infty,
\end{aligned}$$

where we have used the fact that $d_t^2\eta^{t-\nu-1} \to 0$ and $b_{t,1} \to \phi \neq 0$ as $t \to \infty$.

The integrability of $|\Psi_t(\alpha)|$ implies the uniformly boundedness of the densities $f_t(x)$ due to the inverse formula. Similarly, the integrability of $|\alpha||\Psi_t(\alpha)|$ gives the uniformly boundedness of the partial derivatives of $f_t(x)$. In fact

$$\begin{aligned}
\left|\frac{d}{dx}f_t(x)\right| =&\frac{1}{2\pi}\left|\frac{d}{dx}\int e^{-i\alpha x}\Psi_t(\alpha)d\alpha\right| = \frac{1}{2\pi}\left|\int (-i\alpha)e^{-i\alpha x}\Psi_t(\alpha)d\alpha\right| \\
\leq&\frac{1}{2\pi}\int |\alpha||\Psi_t(\alpha)|d\alpha \leq C.
\end{aligned}$$

It follows immediately from the mean value theorem that $\sup_x |f_t(x+\triangle x) - f_t(x)| \leq C|\triangle x|$. Thus, (1) is valid.

The assertions of (2) follow in the same fashion. □

Proof of Lemma A.5 The proof is similar to that of Lemma A.4, so it is omitted. □

Proof of Lemma A.6 (1) The assertions of (A) are proven first. Note that $z_t = \psi(\varepsilon_{t,m_0})$ where $\varepsilon_{t,m_0} \equiv (\epsilon_t, \cdots, \epsilon_{t-m_0+1})$. Let $p(v, w_t, \cdots, w_{t-m_0+1})$ be the joint density of $(z_t, \epsilon_t, \cdots, \epsilon_{t-m_0+1})$.

Observe that

$$
\begin{aligned}
&P(d_t^{-1}x_t < x, z_t < z, \epsilon_j < u)\\
=&P(d_t^{-1}x_{t(m_0)} < x - d_t^{-1}\lambda(\varepsilon_{t,m_0}), z_t < z, \epsilon_j < u)\\
=&\int\cdots\int_{v<z,w_j<u} P(d_t^{-1}x_{t(m_0)} < x - d_t^{-1}\lambda(w))p(v, w_t, \cdots, w_{t-m_0+1})dv \prod_{\ell=t-m_0+1}^{t} dw_\ell,
\end{aligned}
$$

where $w = (w_t, \cdots, w_{t-m_0+1})$, $x_{t(m_0)}$ and $\lambda(w)$ are defined by (A.7) and (A.8) in Appendix A.

Recall that $d_t^{-1}x_{t(m_0)}$ has density $f_{t(m_0)}(x)$ that satisfies the Lipschitz condition. We have

$$
\begin{aligned}
p_t(x, z, u) =& \frac{\partial^{d+2}}{\partial x\partial z\partial u} P(d_t^{-1}x_t < x, z_t < z, \epsilon_j < u)\\
=&\frac{\partial^{d+2}}{\partial x\partial z\partial u}\int\cdots\int_{v<z,w_j<u} P(d_t^{-1}x_{t(m_0)} < x - d_t^{-1}\lambda(w))p(v, w_t, \cdots, w_{t-m_0+1})dv \prod_{\ell=t-m_0+1}^{t} dw_\ell\\
=&\frac{\partial^{d+1}}{\partial z\partial u}\int\cdots\int_{v<z,w_j<u} f_{t(m_0)}(x - d_t^{-1}\lambda(w))p(v, w_t, \cdots, w_{t-m_0+1})dv \prod_{\ell=t-m_0+1}^{t} dw_\ell\\
=&\int\cdots\int f_{t(m_0)}(x - d_t^{-1}\lambda(w_t, \cdots, u, \cdots, w_{t-m_0+1}))\\
&\quad\times p(z, w_t, \cdots, u, \cdots, w_{t-m_0+1}) \prod_{\ell=t-m_0+1,\neq j}^{t} dw_\ell\\
=&f_{t(m_0)}(x)\int\cdots\int p(z, w_t, \cdots, u, \cdots, w_{t-m_0+1}) \prod_{\ell=t-m_0+1,\neq j}^{t} dw_\ell\\
&+\int\cdots\int [f_{t(m_0)}(x - d_t^{-1}\lambda(w_t, \cdots, u, \cdots, w_{t-m_0+1})) - f_{t(m_0)}(x)]\\
&\quad\times p(z, w_t, \cdots, u, \cdots, w_{t-m_0+1}) \prod_{\ell=t-m_0+1,\neq j}^{t} dw_\ell\\
=&f_{t(m_0)}(x)q(z, u) + \int\cdots\int [f_{t(m_0)}(x - d_t^{-1}\lambda(w_t, \cdots, u, \cdots, w_{t-m_0+1})) - f_{t(m_0)}(x)]\\
&\quad\times p(z, w_t, \cdots, u, \cdots, w_{t-m_0+1}) \prod_{\ell=t-m_0+1,\neq j}^{t} dw_\ell.
\end{aligned}
$$

It follows from the Lipschitz condition for $f_{t(m_0)}(x)$ and the linearity of $\lambda(\cdot)$ that

$$
\begin{aligned}
&|p_t(x, z, u) - f_{t(m_0)}(x)q(z, u)|\\
\leq&Cd_t^{-1}\int\cdots\int |\lambda(w_t, \cdots, u, \cdots, w_{t-m_0+1})|p(z, w_t, \cdots, u, \cdots, w_{t-m_0+1}) \prod_{\ell=t-d+1,\neq j}^{t} dw_\ell\\
\leq&Cd_t^{-1}\int\cdots\int \left(|u| + \sum_{\ell=t-m_0+1,\neq j}^{t} |w_\ell|\right) p(z, w_t, \cdots, u, \cdots, w_{t-m_0+1}) \prod_{\ell=t-m_0+1,\neq j}^{t} dw_\ell\\
=&Cd_t^{-1}\Big[\int\cdots\int |u|p(z, w_t, \cdots, u, \cdots, w_{t-m_0+1}) \prod_{\ell=t-m_0+1,\neq j}^{t} dw_\ell\\
&\quad+\sum_{\ell=t-m_0+1,\neq j}^{t}\int\cdots\int |w_\ell|p(z, w_t, \cdots, u, \cdots, w_{t-m_0+1}) \prod_{\ell=t-m_0+1,\neq j}^{t} dw_\ell\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&Cd_t^{-1}\left[|u|q(z,u) + \sum_{\ell=t-m_0+1,\neq j}^{t} \int |w_\ell| q(z,u,w_\ell)dw_\ell\right] \\
=&Cd_t^{-1}\nu(z,u).
\end{aligned}
$$

Here, $\nu(z,u) = |u|q(z,u) + \sum_{\ell=t-m_0+1,\neq j}^{t} \int |w_\ell|q(z,u,w_\ell)dw_\ell$ is an integrable function for (z,u). To see this, note that the marginal density $\int q(z,u)dz = f_\epsilon(u)$ and $\int p(z,u,w_\ell)dz = f_\epsilon(w_\ell)f_\epsilon(u)$, and hence

$$
\begin{aligned}
\iint \nu(z,u)dzdu =& \int |u|f_\epsilon(u)du + \sum_{\ell=t-m_0+1,\neq j}^{t} \iint |w_\ell| f_\epsilon(w_\ell)f_\epsilon(u)dudw_\ell \\
=& \sum_{\ell=t-m_0+1}^{t} \int |w_\ell| f_\epsilon(w_\ell)dw_\ell = m_0 \cdot E|\epsilon_0| < \infty.
\end{aligned}
$$

Also, in view of Assumption 2.1, the moments $\iint |u|^i \nu(z,u)dzdu$ exist up to the third order.

The proof of (B) and (C) follows in the same fashion as (A).

(2) To prove (A), observe that

$$
\begin{aligned}
&P(d_t^{-1}x_t < x, z_t < z) = P(d_t^{-1}x_{t(m_0)} < x - d_t^{-1}\lambda(\varepsilon_{t,m_0}), z_t < z) \\
=& \int \cdots \int^z P(d_t^{-1}x_{t(m_0)} < x - d_t^{-1}\lambda(w))p(v,w_t,\cdots,w_{t-m_0+1})dv \prod_{\ell=t-m_0+1}^{t} dw_\ell,
\end{aligned}
$$

where $p(v,w_t,\cdots,w_{t-m_0+1})$ is the joint density of $(z_t,\epsilon_t,\cdots,\epsilon_{t-m_0+1})$. We have

$$
\begin{aligned}
q_t(x,z) =& \frac{\partial^{d+1}}{\partial x \partial z} P(d_t^{-1}x_t < x, z_t < z) \\
=& \frac{\partial^{d+1}}{\partial x \partial z} \int \cdots \int^z P(d_t^{-1}x_{t(m_0)} < x - d_t^{-1}\lambda(w)) \\
& \times p(v,w_t,\cdots,w_{t-m_0+1})dv \prod_{\ell=t-m_0+1}^{t} dw_\ell \\
=& \frac{\partial^d}{\partial z} \int \cdots \int^z f_{t(m_0)}(x - d_t^{-1}\lambda(w))p(v,w_t,\cdots,w_{t-m_0+1})dv \prod_{\ell=t-m_0+1}^{t} dw_\ell \\
=& \int \cdots \int f_{t(m_0)}(x - d_t^{-1}\lambda(w))p(z,w_t,\cdots,w_{t-m_0+1}) \prod_{\ell=t-m_0+1}^{t} dw_\ell \\
=& f_{t(m_0)}(x) \int \cdots \int p(z,w_t,\cdots,w_{t-m_0+1}) \prod_{\ell=t-m_0+1}^{t} dw_\ell \\
& + \int \cdots \int [f_{t(m_0)}(x - d_t^{-1}\lambda(w)) - f_{t(m_0)}(x)]p(z,w_t,\cdots,w_{t-m_0+1}) \prod_{\ell=t-m_0+1}^{t} dw_\ell \\
=& f_{t(m_0)}(x)p(z) \\
& + \int \cdots \int [f_{t(m_0)}(x - d_t^{-1}\lambda(w)) - f_{t(m_0)}(x)]p(z,w_t,\cdots,w_{t-m_0+1}) \prod_{\ell=t-m_0+1}^{t} dw_\ell,
\end{aligned}
$$

which implies that

$$
|q_t(x,z) - f_{t(m_0)}(x)p(z)|
$$

$$
\begin{aligned}
&\leq C d_t^{-1} \int \cdots \int |\lambda(w))| p(z, w_t, \cdots, w_{t-m_0+1}) \prod_{\ell=t-d+1}^{t} dw_\ell \\
&\leq C d_t^{-1} \int \cdots \int \sum_{\ell=t-m_0+1}^{t} |w_\ell| \cdot p(z, w_t, \cdots, w_{t-m_0+1}) \prod_{\ell=t-m_0+1}^{t} dw_\ell \\
&= C d_t^{-1} \sum_{\ell=t-m_0+1}^{t} \int \cdots \int |w_\ell| \cdot p(z, w_t, \cdots, w_{t-m_0+1}) \prod_{\ell=t-m_0+1}^{t} dw_\ell \\
&= C d_t^{-1} \sum_{\ell=t-m_0+1}^{t} \int |w_\ell| \cdot q(z, w_\ell) dw_\ell = C d_t^{-1} \mu(z)
\end{aligned}
$$

where $q(z, w_\ell)$ is the joint density of (z_t, ϵ_ℓ). Clearly,

$$
\int \mu(z) dz = \sum_{\ell=t-m_0+1}^{t} \int |w_\ell| f_\epsilon(w_\ell) dw_\ell = m_0 \cdot E|\epsilon_0|.
$$

The assertions of (B) and (C) follow by using the densities $f_{ts(m_0)}(x)$ of $d_{ts}^{-1} x_{ts(m_0)}$ and $f_{ts(m_0)/j}(x)$ of $d_{ts}^{-1} x_{ts(m_0)/j}$, respectively. □

Proof of Lemma A.1. Recall that under H_0:

$$
\widehat{e}_t = y_t - g(x_t, z_t; \widehat{\theta}) = e_t + g(x_t, z_t; \theta_0) - g(x_t, z_t; \widehat{\theta}) \equiv e_t + r_n(x_t, z_t), \tag{C.1}
$$

where $r_n(x, z) = g(x, z; \theta_0) - g(x, z; \widehat{\theta})$. Recall also that $K(x, z) = K_1(x) K_2(z)$.

We may prove the two equations in (A.1) simultaneously by considering

$$
\begin{aligned}
\widehat{T}_n \equiv& \frac{d_n}{n h_1 h_2^d} \sum_{t=1}^{n} \sum_{s=1}^{n} \widehat{e}_s \widehat{e}_t \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
&\times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x) \pi_2(z) dz dx \\
=& \frac{d_n}{n h_1 h_2^d} \sum_{t=1}^{n} \sum_{s=1}^{n} (e_t + r_n(x_t, z_t))(e_s + r_n(x_s, z_s)) \\
&\times \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x) \pi_2(z) dz dx \\
\equiv& \widehat{T}_{1n} + \widehat{T}_{2n} + \widehat{T}_{3n} + \widehat{T}_{4n},
\end{aligned} \tag{C.2}
$$

where

$$
\begin{aligned}
\widehat{T}_{1n} =& \frac{d_n}{n h_1 h_2^d} \sum_{t=1}^{n} \sum_{s=1}^{n} e_s e_t \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
&\times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x) \pi_2(z) dz dx \\
\widehat{T}_{2n} =& \frac{d_n}{n h_1 h_2^d} \sum_{t=1}^{n} \sum_{s=1}^{n} e_s r_n(x_t, z_t) \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
&\times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x) \pi_2(z) dz dx \\
\widehat{T}_{3n} =& \frac{d_n}{n h_1 h_2^d} \sum_{t=1}^{n} \sum_{s=1}^{n} e_t r_n(x_s, z_s) \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right)
\end{aligned}
$$

$$\times \ K_1\left(\frac{x_s-x}{h_1}\right)K_2\left(\frac{z_s-z}{h_2}\right)\pi_1(x)\pi_2(z)dzdx$$

$$\widehat{T}_{4n} = \frac{d_n}{nh_1h_2^d}\sum_{t=1}^n\sum_{s=1}^n r_n(x_t,z_t)r_n(x_s,z_s)\iint K_1\left(\frac{x_t-x}{h_1}\right)K_2\left(\frac{z_t-z}{h_2}\right)$$
$$\times \ K_1\left(\frac{x_s-x}{h_1}\right)K_2\left(\frac{z_s-z}{h_2}\right)\pi_1(x)\pi_2(z)dzdx$$

and we will show $\widehat{T}_{jn} = o_P(1)$ for $j = 2, 3, 4$ as $n \to \infty$.

By Assumption 2.2(ii), $\|\widehat{\theta} - \theta_0\| = o_P(1)$, as $n \to \infty$. We thus consider the case where $\|\widehat{\theta} - \theta_0\| < \epsilon$ holds in probability in the following derivations. Using Assumptions 2.2(i) and 2.3(iii) in particular, we then have $|r_n(x_t, z_t)| = |g(x_t, z_t; \theta_0) - g(x_t, z_t; \widehat{\theta})| \le \|G_1(x_t, z_t; \theta_0)\| \, \|\widehat{\theta} - \theta_0\| + \|G_2(\cdot, \cdot; \theta_0)\|\|\widehat{\theta} - \theta_0\|^2$, and accordingly, $|\widehat{T}_{4n}| \le \widehat{T}_{4n1} + \widehat{T}_{4n2}$ where we define

$$\widehat{T}_{4n1} \equiv \|\widehat{\theta} - \theta_0\|^2 \frac{d_n(1 + o_P(1))}{n}\sum_{t=1}^n\sum_{s=1}^n \|G_1(x_t, z_t; \theta_0)\| \cdot \|G_1(x_s, z_s; \theta_0)\|\pi_1(x_s)\pi_2(z_s)$$
$$\times L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right) \tag{C.3}$$

and $\widehat{T}_{4n2}$ is the remainder term that involves $\|G_2(\cdot, \cdot; \theta_0)\|\|\widehat{\theta} - \theta_0\|^2$, which is of an order higher than $\widehat{T}_{4n1}$.

Notice that in $\widehat{T}_{4n1}$ we only need to consider s and t large enough such that $s > m_n$ and $t - s > m_n$ say, where $m_n \to \infty$ and $m_n^4/n \to 0$ as $n \to \infty$, since we could control all the rest terms as small as we wish.

Straightforward calculation as for Π_{2n} in the derivation of (3.6) and (3.7) we have as $n \to \infty$

$$\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n} E\left[\|G_1(x_t, z_t; \theta_0)\| \cdot \|G_1(x_s, z_s; \theta_0)\|\pi_1(x_s)\pi_2(z_s)L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\right]$$
$$= O(1)nh_1h_2^d,$$

which, along with Assumption 2.2(ii), implies that $\widehat{T}_{4n} = o_P(1)$.

To show that $\widehat{T}_{jn} = o_P(1)$ for $j = 2, 3$, we will need to repeatedly use Assumption 2.3. Similarly to the derivation of (C.3), we have

$$\widehat{T}_{2n} = \frac{d_n(1 + o_P(1))}{n}\sum_{t=1}^n\sum_{s=1}^n e_t r_n(x_s, z_s; \theta_0) \cdot L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\pi_1(x_s)\pi_2(z_s)$$
$$= \frac{d_n}{n}(\theta_0 - \widehat{\theta})\sum_{t=1}^n\left(\sum_{s=1}^n G_1(x_s, z_s; \theta_0)L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\pi_1(x_s)\pi_2(z_s)\right)e_t$$
$$+ \frac{d_n}{n}\sum_{t=1}^n\sum_{s=1}^n\left(r_n(x_s, z_s; \theta_0) - G_1(x_s, z_s; \theta_0)^\tau(\theta_0 - \widehat{\theta})\right)e_t$$
$$\times \ L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\pi_1(x_s)\pi_2(z_s)$$
$$\equiv \frac{d_n}{n}(\theta_0 - \widehat{\theta}) \cdot I_{1n} + \frac{d_n}{n} \cdot I_{2n}. \tag{C.4}$$

Observe that

$$E\left[I_{1n}^2\right] = E\left[\sum_{t=1}^n\left(\sum_{s=1}^n G_1(x_s, z_s; \theta_0)L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\pi_1(x_s)\pi_2(z_s)\right)e_t\right]^2$$

$$
\begin{aligned}
=&\sum_{t=1}^{n} E\left(\left[\sum_{s=1}^{n} G_1(x_s,z_s;\theta_0)L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\pi_1(x_s)\pi_2(z_s)\right]^2 e_t^2\right)\\
&+2\sum_{t_1=2}^{n}\sum_{t_2=1}^{t_1-1}\sum_{s_1=1}^{n} E\left[G_1(x_{s_1},z_{s_1};\theta_0)L_1\left(\frac{x_{t_1}-x_{s_1}}{h_1}\right)L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)\pi_1(x_{s_1})\pi_2(z_{s_1})e_{t_1}\right.\\
&\quad\left.\times\sum_{s_2=1}^{n} G_1(x_{s_2},z_{s_2};\theta_0)L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\pi_1(x_{s_2})\pi_2(z_{s_2})e_{t_2}\right]\\
=&\sum_{t=1}^{n}\sum_{s=1}^{n} E\left(\left[G_1(x_s,z_s;\theta_0)L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\pi_1(x_s)\pi_2(z_s)\right]^2 e_t^2\right)\\
&+2\sum_{t=1}^{n}\sum_{s_1=2}^{n} E\left[e_t^2 G_1(x_{s_1},z_{s_1};\theta_0)L_1\left(\frac{x_t-x_{s_1}}{h_1}\right)L_2\left(\frac{z_t-z_{s_1}}{h_2}\right)\pi_1(x_{s_1})\pi_2(z_{s_1})\right.\\
&\quad\left.\times\sum_{s_2=1}^{s_1-1} G_1(x_{s_2},z_{s_2};\theta_0)L_1\left(\frac{x_t-x_{s_2}}{h_1}\right)L_2\left(\frac{z_t-z_{s_2}}{h_2}\right)\pi_1(x_{s_2})\pi_2(z_{s_2})\right]\\
&+2\sum_{t_1=2}^{n}\sum_{t_2=1}^{t_1-1}\sum_{s=1}^{n} E\left[e_{t_1}e_{t_2}G_1^2(x_s,z_s;\theta_0)\pi_1^2(x_s)\pi_2^2(z_s)L_1\left(\frac{x_{t_1}-x_s}{h_1}\right)L_2\left(\frac{z_{t_1}-z_s}{h_2}\right)\right.\\
&\quad\left.\times L_1\left(\frac{x_{t_2}-x_s}{h_1}\right)L_2\left(\frac{z_{t_2}-z_s}{h_2}\right)\right]\\
&+4\sum_{t_1=2}^{n}\sum_{t_2=1}^{t_1-1}\sum_{s_1=2}^{n} E\left[e_{t_1}e_{t_2}G_1(x_{s_1},z_{s_1};\theta_0)L_1\left(\frac{x_{t_1}-x_{s_1}}{h_1}\right)L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)\pi_1(x_{s_1})\pi_2(z_{s_1})\right.\\
&\quad\left.\times\sum_{s_2=1}^{s_1-1} G_1(x_{s_2},z_{s_2};\theta_0)L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\pi_1(x_{s_2})\pi_2(z_{s_2})\right]\\
\equiv& J_1+J_2+J_3+J_4,\quad \text{say.}
\end{aligned}
$$

Using the decomposition for x_t, the densities in Lemma A.4 and the conditional argument (the same calculation as that in the proof of $S_{2n}=o_P(1)$ in Theorem 2.1 below) yields

$$
\begin{aligned}
J_1 &= O(1) n h_1 h_2^d, & J_2 &= O(1) n^{3/2} h_1^2 h_2^{2d},\\
J_3 &= O(1)\sqrt{n}\ln(n) h_1^2 h_2^{2d}, & J_4 &= O(1) n\ln(n) h_1^2 h_2^{2d},
\end{aligned}
$$

showing that the first part of (C.4) is $o_P(1)$.

Moreover, by Assumption 2.2(i) we have

$$
\begin{aligned}
|I_{2n}| \le& \sum_{t=1}^{n}\sum_{s=1}^{n}\left|r_n(x_s,z_s;\theta_0)-G_1(x_s,z_s;\theta_0)\left(\theta_0-\widehat{\theta}\right)\right| L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\\
&\times \pi_1(x_s)\pi_2(z_s)|e_t|\\
\le& \|\widehat{\theta}-\theta_0\|^2\cdot\sum_{t=1}^{n}\sum_{s=1}^{n}\|G_2(x_s,z_s;\theta_0)\| L_1\left(\frac{x_t-x_s}{h_1}\right)L_2\left(\frac{z_t-z_s}{h_2}\right)\pi_1(x_s)\pi_2(z_s)|e_t|\\
\equiv& \|\widehat{\theta}-\theta_0\|^2\cdot I_{3n}, \qquad\qquad (C.5)
\end{aligned}
$$

in view of the fact that both $L_i(\cdot)$ and $\pi_i(\cdot)$ are positive. A simple calculation by virtue of the structure of x_t, z_t and e_t gives $E[I_{3n}] = O(1)Cnh_1h_2^d$ which, in view of Assumption 2.2 (ii), implies that the second part of (C.4) is $o_P(1)$ as well, and hence $\widehat{T}_{2n} = o_P(1)$. The same conclusion is manifestly true for $\widehat{T}_{3n}$. Therefore, we have shown under H_0 as $n \to \infty$

$$\widehat{T}_n = \frac{d_n}{nh_1h_2^d} \sum_{t=1}^{n}\sum_{s=1}^{n} e_s e_t \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z)dzdx + o_P(1) \tag{C.6}$$

This completes the proof of Lemma A.1. □

Proof of Lemma A.2. We only prove equation (A.3), as the proof of (A.2) follows similarly. Notice that under H_1,

$$\widehat{e}_t = e_t + r_n(x_t, z_t; \theta_0) + \Delta_n(x_t, z_t) \tag{C.7}$$

where $r_n(x, z; \theta_0) = g(x, z; \theta_0) - g(x, z; \widehat{\theta})$ and $\Delta_n(x, z) = \delta_n \Delta(x, z)$ as defined in Assumption 3.1.

For $\widehat{T}_n$ defined in (C.2), we have under H_1:

$$\begin{aligned}
\widehat{T}_n =& \frac{d_n}{nh_1h_2^d} \sum_{t=1}^{n}\sum_{s=1}^{n} \widehat{e}_s \widehat{e}_t \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
&\times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z)dzdx \\
=& \sum_{j=1}^{4} \widehat{T}_{jn} + \frac{d_n}{nh_1h_2^d} \sum_{t=1}^{n}\sum_{s=1}^{n} \Delta_n(x_s, z_s)\Delta_n(x_t, z_t) \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
&\times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z)dzdx \\
&+ \frac{d_n}{nh_1h_2^d} \sum_{t=1}^{n}\sum_{s=1}^{n} \Delta_n(x_s, z_s) r_n(x_t, z_t; \theta_0) \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
&\times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z)dzdx \\
&+ \frac{d_n}{nh_1h_2^d} \sum_{t=1}^{n}\sum_{s=1}^{n} \Delta_n(x_t, z_t) r_n(x_s, z_s; \theta_0) \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
&\times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z)dzdx \\
&+ \frac{d_n}{nh_1h_2^d} \sum_{t=1}^{n}\sum_{s=1}^{n} \Delta_n(x_t, z_t) e_s \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
&\times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z)dzdx \\
&+ \frac{d_n}{nh_1h_2^d} \sum_{t=1}^{n}\sum_{s=1}^{n} \Delta_n(x_s, z_s) e_t \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
&\times K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z)dzdx \equiv \sum_{j=1}^{4} \widehat{T}_{jn} + \sum_{k=5}^{9} \widehat{T}_{kn},
\end{aligned} \tag{C.8}$$

where $\widehat{T}_{jn}$ for $1 \le j \le 4$ are the same as in (C.2).

In view of the proof of (C.6) and equation (A.3), in order to complete the proof of Lemma A.2, we need only to deal with $\sum_{k=6}^{9} \widehat{T}_{kn}$. Precisely, we are about to show that under H_1:

$$\widehat{T}_{kn} = o_P\left(\delta_n^2 \sqrt{n} h_1 h_2^d\right) \quad \text{for } k = 6, \cdots, 9 \tag{C.9}$$

noting that $\widehat{T}_{5n} = O_P\left(\delta_n^2\sqrt{n}h_1h_2^d\right)$.

To complete the proof of (C.9), obviously we need only to deal with $\widehat{T}_{6n}$ and $\widehat{T}_{9n}$. Using similar arguments to those used in the derivation of (C.3), we have as $n \to \infty$

$$
\begin{aligned}
|\widehat{T}_{6n}| \leq \delta_n \|\widehat{\theta} - \theta_0\| \frac{d_n}{n} & \sum_{t=1}^{n}\sum_{s=1}^{n} \|G_1(x_t, z_t; \theta_0)\| \cdot |\Delta(x_s, z_s)| \pi_1(x_s)\pi_2(z_s) \\
& \times L_1\left(\frac{x_t - x_s}{h_1}\right) L_2\left(\frac{z_t - z_s}{h_2}\right), \quad \text{and} \\
\sum_{t=1}^{n}\sum_{s=1}^{n} E\Big[\|G_1(x_t, z_t; \theta_0)\| \cdot & |\Delta(x_s, z_s)| \pi_1(x_s)\pi_2(z_s) L_1\left(\frac{x_t - x_s}{h_1}\right) L_2\left(\frac{z_t - z_s}{h_2}\right)\Big] \\
& \leq C n h_1 h_2^d,
\end{aligned}
\tag{C.10}
$$

which, along with $\frac{\|\widehat{\theta}-\theta_0\|\sqrt{\sqrt{n}h_1h_2^d}}{\delta_n\sqrt{\sqrt{n}h_1h_2^d}} = o_P(1)$ by Assumption 2.3(ii) (the numerator is $o_P(1)$) and by Assumption 3.1 (the denominator $\lim_{n\to\infty} \delta_n^2\sqrt{n}h_1h_2^d = \infty$), implies that $\widehat{T}_{6n} = o_P\left(\delta_n^2 n h_1 h_2^d\right)$.

Meanwhile, in order to deal with $\widehat{T}_{9n}$, it suffices to show that as $n \to \infty$

$$
E\left[\delta_n \sum_{t=1}^{n}\left(\sum_{s=1}^{n} \Delta(x_s, z_s)\pi_1(x_s)\pi_2(z_s) L_1\left(\frac{x_t - x_s}{h_1}\right) L_2\left(\frac{z_t - z_s}{h_2}\right)\right) e_t\right]^2 \leq C\ \delta_n^2 n^{3/2} h_1^2 h_2^{2d}, \tag{C.11}
$$

which follows similarly from the proof of $I_{1n} = o_P(1)$ in Lemma A.1, taking into account the condition for $\Delta(x, z)$ in Assumption 3.1. Equation (C.11) then implies that as $n \to \infty$

$$
\widehat{T}_{9n} = O_P\left(\delta_n n^{1/4} h_1 h_2^d\right) = o_P\left(\delta_n^2 \sqrt{n} h_1 h_2^d\right), \tag{C.12}
$$

which, along with the proofs of (C.2) and (C.8)–(C.12), shows that under H_1, as $n \to \infty$

$$
\begin{aligned}
\widehat{T}_n =& \frac{d_n}{n h_1 h_2^d} \sum_{t=1}^{n}\sum_{s=1}^{n} \widehat{e}_s \widehat{e}_t \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
& \times\ K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z) dz dx \\
=& \frac{d_n}{n h_1 h_2^d} \sum_{t=1}^{n}\sum_{s=1}^{n} \Delta_n(x_s, z_s)\Delta_n(x_t, z_t) \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) \\
& \times\ K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z) dz dx\ (1 + o_P(1)),
\end{aligned}
\tag{C.13}
$$

which completes the proof of Lemma A.2. □

PROOF OF LEMMA A.3. Let $\tau_t = e_t^2\varpi_2(z_t) - E[e_t^2\varpi_2(z_t)]$. Then

$$
\frac{d_n}{n}\sum_{t=1}^{n} \varpi_1(x_t) e_t^2 \varpi_2(z_t) = E[e_t^2\varpi_2(z_t)] \frac{d_n}{n}\sum_{t=1}^{n} \varpi_1(x_t) + \frac{d_n}{n}\sum_{t=1}^{n} \varpi_1(x_t)\tau_t.
$$

It follows from Wang and Phillips (2009a) that $\frac{d_n}{n}\sum_{t=1}^{n} \varpi_1(x_t) \to_D L_B(1,0)\int \varpi_1(x)dx$. Thus, to fulfill Lemma A.3, it suffices to show that

$$
A_n \equiv \frac{d_n}{n}\sum_{t=1}^{n} \varpi_1(x_t)\tau_t = o_P(1).
$$

Notice that $A_n^2 = \frac{d_n^2}{n^2}\sum_{t=1}^{n}[\varpi_1(x_t)\tau_t]^2 + 2\frac{d_n^2}{n^2}\sum_{t=2}^{n}\sum_{s=1}^{t-1} \varpi_1(x_t)\varpi_1(x_s)\tau_t\tau_s \equiv A_{1n} + 2A_{2n}$.

We first show that $E[A_{1n}] = o(1)$. Using Cauchy-Schwarz inequality and the boundedness of the density for $d_t^{-1}x_t$ for $t \geq m_n$ where m_n is a sequence such that $m_n^2/n \to 0$ and $m_n \to \infty$ as $n \to \infty$,

$$
\begin{aligned}
E[A_{1n}] =& o(1) + \frac{d_n^2}{n^2} \sum_{t=m_n}^{n} E[\varpi_1(x_t)\tau_t]^2 \leq \frac{d_n^2}{n^2} \sum_{t=m_n}^{n} \left(E[\varpi_1^4(x_t)]E[\tau_t^4]\right)^{1/2} \\
\leq& O(1)\frac{1}{n} \sum_{t=m_n}^{n} \left(\int \varpi_1^4(d_t x) f_t(x) dx\right)^{1/2} = O(1)\frac{1}{n} \sum_{t=m_n}^{n} \left(\frac{1}{d_t} \int \varpi_1^4(x) f_t\left(\frac{1}{d_t}x\right) dx\right)^{1/2} \\
\leq& O(1)\frac{1}{n} \sum_{t=m_n}^{n} \frac{1}{\sqrt[4]{t}} \left(\int \varpi_1^4(x) dx\right)^{1/2} = O(1)\frac{1}{n} n^{3/4} = o(1).
\end{aligned}
$$

Next, we shall show that $E[A_{2n}] = o(1)$. In A_{2n}, however, we only need to consider the terms where $t - s$ large, while the others are negligible. To this end, let m_n be a quantity such that $m_n \to \infty$ but $m_n/\sqrt[4]{n} \to 0$ as $n \to \infty$. We shall consider that $t - s > m_n$ and $s > m_n$. Observe that

$$
\begin{aligned}
e_t^2 =& \left(\sum_{j=0}^{\infty} \rho_j \epsilon_{t-j}\right)^2 = \sum_{j=0}^{\infty} (\rho_j \epsilon_{t-j})^2 + \sum_{j_1=0}^{\infty} \sum_{j_2=0, \neq j_1}^{\infty} \rho_{j_1} \epsilon_{t-j_1} \rho_{j_2} \epsilon_{t-j_2} \\
=& \sum_{j=-\infty}^{t} \rho_{t-j}^2 \epsilon_j^2 + 2 \sum_{j_1=-\infty}^{t} \sum_{j_2=-\infty}^{j_1-1} \rho_{t-j_1} \epsilon_{j_1} \rho_{t-j_2} \epsilon_{j_2}.
\end{aligned}
$$

It follows from Assumption 2.1(ii) that

$$
\begin{aligned}
E[e_t^2\ \varpi_2(z_t)] =& \sum_{j=-\infty}^{t} \rho_{t-j}^2 E[\epsilon_j^2 \varpi_2(z_t)] + 2 \sum_{j_1=-\infty}^{t} \sum_{j_2=-\infty}^{j_1-1} \rho_{t-j_1} \rho_{t-j_2} E[\epsilon_{j_1} \epsilon_{j_2} \varpi_2(z_t)] \\
=& \sum_{j=-\infty}^{t} \rho_{t-j}^2 E[\epsilon_j^2 \varpi_2(z_t)] + 2 \sum_{j_1=t-m_0+2}^{t} \sum_{j_2=t-m_0+1}^{j_1-1} \rho_{t-j_1} \rho_{t-j_2} E[\epsilon_{j_1} \epsilon_{j_2} \varpi_2(z_t)]
\end{aligned}
$$

and then

$$
\begin{aligned}
\tau_t =& \sum_{j=-\infty}^{t} \rho_{t-j}^2 \{\epsilon_j^2 \varpi_2(z_t) - E[\epsilon_j^2 \varpi_2(z_t)]\} \\
&+ 2 \sum_{j_1=t-m_0+2}^{t} \sum_{j_2=t-m_0+1}^{j_1-1} \rho_{t-j_1} \rho_{t-j_2} \{\epsilon_{j_1} \epsilon_{j_2} \varpi_2(z_t) - E[\epsilon_{j_1} \epsilon_{j_2} \varpi_2(z_t)]\} \\
&+ 2 \sum_{j_1=-\infty}^{t-m_0} \sum_{j_2=-\infty}^{j_1-1} \rho_{t-j_1} \rho_{t-j_2} \epsilon_{j_1} \epsilon_{j_2} \varpi_2(z_t).
\end{aligned}
$$

It is evident that omission of the second term does not do any harm for the proof of $E[A_{2n}] = o(1)$ because it is a sum of $m_0(m_0 - 1)$ terms. For simplicity, it is neglected hereafter.

Thus,

$$
\begin{aligned}
A_{2n} =& \frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} \varpi_1(x_t) \varpi_1(x_s) \tau_t \tau_s \\
=& \frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} \sum_{j=-\infty}^{t} \varpi_1(x_t) \varpi_1(x_s) \rho_{t-j}^2 \{\epsilon_j^2 \varpi_2(z_t) - E[\epsilon_j^2 \varpi_2(z_t)]\} \tau_s \\
&+ 2 \frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} \sum_{j_1=-\infty}^{t} \sum_{j_2=-\infty}^{j_1-1} \varpi_1(x_t) \varpi_1(x_s) \rho_{t-j_1} \rho_{t-j_2} \epsilon_{j_1} \epsilon_{j_2} \varpi_2(z_t) \tau_s
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j=s+1}^{t}\varpi_1(x_t)\varpi_1(x_s)\rho_{t-j}^2\{\epsilon_j^2\varpi_2(z_t)-E[\epsilon_j^2\varpi_2(z_t)]\}\tau_s\\
&+\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j=-\infty}^{s}\varpi_1(x_t)\varpi_1(x_s)\rho_{t-j}^2\{\epsilon_j^2\varpi_2(z_t)-E[\epsilon_j^2\varpi_2(z_t)]\}\tau_s\\
&+2\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j_1=s+2}^{t-m_0}\sum_{j_2=s+1}^{j_1-1}\varpi_1(x_t)\varpi_1(x_s)\rho_{t-j_1}\rho_{t-j_2}\epsilon_{j_1}\epsilon_{j_2}\varpi_2(z_t)\tau_s\\
&+2\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j_1=s+1}^{t-m_0}\sum_{j_2=-\infty}^{s}\varpi_1(x_t)\varpi_1(x_s)\rho_{t-j_1}\rho_{t-j_2}\epsilon_{j_1}\epsilon_{j_2}\varpi_2(z_t)\tau_s\\
&+2\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j_1=-\infty}^{s}\sum_{j_2=-\infty}^{j_1-1}\varpi_1(x_t)\varpi_1(x_s)\rho_{t-j_1}\rho_{t-j_2}\epsilon_{j_1}\epsilon_{j_2}\varpi_2(z_t)\tau_s\\
\equiv&A_{2n1}+\cdots+A_{2n5},\quad \text{say.}
\end{aligned}
$$

Observe further that

$$
\begin{aligned}
E[A_{2n1}]=&\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j=s+1}^{t}E\left[\varpi_1(x_t)\varpi_1(x_s)\rho_{t-j}^2\{\epsilon_j^2\psi_2(z_t)-E[\epsilon_j^2\varpi_2(z_t)]\}\tau_s\right]\\
=&\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j=t-m_0+1}^{t}E\left[\varpi_1(x_t)\varpi_1(x_s)\rho_{t-j}^2\{\epsilon_j^2\psi_2(z_t)-E[\epsilon_j^2\varpi_2(z_t)]\}\tau_s\right]\\
&+\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j=s+1}^{t-m_0}\rho_{t-j}^2E\left[\varpi_1(x_t)\varpi_1(x_s)\{\epsilon_j^2\psi_2(z_t)-E[\epsilon_j^2\varpi_2(z_t)]\}\tau_s\right]\\
\equiv&T_1+T_2.
\end{aligned}
$$

We calculate T_1 first. By Lemma A.6 and $x_t=x_{ts}+x_{ts}^*$, for $t-m_0+1\le j\le t$,

$$
\begin{aligned}
&E\left[\varpi_1(x_t)\{\epsilon_j^2\varpi_2(z_t)-E[\epsilon_j^2\varpi_2(z_t)]\}|\mathcal{F}_s\right]\\
=&\iiint\varpi_1(d_{ts}x+x_{ts}^*)\{u^2\varpi_2(z)-E[\epsilon_j^2\varpi_2(z_t)]\}p_{ts}(x,z,u)dxdzdu\\
=&d_{ts}^{-1}\iiint\varpi_1(x)\{u^2\varpi_2(z)-E[\epsilon_j^2\varpi_2(z_t)]\}p_{ts}\left(\frac{x-x_{ts}^*}{d_{ts}},z,u\right)dxdzdu\\
=&d_{ts}^{-1}\iiint\varpi_1(x)\{u^2\varpi_2(z)-E[\epsilon_j^2\varpi_2(z_t)]\}\\
&\times\left[p_{ts}\left(\frac{x-x_{ts}^*}{d_{ts}},z,u\right)-f_{ts(m_0)}\left(\frac{x-x_{ts}^*}{d_{ts}}\right)q(z,u)\right]dxdzdu
\end{aligned}
$$

due to the vanish of the integral related to $q(z,u)$, which by Lemma A.6 gives

$$
\begin{aligned}
&\left|E\left[\varpi_1(x_t)\{\epsilon_j^2\varpi_2(z_t)-E[\epsilon_j^2\varpi_2(z_t)]\}|\mathcal{F}_s\right]\right|\\
\le&Cd_{ts}^{-2}\int|\varpi_1(x)|dx\iint\nu(z,u)dzdu=Cd_{t-s}^{-2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|T_1|\le&\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j=t-m_0+1}^{t}\rho_{t-j}^2|E\left[\varpi_1(x_t)\varpi_1(x_s)\{\epsilon_j^2\varpi_2(z_t)-E[\epsilon_j^2\varpi_2(z_t)]\}\tau_s\right]|\\
=&\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\sum_{j=t-m_0+1}^{t}\rho_{t-j}^2\left|E\left(\varpi_1(x_s)\tau_sE\left[\varpi_1(x_t)\{\epsilon_j^2\varpi_2(z_t)-E[\epsilon_j^2\varpi_2(z_t)]\}\Big|\mathcal{F}_s\right]\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} \sum_{j=t-m_0+1}^{t} \rho_{t-j}^2 E\left(|\varpi_1(x_s)\tau_s| \left|E\left[\varpi_1(x_t)\{\epsilon_j^2\varpi_2(z_t) - E[\epsilon_j^2\varpi_2(z_t)]\}|\mathcal{F}_s\right]\right|\right) \\
\leq & O(1)\frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} d_{ts}^{-2} E|\varpi_1(x_s)\tau_s| \sum_{j=t-m_0+1}^{t} |\rho_{t-j}|^2 \\
\leq & O(1)\frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} d_{ts}^{-2}\left(E\varpi_1^2(x_s)E\tau_s^2\right)^{1/2} \leq O(1)\frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} d_{ts}^{-2} d_s^{-1/2} \|\varpi_1(x)\|_{L^2} \\
= & O(1)\frac{1}{n} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} \frac{1}{t-s}\frac{1}{\sqrt[4]{s}} = O(1)\frac{1}{n} n^{3/4}\ln(n) = o(1),
\end{aligned}
$$

where $\|\varpi_1(x)\|_{L^2} = \left(\int \varpi_1^2(x)dx\right)^{1/2}$ is the function norm in $L^2(\mathbb{R})$.

Now we turn to prove $T_2 = o(1)$. Notice that, for $s+1 \leq j \leq t-m_0$, ϵ_j is independent of z_t. Thus, by Lemma A.6,

$$
\begin{aligned}
& E\left[\varpi_1(x_t)\{\epsilon_j^2\varpi_2(z_t) - \sigma_0^2 E[\varpi_2(z_t)]\}|\mathcal{F}_s\right] \\
= & E\left[\varpi_1(x_{ts/j} + b_{t,j}\epsilon_j + x_{ts}^*)\{\epsilon_j^2\varpi_2(z_t) - \sigma_0^2 E[\varpi_2(z_t)]\}|\mathcal{F}_s\right] \\
= & \iiint \varpi_1(d_{ts}x + b_{t,j}u + x_{ts}^*)\{u^2\varpi_2(z) - \sigma_0^2 E[\varpi_2(z_t)]\} q_{ts/j}(x,z)dxdz f_\epsilon(u)du \\
= & d_{ts}^{-1} \iiint \{u^2\varpi_2(z) - \sigma_0^2 E[\varpi_2(z_t)]\}\varpi_1(x)\, q_{ts/j}\left(\frac{x - b_{t,j}u - x_{ts}^*}{d_{ts}}, z\right) dxdz\, f_\epsilon(u)du \\
= & d_{ts}^{-1} \iiint \{u^2\varpi_2(z) - \sigma_0^2 E[\varpi_2(z_t)]\}\varpi_1(x) \\
& \times \left[q_{ts/j}\left(\frac{x - b_{t,j}u - x_{ts}^*}{d_{ts}}, z\right) - f_{ts(m_0)}\left(\frac{x - x_{ts}^*}{d_{ts}}\right)p(z)\right] dxdz\, f_\epsilon(u)du \\
= & d_{ts}^{-1} \iiint \{u^2\varpi_2(z) - \sigma_0^2 E[\varpi_2(z_t)]\}\varpi_1(x) \\
& \times \left[q_{ts/j}\left(\frac{x - b_{t,j}u - x_{ts}^*}{d_{ts}}, z\right) - f_{ts(m_0)}\left(\frac{x - b_{t,j}u - x_{ts}^*}{d_{ts}}\right)p(z)\right. \\
& \left. + f_{ts(m_0)}\left(\frac{x - b_{t,j}u - x_{ts}^*}{d_{ts}}\right)p(z) - f_{ts(m_0)}\left(\frac{x - x_{ts}^*}{d_{ts}}\right)p(z)\right] dxdz f_\epsilon(u)du
\end{aligned}
$$

which, along with Lemma A.6 and Lemma A.5, yields that

$$
\begin{aligned}
& |E\left[\varpi_1(x_t)\{\epsilon_j^2\varpi_2(z_t) - \sigma_0^2 E[\varpi_2(z_t)]\}|\mathcal{F}_s\right]| \\
\leq & d_{ts}^{-2} \int |\varpi_1(x)|dx \iint |\{u^2\varpi_2(z) - \sigma_0^2 E[\varpi_2(z_t)]\}|(C_1\mu(z) + C_2|u|p(z))f_\epsilon(u)dzdu \\
= & C d_{ts}^{-2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|T_2| \leq & \frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} \sum_{j=s+1}^{t-m_0} \rho_{t-j}^2 |E\left[\varpi_1(x_t)\varpi_1(x_s)\{\epsilon_j^2\psi_2(z_t) - E[\epsilon_j^2\varpi_2(z_t)]\}\tau_s\right]| \\
\leq & O(1)\frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} d_{ts}^{-2} E|\varpi_1(x_s)\tau_s| = o(1),
\end{aligned}
$$

similar to T_1. This shows $E[A_{2n1}] = o(1)$.

It is evident that $E[A_{1ni}] = o(1)$, $i = 2, \cdots, 5$, can be shown in the same fashion or in an easier way since in A_{2n2} and A_{2n5} one may make use of the convergence of $\rho_{t-j} \to 0$ for $j \leq s$ stipulated in Assumption 2.1. Hence, the proof of Lemma A.3 is completed. □

Appendix D Proofs of Theorem 2.1 and Corollary 2.1

PROOF OF THEOREM 2.1. Note by Lemma A.1 that $\widehat{L}_n = \widehat{S}_{1n} + \widehat{S}_{2n} = S_{1n} + S_{2n} + o_P(1)$. In order to prove Theorem 2.1, it suffices to show that as $n \to \infty$

$$S_{1n} \to_D L_B(1,0) \cdot E\left[e_1^2 \pi_2(z_1)\right] \iint K^2(u,v)dvdu, \quad \text{(D.1)}$$

$$S_{2n} = o_P(1). \quad \text{(D.2)}$$

We start with the proof of (D.1). Under Assumptions 2.1 and 2.3, using Lemma A.3, we have as $n \to \infty$

$$\begin{aligned}
S_{1n} =& \frac{d_n}{n}\sum_{t=1}^n e_t^2 \iint K^2(u,v)\pi_1(x_t - uh_1)\pi_2(z_t - vh_2)dudv \\
=& \frac{d_n}{n}\sum_{t=1}^n e_t^2 \iint K^2(u,v)\pi_1(x_t)\pi_2(z_t)dudv \\
&+ \frac{d_n}{n}\sum_{t=1}^n e_t^2 \iint K^2(u,v)\left(\pi_1(x_t - uh_1) - \pi_1(x_t)\right)\left(\pi_2(z_t - vh_2) - \pi_2(z_t)\right)dudv \\
&+ \frac{d_n}{n}\sum_{t=1}^n e_t^2 \iint K^2(u,v)\left(\pi_1(x_t - uh_1) - \pi_1(x_t)\right)\pi_2(z_t)dvdu \\
&+ \frac{d_n}{n}\sum_{t=1}^n e_t^2 \iint K^2(u,v)\pi_1(x_t)\left(\pi_2(z_t - vh_2) - \pi_2(z_t)\right)dudv \\
=& \frac{d_n}{n}\sum_{t=1}^n e_t^2 \pi_1(x_t)\pi_2(z_t) \iint K^2(u,v)dudv \\
&+ O(h_1 h_2^d) \cdot \frac{d_n}{n}\sum_{t=1}^n e_t^2 D_1(x_t)D_2(z_t) \iint |u||v|K^2(u,v)dvdu \\
&+ O(h_1) \cdot \frac{d_n}{n}\sum_{t=1}^n e_t^2 D_1(x_t)\pi_2(z_t) \iint |u|K^2(u,v)dvdu \\
&+ O(h_2^d) \cdot \frac{d_n}{n}\sum_{t=1}^n e_t^2 \pi_1(x_t)D_2(z_t) \iint |v|K^2(u,v)dvdu \\
=& \frac{d_n}{n}\sum_{t=1}^n e_t^2 \pi_1(x_t)\pi_2(z_t) \iint K^2(u,v)dudv + o_P(1) \to_D L_B(1,0) \cdot E\left[e_1^2\pi_2(z_1)\right] \cdot \iint K^2(u,v)dudv,
\end{aligned}$$

which completes the proof of (D.1).

We then prove (D.2). Denote $S_{2n} = \frac{2d_n}{nh_1h_2^d}\sum_{t=2}^n\sum_{s=1}^{t-1} B(t,s)e_te_s$, where, recalling that $K(u,v) = K_1(u)K_2(v)$,

$$B(t,s) \equiv \iint K_1\left(\frac{x_t - x}{h_1}\right) K_2\left(\frac{z_t - z}{h_2}\right) K_1\left(\frac{x_s - x}{h_1}\right) K_2\left(\frac{z_s - z}{h_2}\right) \pi_1(x)\pi_2(z)dzdx. \quad \text{(D.3)}$$

Similarly to the proof of (D.1), we have

$$\begin{aligned}
B(t,s) =& h_1h_2^d \int K_1\left(\frac{x_t - x_s}{h_1} + u\right) K_1(u)\,\pi_1(x_s - uh_1)du \\
&\times \int K_2\left(\frac{z_t - z_s}{h_2} + v\right) K_2(v)\pi_2(z_s - vh_2)dv \\
=& h_1h_2^d \pi_1(x_s)\,\pi_2(z_s) \cdot L_1\left(\frac{x_t - x_s}{h_1}\right) L_2\left(\frac{z_t - z_s}{h_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ h_1 h_2^d \cdot \int K_1\left(\frac{x_t - x_s}{h_1} + u\right) K_1(u)\left(\pi_1(x_s - uh_1) - \pi_1(x_s)\right) du \\
&\times \int K_2\left(\frac{z_t - z_s}{h_2} + v\right) K_2(v)\left(\pi_2(z_s - vh_2) - \pi_2(z_s)\right) dv \\
&+ h_1 h_2^d \pi_2(z_s) L_2\left(\frac{z_t - z_s}{h_2}\right) \int K_1\left(\frac{x_t - x_s}{h_1} + u\right) K_1(u)\left(\pi_1(x_s - uh_1) - \pi_1(x_s)\right) du \\
&+ h_1 h_2^d \pi_1(x_s) L_1\left(\frac{x_t - x_s}{h_1}\right) \int K_2\left(\frac{z_t - z_s}{h_2} + v\right) K_2(v)\left(\pi_2(z_s - vh_2) - \pi_2(z_s)\right) dv \\
\equiv\ & h_1 h_2^d \left(B_1(t,s) + B_2(t,s) + B_3(t,s) + B_4(t,s)\right),
\end{aligned}
$$

where $L_i(v) = \int K_i(u+v)K_i(u)du$ for $i = 1, 2$, and

$$
\begin{aligned}
B_1(t,s) =& \pi_1(x_s)\ \pi_2(z_s) \cdot L_1\left(\frac{x_t - x_s}{h_1}\right) L_2\left(\frac{z_t - z_s}{h_2}\right) \\
B_2(t,s) =& \int K_1\left(\frac{x_t - x_s}{h_1} + u\right) K_1(u)\left(\pi_1(x_s - uh_1) - \pi_1(x_s)\right) du \\
&\times \int K_2\left(\frac{z_t - z_s}{h_2} + v\right) K_2(v)\left(\pi_2(z_s - vh_2) - \pi_2(z_s)\right) dv \\
B_3(t,s) =& \pi_2(z_s) L_2\left(\frac{z_t - z_s}{h_2}\right) \int K_1\left(\frac{x_t - x_s}{h_1} + u\right) K_1(u)\left(\pi_1(x_s - uh_1) - \pi_1(x_s)\right) du \\
B_4(t,s) =& \pi_1(x_s) L_1\left(\frac{x_t - x_s}{h_1}\right) \int K_2\left(\frac{z_t - z_s}{h_2} + v\right) K_2(v)\left(\pi_2(z_s - vh_2) - \pi_2(z_s)\right) dv.
\end{aligned}
\tag{D.4}
$$

Then, we have

$$
\begin{aligned}
S_{2n} =& \frac{2d_n}{n} \sum_{t=2}^{n} \sum_{s=1}^{t-1} B_1(t,s) e_t e_s + \frac{2d_n}{n} \sum_{t=2}^{n} \sum_{s=1}^{t-1} B_2(t,s) e_t e_s \\
&+ \frac{2d_n}{n} \sum_{t=2}^{n} \sum_{s=1}^{t-1} B_3(t,s) e_t e_s + \frac{2d_n}{n} \sum_{t=2}^{n} \sum_{s=1}^{t-1} B_4(t,s) e_t e_s \\
\equiv& 2(B_{n1} + \cdots + B_{n4}), \quad \text{say.}
\end{aligned}
\tag{D.5}
$$

We shall investigate each term in (D.5). For the first one, observe that

$$
\begin{aligned}
B_{n1}^2 =& \left(\frac{d_n}{n} \sum_{t=2}^{n} \sum_{s=1}^{t-1} B_1(t,s) e_s e_t\right)^2 \\
=& \frac{d_n^2}{n^2} \sum_{t=2}^{n} \left(\sum_{s=1}^{t-1} B_1(t,s) e_s e_t\right)^2 + 2\frac{d_n^2}{n^2} \sum_{t_1=3}^{n} \sum_{t_2=2}^{t_1-1} \sum_{s_1=1}^{t_1-1} B_1(t_1,s_1) e_{s_1} e_{t_1} \sum_{s_2=1}^{t_2-1} B_1(t_2,s_2) e_{s_2} e_{t_2} \\
=& \frac{d_n^2}{n^2} \sum_{t=2}^{n} \sum_{s=1}^{t-1} \left(B_1(t,s) e_s e_t\right)^2 + 2\frac{d_n^2}{n^2} \sum_{t=2}^{n} \sum_{s_1=2}^{t-1} \sum_{s_2=1}^{s_1-1} B_1(t,s_1) B_1(t,s_2) e_{s_1} e_{s_2} e_t^2 \\
&+ 4\frac{d_n^2}{n^2} \sum_{t_1=3}^{n} \sum_{t_2=2}^{t_1-1} \sum_{s_2=1}^{t_2-1} B_1(t_1,t_2) e_{t_1} B_1(t_2,s_2) e_{s_2} e_{t_2}^2 \\
&+ 4\frac{d_n^2}{n^2} \sum_{t_1=3}^{n} \sum_{t_2=2}^{t_1-1} \sum_{s_2=1}^{t_2-1} B_1(t_1,s_2) B_1(t_2,s_2) e_{t_1} e_{s_2}^2 e_{t_2} \\
&+ 2\frac{d_n^2}{n^2} \sum_{t_1=4}^{n} \sum_{t_2=3}^{t_1-1} \sum_{s_1=2}^{t_2-1} \sum_{s_2=1}^{s_1-1} B_1(t_1,s_1) B_1(t_2,s_2) e_{s_1} e_{t_1} e_{s_2} e_{t_2}
\end{aligned}
$$

$$:= B_{n11} + \cdots + B_{n15}, \quad \text{say.}$$

We shall show that $E[B_{n1i}] = o(1)$ for all $i = 1, \cdots, 5$, starting with $E[B_{n11}] = o(1)$ first. In view of the boundedness of $L_2(\cdot)$ and $\pi_2(\cdot)$, we have

$$\begin{aligned}
E[B_{n11}] =& \frac{d_n^2}{n^2} \sum_{t=2}^{n} \sum_{s=1}^{t-1} E\left(B_1(t,s) e_s e_t\right)^2 \\
\le& C \frac{d_n^2}{n^2} \sum_{t=2}^{n} \sum_{s=1}^{t-1} E\left[\pi_1^2(x_s) \cdot L_1^2\left(\frac{x_t - x_s}{h_1}\right) e_s^2 e_t^2\right] \\
=& o(1) + C \frac{d_n^2}{n^2} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_n} E\left\{E\left[L_1^2\left(\frac{x_t - x_s}{h_1}\right) e_t^2 \middle| \mathcal{F}_s\right] \pi_1^2(x_s) e_s^2\right\}
\end{aligned} \tag{D.6}$$

where m_n is a sequence such that $m_n \to \infty$ and $m_n^2/n \to 0$ as $n \to \infty$.

Note that $e_t = \sum_{j=-\infty}^{t} \rho_{t-j} \epsilon_j$, and recall the structure of x_t from Appendix A, $x_t - x_s = x_{ts} + \bar{x}_{ts}$ where $\bar{x}_{ts}$ is adapted with $\mathcal{F}_s$. Using the densities in Lemma A.5 we have

$$\begin{aligned}
& E\left[L_1^2\left(\frac{x_t - x_s}{h_1}\right) e_t^2 \middle| \mathcal{F}_s\right] \\
=& \sum_{j=-\infty}^{t} \rho_{t-j}^2 E\left[L_1^2\left(\frac{x_{ts} + \bar{x}_{ts}}{h_1}\right) \epsilon_j^2 \middle| \mathcal{F}_s\right] \\
& + 2 \sum_{j_1=-\infty}^{t} \sum_{j_2=-\infty}^{j_1-1} \rho_{t-j_1} \rho_{t-j_2} E\left[L_1^2\left(\frac{x_{ts} + \bar{x}_{ts}}{h_1}\right) \epsilon_{j_1} \epsilon_{j_2} \middle| \mathcal{F}_s\right] \\
=& \sum_{j=s+1}^{t} \rho_{t-j}^2 E\left[L_1^2\left(\frac{x_{ts/j} + b_{t,j}\epsilon_j + \bar{x}_{ts}}{h_1}\right) \epsilon_j^2 \middle| \mathcal{F}_s\right] \\
& + \sum_{j=-\infty}^{s} \rho_{t-j}^2 \epsilon_j^2 E\left[L_1^2\left(\frac{x_{ts} + \bar{x}_{ts}}{h_1}\right) \middle| \mathcal{F}_s\right] \\
& + 2 \sum_{j_1=s+2}^{t} \sum_{j_2=s+1}^{j_1-1} \rho_{t-j_1} \rho_{t-j_2} E\left[L_1^2\left(\frac{x_{ts/j_1j_2} + b_{t,j_1}\epsilon_{j_1} + b_{t,j_2}\epsilon_{j_2} + \bar{x}_{ts}}{h_1}\right) \epsilon_{j_1} \epsilon_{j_2} \middle| \mathcal{F}_s\right] \\
& + 2 \sum_{j_1=s+1}^{t} \sum_{j_2=-\infty}^{s} \rho_{t-j_1} \rho_{t-j_2} \epsilon_{j_2} E\left[L_1^2\left(\frac{x_{ts/j_1} + b_{t,j_1}\epsilon_{j_1} + \bar{x}_{ts}}{h_1}\right) \epsilon_{j_1} \middle| \mathcal{F}_s\right] \\
& + 2 \sum_{j_1=-\infty}^{s} \sum_{j_2=-\infty}^{j_1-1} \rho_{t-j_1} \rho_{t-j_2} \epsilon_{j_1} \epsilon_{j_2} E\left[L_1^2\left(\frac{x_{ts} + \bar{x}_{ts}}{h_1}\right) \middle| \mathcal{F}_s\right] \\
=& \sum_{j=s+1}^{t} \rho_{t-j}^2 \iint L_1^2\left(\frac{d_{ts}x + b_{t,j}u + \bar{x}_{ts}}{h_1}\right) u^2 f_{ts/j}(x) f_\epsilon(u) dx du \\
& + \sum_{j=-\infty}^{s} \rho_{t-j}^2 \epsilon_j^2 \int L_1^2\left(\frac{d_{ts}x + \bar{x}_{ts}}{h_1}\right) f_{ts}(x) dx \\
& + 2 \sum_{j_1=s+2}^{t} \sum_{j_2=s+1}^{j_1-1} \rho_{t-j_1} \rho_{t-j_2} \iiint L_1^2\left(\frac{d_{ts}x + b_{t,j_1}u + b_{t,j_2}v + \bar{x}_{ts}}{h_1}\right) uv \\
& \quad \times f_{ts/j_1j_2}(x) f_\epsilon(u) f_\epsilon(v) dx du dv \\
& + 2 \sum_{j_1=s+1}^{t} \sum_{j_2=-\infty}^{s} \rho_{t-j_1} \rho_{t-j_2} \epsilon_{j_2} \iint L_1^2\left(\frac{d_{ts}x + b_{t,j_1}u + \bar{x}_{ts}}{h_1}\right) u f_{ts/j_1}(x) f_\epsilon(u) dx du
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{j_1=-\infty}^{s}\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}\epsilon_{j_1}\epsilon_{j_2}\int L_1^2\left(\frac{d_{ts}x+\bar{x}_{ts}}{h_1}\right)f_{ts}(x)dx\\
=&h_1d_{ts}^{-1}\sum_{j=s+1}^{t}\rho_{t-j}^2\iint L_1^2(x)u^2f_{ts/j}\left(\frac{h_1x-b_{t,j}u-\bar{x}_{ts}}{d_{ts}}\right)f_\epsilon(u)dxdu\\
&+h_1d_{ts}^{-1}\sum_{j=-\infty}^{s}\rho_{t-j}^2\epsilon_j^2\int L_1^2(x)f_{ts}\left(\frac{h_1x-\bar{x}_{ts}}{d_{ts}}\right)dx\\
&+2h_1d_{ts}^{-1}\sum_{j_1=s+2}^{t}\sum_{j_2=s+1}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}\iiint L_1^2(x)uv\\
&\quad\times f_{ts/j_1j_2}\left(\frac{h_1x-b_{t,j_1}u-b_{t,j_2}v-\bar{x}_{ts}}{d_{ts}}\right)f_\epsilon(u)f_\epsilon(v)dxdudv\\
&+2h_1d_{ts}^{-1}\sum_{j_1=s+1}^{t}\sum_{j_2=-\infty}^{s}\rho_{t-j_1}\rho_{t-j_2}\epsilon_{j_2}\iint L_1^2(x)uf_{ts/j_1}\left(\frac{h_1x-b_{t,j_1}u-\bar{x}_{ts}}{d_{ts}}\right)f_\epsilon(u)dxdu\\
&+2h_1d_{ts}^{-1}\sum_{j_1=-\infty}^{s}\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}\epsilon_{j_1}\epsilon_{j_2}\int L_1^2(x)f_{ts}\left(\frac{h_1x-\bar{x}_{ts}}{d_{ts}}\right)dx\\
\equiv&I_1+\cdots+I_5,\quad\text{say,}
\end{aligned}
$$

where x_{ts/j_1j_2} and $f_{ts/j_1j_2}(\cdot)$ are defined similar to $x_{ts/j}$ and $f_{ts/j}(\cdot)$, and it is clear that $f_{ts/j_1j_2}(\cdot)$ possess the properties of $f_{ts/j}(\cdot)$, namely, uniform boundedness and Lipschitz condition.

Apparently, almost surely $I_1\le Ch_1d_{ts}^{-1}$; $I_2\equiv I_2^*\sum_{j=-\infty}^{s}\rho_{t-j}^2\epsilon_j^2$ and $I_2^*\le Ch_1d_{ts}^{-1}$; $|I_3|\le Ch_1d_{ts}^{-1}$; $I_4\equiv I_4^*\sum_{j_2=-\infty}^{s}\rho_{t-j_2}\epsilon_{j_2}$ and $|I_4^*|\le Ch_1d_{ts}^{-1}$; $I_5\equiv I_5^*\sum_{j_1=-\infty}^{s}\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}\epsilon_{j_1}$ and $|I_5^*|\le Ch_1d_{ts}^{-1}$.

In view of (D.6), we need to calculate expectations $E[I_k\pi_1^2(x_s)e_s^2]$ for $k=1,\cdots,5$. Repeating a similar procedure, it is not difficult to have $E[I_1\pi_1^2(x_s)e_s^2]\le Ch_1d_{ts}^{-1}d_s^{-1}$; $E[I_2\pi_1^2(x_s)e_s^2]$ $\le Ch_1d_{ts}^{-1}d_s^{-1}|\rho_{t-s}|$; $|E[I_3\pi_1^2(x_s)e_s^2]|\le Ch_1d_{ts}^{-1}d_s^{-1}$; $|E[I_4\pi_1^2(x_s)e_s^2]|\le Ch_1d_{ts}^{-1}d_s^{-1}(t-s)^{-1}$; $|E[I_5\pi_1^2(x_s)e_s^2]|\le Ch_1d_{ts}^{-1}d_s^{-1}(t-s)^{-2}$, where we have used the convergence of $\sum_j j|\rho_j|<\infty$ stipulated in Assumption 2.1.

It then follows that

$$
\begin{aligned}
E[B_{n11}]=&o(1)+C\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}E\left\{E\left[L_1^2\left(\frac{x_t-x_s}{h_1}\right)e_t^2\Big|\mathcal{F}_s\right]\pi_1^2(x_s)e_s^2\right\}\\
=&o(1)+C\frac{1}{n}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}E\left\{[I_1+\cdots+I_5]\,\pi_1^2(x_s)e_s^2\right\}\\
\le&o(1)+Ch_1\frac{1}{n}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}d_{ts}^{-1}d_s^{-1}=o(1)+Ch_1=o(1),
\end{aligned}
$$

where the higher-order terms are abandoned. This finishes the proof of $E[B_{n11}]=o(1)$.

To prove $E[B_{n1i}]=o(1)$, $i=2,\cdots,5$, because of similarity, in what follows we only show that $E[B_{n15}]=o(1)$ that is the toughest one. As explained before, we only consider in the following that $s_i,t_i>m_n$, $i=1,2$, and their differences are greater than m_n as well. Here, $m_n\to\infty$ and $m_n/n^{1/4}\to 0$ as $n\to\infty$.

Ignoring the constant,

$$
B_{n15}=\frac{d_n^2}{n^2}\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_1-1}\sum_{s_2=1}^{s_1-1}B_1(t_1,s_1)B_1(t_2,s_2)e_{s_1}e_{t_1}e_{s_2}e_{t_2}
$$

$$=2\frac{d_n^2}{n^2}\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_1-1}\sum_{s_2=1}^{s_1-1}\pi_1(x_{s_1})\pi_2(z_{s_1})L_1\left(\frac{x_{t_1}-x_{s_1}}{h_1}\right)L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)$$
$$\times\pi_1(x_{s_2})\pi_2(z_{s_2})L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{t_1}e_{t_2}e_{s_1}e_{s_2},$$

which, in order to calculate $E[B_{n15}]$, motivates the following route map via conditional expectations,

$$E[B_{n15}]=E[E[E[E[B_{n15}|\mathcal{F}_{t_2}]|\mathcal{F}_{s_1}]|\mathcal{F}_{s_2}]]. \tag{D.7}$$

Accordingly, the following calculation is split into four steps: **Step I**–**Step IV**. That is, each step corresponds to the calculation of a conditional expectation or the final expectation.

To make our calculation concise, we introduce *super rate* first. In what follows, we say a term has *super rate* if its expectation or conditional expectation in any step is proportional to either $d_{t_1t_2}^{-2}$, $d_{t_2s_1}^{-2}$, $d_{s_1s_2}^{-2}$ or $d_{s_2}^{-2}$ (or even faster than these); we say a term has *double super rates* if its expectation or conditional expectation is at least proportional to the product of any two super rates. It would be clear later from the following derivation (precisely, Step III) that if a term has double super rates it is negligible for further calculation. Since B_{n15} has a normalizer n^{-1} and, as can be seen from B_{n11}, each term will have factor $h_1^2h_2^{2d}$, the term that has double super rates is clearly $o_P(1)$.

Step I. Putting the known variables in $E[B_{n15}|\mathcal{F}_{t_2}]$ aside first, it follows from Lemma A.6 that

$$\begin{aligned}
&E\left[L_1\left(\frac{x_{t_1}-x_{s_1}}{h_1}\right)L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)e_{t_1}\Big|\mathcal{F}_{t_2}\right]\\
=&\sum_{j=t_1-m_0+1}^{t_1}\rho_{t_1-j}E\left[L_1\left(\frac{x_{t_1t_2}+x_{t_1t_2}^*-x_{s_1}}{h_1}\right)L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)\epsilon_j\Big|\mathcal{F}_{t_2}\right]\\
&+\sum_{j=t_2+1}^{t_1-m_0}\rho_{t_1-j}E\left[L_1\left(\frac{x_{t_1t_2/j}+b_{t_1,j}\epsilon_j+x_{t_1t_2}^*-x_{s_1}}{h_1}\right)L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)\epsilon_j\Big|\mathcal{F}_{t_2}\right]\\
&+\sum_{j=-\infty}^{t_2}\rho_{t_1-j}\epsilon_jE\left[L_1\left(\frac{x_{t_1t_2}+x_{t_1t_2}^*-x_{s_1}}{h_1}\right)L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)\Big|\mathcal{F}_{t_2}\right]\\
=&\sum_{j=t_1-m_0+1}^{t_1}\rho_{t_1-j}\iiint L_1\left(\frac{d_{t_1t_2}x+x_{t_1t_2}^*-x_{s_1}}{h_1}\right)L_2\left(\frac{z-z_{s_1}}{h_2}\right)up_{t_1t_2}(x,z,u)dxdzdu\\
&+\sum_{j=t_2+1}^{t_1-m_0}\rho_{t_1-j}\iiint L_1\left(\frac{d_{t_1t_2}x+b_{t_1,j}u+x_{t_1t_2}^*-x_{s_1}}{h_1}\right)L_2\left(\frac{z-z_{s_1}}{h_2}\right)uq_{t_1t_2/j}(x,z)f_\epsilon(u)dudzdx\\
&+\sum_{j=-\infty}^{t_2}\rho_{t_1-j}\epsilon_j\iint L_1\left(\frac{d_{t_1t_2}x+x_{t_1t_2}^*-x_{s_1}}{h_1}\right)L_2\left(\frac{z-z_{s_1}}{h_2}\right)q_{t_1t_2}(x,z)dxdz\\
=&h_1h_2^dd_{t_1t_2}^{-1}\sum_{j=t_1-m_0+1}^{t_1}\rho_{t_1-j}\iiint L_1(x)L_2(z)u\;p_{t_1t_2}\left(\frac{h_1x-x_{t_1t_2}^*+x_{s_1}}{d_{t_1t_2}},h_2z+z_{s_1},u\right)dxdzdu\\
&+h_1h_2^dd_{t_1t_2}^{-1}\sum_{j=t_2+1}^{t_1-m_0}\rho_{t_1-j}\iiint L_1(x)L_2(z)u\\
&\quad\times q_{t_1t_2/j}\left(\frac{h_1x-b_{t_1,j}u-x_{t_1t_2}^*+x_{s_1}}{d_{t_1t_2}},h_2z+z_{s_1}\right)f_\epsilon(u)dudzdx\\
&+h_1h_2^dd_{t_1t_2}^{-1}\sum_{j=-\infty}^{t_2}\rho_{t_1-j}\epsilon_j\iint L_1(x)L_2(z)q_{t_1t_2}\left(\frac{h_1x-x_{t_1t_2}^*+x_{s_1}}{d_{t_1t_2}},h_2z+z_{s_1}\right)dxdz
\end{aligned}$$

$$
\begin{aligned}
=&h_1 h_2^d d_{t_1t_2}^{-1} \sum_{j=t_1-m_0+1}^{t_1} \rho_{t_1-j} \iiint dxdzdu\ L_1(x)L_2(z)u \\
&\times \left[p_{t_1t_2}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}, h_2z + z_{s_1}, u\right) - f_{t_1t_2(m_0)}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}\right)\kappa_j(h_2z + z_{s_1}, u)\right] \\
&+ h_1 h_2^d d_{t_1t_2}^{-1} \int L_1(x) f_{t_1t_2(m_0)}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}\right) dx \\
&\quad \times \sum_{j=t_1-m_0+1}^{t_1} \rho_{t_1-j} \iint L_2(z)u\kappa_j(h_2z + z_{s_1}, u)dzdu \\
&+ h_1 h_2^d d_{t_1t_2}^{-1} \sum_{j=t_2+1}^{t_1-m_0} \rho_{t_1-j} \iiint dudzdx\ L_1(x)L_2(z)u\ f_\epsilon(u) \\
&\quad \times \left[q_{t_1t_2/j}\left(\frac{h_1x - b_{t_1,j}u - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}, h_2z + z_{s_1}\right) - q_{t_1t_2/j}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}, h_2z + z_{s_1}\right)\right] \\
&+ h_1 h_2^d d_{t_1t_2}^{-1} \sum_{j=-\infty}^{t_2} \rho_{t_1-j}\epsilon_j \iint L_1(x)L_2(z) q_{t_1t_2}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}, h_2z + z_{s_1}\right) dxdz \\
\equiv& T_1 + T_1^* + T_2 + T_3, \quad \text{say,}
\end{aligned}
$$

where $\kappa_j(z,u)$ is the joint density of (z_{t_1}, ϵ_j) with $t_1 - m_0 + 1 \leq j \leq t_1$, while in the second term $\int u f_\epsilon(u) = E[\epsilon_1] = 0$ has been invoked. For convenience, T_3 is written as $T_3 \equiv T_3' \sum_{j=-\infty}^{t_2} \rho_{t_1-j}\epsilon_j$ where

$$
T_3'(x_{t_1t_2}^* - x_{s_1}, z_{s_1}) \equiv \iint L_1(x)L_2(z) q_{t_1t_2}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}, h_2z + z_{s_1}\right) dxdz. \tag{D.8}
$$

Note that

$$
\begin{aligned}
T_1^* \equiv& h_1 h_2^d d_{t_1t_2}^{-1} \int L_1(x) f_{t_1t_2(m_0)}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}\right) dx \\
&\times \sum_{j=t_1-d+1}^{t_1} \rho_{t_1-j} \iint L_2(z)u\ \kappa_j(h_2z + z_{s_1}, u)dudz.
\end{aligned}
$$

For convenience, denote $T_1^* = U_1(x_{t_1t_2}^* - x_{s_1})V(z_{s_1})$, where

$$
U_1(x_{t_1t_2}^* - x_{s_1}) \equiv h_1 h_2^d d_{t_1t_2}^{-1} \int L_1(x) f_{t_1t_2(m_0)}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}\right) dx \tag{D.9}
$$

$$
V(z_{s_1}) \equiv \sum_{j=t_1-m_0+1}^{t_1} \rho_{t_1-j} \iint L_2(z)u\ \kappa_j(h_2z + z_{s_1}, u)dudz, \tag{D.10}
$$

in which we have suppressed many variables that have nothing to do with the following expectation calculations. In addition, $|U_1(x_{t_1t_2}^* - x_{s_1})| \leq Ch_1h_2^d d_{t_1t_2}^{-1}$ a.s. for some constant C and $V(z_{s_1})$ is a bounded function by virtue of the integrability of $L_2(\cdot)$.

Notice that, by Lemma A.6 again, almost surely,

$$
\begin{aligned}
|T_1| \leq& Ch_1h_2^d d_{t_1t_2}^{-2} \int L_1(x)dx \iint L_2(z)\nu(h_2z + z_{s_1}, u)|u|dzdu \\
\leq& Ch_1h_2^d d_{t_1t_2}^{-2} \int L_1(x)dx \int L_2(z)dz \sup_z \int \nu(z,u)|u|du = Ch_1h_2^d d_{t_1t_2}^{-2},
\end{aligned}
$$

where as $\iint \nu(z,u)|u|dzdu < \infty$, $\int \nu(z,u)|u|du$ is a bounded function in z almost everywhere; and

$$\begin{aligned}
T_2 =& h_1 h_2^d d_{t_1t_2}^{-1} \sum_{j=t_2+1}^{t_1-m_0} \rho_{t_1-j} \iiint dudzdx \; L_1(x)L_2(z)u \; f_\epsilon(u) \\
& \times \Bigg\{ q_{t_1t_2/j}\left(\frac{h_1x - b_{t_1,j}u - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}, h_2z + z_{s_1}\right) \\
& \quad - f_{t_1t_2(m_0)}\left(\frac{h_1x - b_{t_1,j}u - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}\right) p(h_2z + z_{s_1}) \\
& + \left[f_{t_1t_2(m_0)}\left(\frac{h_1x - b_{t_1,j}u - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}\right) - f_{t_1t_2(m_0)}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}\right) \right] p(h_2z + z_{s_1}) \\
& \quad + f_{t_1t_2(m_0)}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}\right) p(h_2z + z_{s_1}) - q_{t_1t_2/j}\left(\frac{h_1x - x_{t_1t_2}^* + x_{s_1}}{d_{t_1t_2}}, h_2z + z_{s_1}\right) \Bigg\}
\end{aligned}$$

implying that, almost surely,

$$\begin{aligned}
|T_2| \le & h_1 h_2^d d_{t_1t_2}^{-2} \iiint dudzdx \; L_1(x)L_2(z)|u| f_\epsilon(u)(C_1\mu(h_2z + z_{s_1}) + C_2|u|p(h_2z + z_{s_1})) \\
\le & h_1 h_2^d d_{t_1t_2}^{-2} \iint L_1(x)L_2(z)dzdx \left(C_1 \sup_z[\mu(z)] \int |u| f_\epsilon(u)du + C_2 \sup_z[p(z)] \int |u|^2 f_\epsilon(u)du \right) \\
= & C h_1 h_2^d d_{t_1t_2}^{-2},
\end{aligned}$$

by the almost everywhere boundedness of $\mu(z)$ and $p(z)$; meanwhile, obviously $|T_3'| \le C h_1 h_2^d d_{t_1t_2}^{-1}$ a.s.

However, these upper bounds for T_1, T_2 and T_3' can not be used at the moment. We need to regard them as functions of $x_{t_1t_2}^* - x_{s_1}$ and z_{s_1} for the further calculation of conditional expectation.

To compute further, let us recall from Appendix A that $x_{t_1t_2}^* = x_{t_2} + \bar{x}_{t_1t_2} = x_{t_2s_1} + x_{t_2s_1}^* + \bar{x}_{t_1t_2} = x_{t_2s_1} + x_{s_1} + \bar{x}_{t_2s_1} + \bar{x}_{t_1t_2}$ where $\bar{x}_{t_2s_1} = O_P(1)$ and $\bar{x}_{t_1t_2} = O_P(1)$. This yields that

$$\begin{aligned}
x_{t_1t_2}^* - x_{s_1} =& x_{t_2s_1} + \bar{x}_{t_2s_1} + \bar{x}_{t_1t_2} \\
=& x_{t_2s_1} + \sum_{j=-\infty}^{s_1} \left(\sum_{i=s_1+1}^{t_2} \phi_{i-j} \right) \epsilon_j + \sum_{j=-\infty}^{t_2} \left(\sum_{i=t_2+1}^{t_1} \phi_{i-j} \right) \epsilon_j \\
=& x_{t_2s_1} + \xi_{t_2s_1} + \sum_{j=-\infty}^{s_1} \left(\sum_{i=s_1+1}^{t_1} \phi_{i-j} \right) \epsilon_j \\
=& x_{t_2s_1} + \xi_{t_2s_1} + \bar{x}_{t_1s_1} = \sum_{j=s_1+1}^{t_2} \tilde{b}_{t_2,j}\epsilon_j + \bar{x}_{t_1s_1} \equiv \tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1}, \qquad \text{(D.11)}
\end{aligned}$$

where $\xi_{t_2s_1} = \sum_{j=s_1+1}^{t_2}(\sum_{i=t_2+1}^{t_1} \phi_{i-j})\epsilon_j = O_P(1)$, a part of $\bar{x}_{t_1t_2}$. Notice that $\tilde{x}_{t_2s_1} = x_{t_2s_1} + \xi_{t_2s_1}$ has information on $[s_1+1, t_2]$, while $\bar{x}_{t_1s_1}$ has information up to s_1, hence they are mutually independent. Since $d_{t_2s_1}^{-1}\tilde{x}_{t_2s_1} = d_{t_2s_1}^{-1}x_{t_2s_1} + o_P(1)$ and $d_{t_2s_1}^{-1}(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1}) = d_{t_2s_1}^{-1}x_{t_2s_1} + o_P(1)$, all these quantities $d_{t_2s_1}^{-1}x_{t_2s_1}$, $d_{t_2s_1}^{-1}\tilde{x}_{t_2s_1}$ and $d_{t_2s_1}^{-1}(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1})$ have asymptotically the same density function. See, for example, Lemma 2 of Renyi (1958, p. 223). Thus, in the calculation of expectation we treat them the same.

Denote $T_1 \equiv T_1(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1}, z_{s_1})$, $T_2 \equiv T_2(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1}, z_{s_1})$ and $T_3' \equiv T_3'(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1}, z_{s_1})$, as well as $U_1 = U_1(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1})$, to spell their arguments out for the following expectation computation.

Step II. Moving to the next step in the route map of (D.7), let us consider the following conditional expectation,

$$E\left[L_1\left(\frac{x_{t_1} - x_{s_1}}{h_1}\right) L_2\left(\frac{z_{t_1} - z_{s_1}}{h_2}\right) L_1\left(\frac{x_{t_2} - x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right) e_{t_1}e_{t_2} \Big| \mathcal{F}_{s_1} \right]$$

$$
\begin{aligned}
=&E\left\{E\left[L_1\left(\frac{x_{t_1}-x_{s_1}}{h_1}\right)L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)e_{t_1}\Big|\mathcal{F}_{t_2}\right]L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{t_2}\Big|\mathcal{F}_{s_1}\right\}\\
=&E\left[(T_1+T_1^*+T_2+T_3)L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{t_2}\Big|\mathcal{F}_{s_1}\right]\\
=&E\left[T_1\ L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{t_2}\Big|\mathcal{F}_{s_1}\right]\\
&+E\left[T_1^*\ L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{t_2}\Big|\mathcal{F}_{s_1}\right]\\
&+E\left[T_2\ L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{t_2}\Big|\mathcal{F}_{s_1}\right]\\
&+E\left[T_3'\ L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{t_2}\sum_{j=-\infty}^{t_2}\rho_{t_1-j}\epsilon_j\Big|\mathcal{F}_{s_1}\right].
\end{aligned}
\tag{D.12}
$$

For the first term of (D.12),

$$
\begin{aligned}
&E\left[T_1\ L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{t_2}\Big|\mathcal{F}_{s_1}\right]\\
=&\sum_{j=t_2-m_0+1}^{t_2}\rho_{t_2-j}E\left[T_1\ L_1\left(\frac{x_{t_2s_1}+x^*_{t_2s_1}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\epsilon_j\Big|\mathcal{F}_{s_1}\right]\\
&+\sum_{j=s_1+1}^{t_2-m_0}\rho_{t_2-j}E\left[T_1\ L_1\left(\frac{x_{t_2s_1}+x^*_{t_2s_1}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\epsilon_j\Big|\mathcal{F}_{s_1}\right]\\
&+\sum_{j=-\infty}^{s_1}\rho_{t_2-j}\epsilon_jE\left[T_1\ L_1\left(\frac{x_{t_2s_1}+x^*_{t_2s_1}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\Big|\mathcal{F}_{s_1}\right]\\
=&\sum_{j=t_2-m_0+1}^{t_2}\rho_{t_2-j}E\left[T_1(\tilde{x}_{t_2s_1}+\bar{x}_{t_1s_1},z_{s_1})L_1\left(\frac{x_{t_2s_1}+x^*_{t_2s_1}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\epsilon_j\Big|\mathcal{F}_{s_1}\right]\\
&+\sum_{j=s_1+1}^{t_2-m_0}\rho_{t_2-j}E\Big[T_1(\tilde{x}_{t_2s_1/j}+\tilde{b}_{t_2,j}\epsilon_j+\bar{x}_{t_1s_1},z_{s_1})L_1\left(\frac{x_{t_2s_1/j}+b_{t_2,j}\epsilon_j+x^*_{t_2s_1}-x_{s_2}}{h_1}\right)\\
&\times L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\epsilon_j\Big|\mathcal{F}_{s_1}\Big]\\
&+\sum_{j=-\infty}^{s_1}\rho_{t_2-j}\epsilon_jE\left[T_1(\tilde{x}_{t_2s_1}+\bar{x}_{t_1s_1},z_{s_1})L_1\left(\frac{x_{t_2s_1}+x^*_{t_2s_1}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\Big|\mathcal{F}_{s_1}\right]\\
=&\sum_{j=t_2-m_0+1}^{t_2}\rho_{t_2-j}\iiint T_1(d_{t_2s_1}x,z_{s_1})L_1\left(\frac{d_{t_2s_1}x+x^*_{t_2s_1}-x_{s_2}}{h_1}\right)L_2\left(\frac{z-z_{s_2}}{h_2}\right)u\\
&\times p_{t_2s_1}(x,z,u)dxdzdu\\
&+\sum_{j=s_1+1}^{t_2-m_0}\rho_{t_2-j}\iiint[T_1(d_{t_2s_1}x+\tilde{b}_{t_2,j}u,z_{s_1})L_1\left(\frac{d_{t_2s_1}x+b_{t_2,j}u+x^*_{t_2s_1}-x_{s_2}}{h_1}\right)\\
&\times L_2\left(\frac{z-z_{s_2}}{h_2}\right)q_{t_2s_1/j}(x,z)dxdzuf_\epsilon(u)du\\
&+\sum_{j=-\infty}^{s_1}\rho_{t_2-j}\epsilon_j\iint T_1(d_{t_2s_1}x,z_{s_1})L_1\left(\frac{d_{t_2s_1}x+x^*_{t_2s_1}-x_{s_2}}{h_1}\right)L_2\left(\frac{z-z_{s_2}}{h_2}\right)q_{t_2s_1}(x,z)dxdz\\
=&h_1h_2^dd^{-1}_{t_2s_1}\sum_{j=t_2-m_0+1}^{t_2}\rho_{t_2-j}\iiint T_1(h_1x-x^*_{t_2s_1}+x_{s_2},z_{s_1})L_1(x)L_2(z)u
\end{aligned}
$$

$$
\begin{aligned}
&\times p_{t_2s_1}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}, u\right) dxdzdu \\
&+ \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_2-j} \iiint [T_1(d_{t_2s_1}x + \tilde{b}_{t_2,j}u, z_{s_1}) L_1\left(\frac{d_{t_2s_1}x + b_{t_2,j}u + x^*_{t_2s_1} - x_{s_2}}{h_1}\right) \\
&\times L_2\left(\frac{z - z_{s_2}}{h_2}\right) q_{t_2s_1/j}(x,z) dxdzuf_\epsilon(u)du \\
&+ h_1h_2^d d_{t_2s_1}^{-1} \sum_{j=-\infty}^{s_1} \rho_{t_2-j}\epsilon_j \iint T_1(h_1x + x^*_{t_2s_1} - x_{s_2}, z_{s_1}) L_1(x)L_2(z) \\
&\times q_{t_2s_1}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}\right) dxdz \\
=&h_1h_2^d d_{t_2s_1}^{-1} \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_2-j} \iiint dxdzdu\ T_1(h_1x - x^*_{t_2s_1} + x_{s_2}, z_{s_1}) L_1(x)L_2(z)u \\
&\times \Big[p_{t_2s_1}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}, u\right) \\
&\qquad - f_{t_2s_1(m_0)}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}\right) \kappa_j(h_2z + z_{s_2}, u)\Big] \\
&+ h_1h_2^d d_{t_2s_1}^{-1} \int T_1(h_1x - x^*_{t_2s_1} + x_{s_2}, z_{s_1}) L_1(x) f_{t_2s_1(m_0)}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}\right) dx \\
&\quad \times \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_2-j} \iint L_2(z)u\kappa_j(h_2z + z_{s_2}, u) dzdu \\
&+ h_1h_2^d d_{t_2s_1}^{-1} \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_2-j} \iiint [T_1(-h_1x + x^*_{t_2s_1} - x_{s_2} + (b_{t_2,j} - \tilde{b}_{t_2,j})u, z_{s_1}) L_1(x) \\
&\times L_2(z) q_{t_2s_1/j}\left(\frac{h_1x - b_{t_2,j}u - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}\right) dxdzuf_\epsilon(u)du \\
&+ \sum_{j=-\infty}^{s_1} \rho_{t_2-j}\epsilon_j\ T'_{13} \equiv\ T_{11} + T^{**}_{11} \\
&+ (1 + o(1)) h_1h_2^d d_{t_2s_1}^{-1} \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_2-j} \iiint [T_1(-h_1x + x^*_{t_2s_1} - x_{s_2}, z_{s_1}) L_1(x)L_2(z)uf_\epsilon(u)du \\
&\times \Big[q_{t_2s_1/j}\left(\frac{h_1x - b_{t_2,j}u - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}\right) \\
&\quad - q_{t_2s_1/j}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}\right)\Big] dxdz + T_{13} \equiv\ T_{11} + T^{**}_{11} + T_{12} + T_{13}.
\end{aligned}
$$

where in T_{12} we have used the form of T_1 to remove its variable $(b_{t_2,j} - \tilde{b}_{t_2,j})u$ since the variable is divided by $d_{t_1t_2}$ in T_1 (we do not mention again in the similar situations below); also we have used $\int uf_\epsilon(u)du = 0$.

Observe that $T_{11} = T_{11}(-x^*_{t_2s_1} + x_{s_2}, z_{s_1}, z_{s_2})$, where z_{s_1} is comprised in T_1, and by virtue of Lemma A.6 and the bound of T_1 we have

$$
\begin{aligned}
T_{11} \le& Ch_1^2h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2} \iiint L_1(x)L_2(z)|u|\nu(h_2z + z_{s_2}, u) dxdzdu \\
\le& Ch_1^2h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2} \int L_1(x)dx \int L_2(z)dz \sup_z \int |u|\nu(z,u)du \\
=& Ch_1^2h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2}, \quad a.s.
\end{aligned}
$$

possessing double super rates.

Denote $T_{11}^{**} = U_2(x_{t_2s_1}^* - x_{s_2}, z_{s_1})V(z_{s_2})$ where

$$U_2(x_{t_2s_1}^* - x_{s_2}, z_{s_1}) \equiv h_1 h_2^d d_{t_2s_1}^{-1} \int T_1(h_1x - x_{t_2s_1}^* + x_{s_2}, z_{s_1}) L_1(x) \times f_{t_2s_1(d)}\left(\frac{h_1x - x_{t_2s_1}^* + x_{s_2}}{d_{t_2s_1}}\right) dx, \tag{D.13}$$

$$V(z_{s_2}) \equiv \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_2-j} \iint L_2(z)u\ \kappa_j(h_2z + z_{s_2}, u)dzdu. \tag{D.14}$$

Moreover,

$$\begin{aligned}|T_{12}| \leq & Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2} \iiint L_1(x)L_2(z)|u|(C_1\mu(h_2z + z_{s_2}) + C_2|u|)f_\epsilon(u)dudxdz \\ = & Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}, \quad a.s.\end{aligned}$$

as derived for T_2, admitting double super rates as well.

For T_{13}, note that $T_{13} \equiv T_{13}' \sum_{j=-\infty}^{s_1} \rho_{t_2-j}\epsilon_j$, and notice that $|T_{13}'| \leq Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-1}$ and T_{13}' is a function of $(x_{t_2s_1}^* - x_{s_2}, z_{s_1})$. Denote $T_{13}' \equiv T_{13}'(x_{t_2s_1}^* - x_{s_2}, z_{s_1})$.

For the second term of (D.12),

$$\begin{aligned}& E\left[T_1^*\ L_1\left(\frac{x_{t_2} - x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right) e_{t_2}\Big|\mathcal{F}_{s_1}\right] \\ = & \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_2-j} E\left[U_1(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1})V(z_{s_1})L_1\left(\frac{x_{t_2s_1} + x_{t_2s_1}^* - x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right)\epsilon_j\Big|\mathcal{F}_{s_1}\right] \\ & + \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_2-j} E\left[U_1(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1})V(z_{s_1})L_1\left(\frac{x_{t_2s_1} + x_{t_2s_1}^* - x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right)\epsilon_j\Big|\mathcal{F}_{s_1}\right] \\ & + \sum_{j=-\infty}^{s_1} \rho_{t_2-j}\epsilon_j E\left[U_1(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1})V(z_{s_1})L_1\left(\frac{x_{t_2s_1} + x_{t_2s_1}^* - x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right)\Big|\mathcal{F}_{s_1}\right] \\ \equiv & T_{11}^* + T_{12}^* + T_{13}^*.\end{aligned}$$

Hence, by Lemma A.6,

$$\begin{aligned}T_{11}^* = & \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_2-j} E\left[U_1(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1})V(z_{s_1})L_1\left(\frac{x_{t_2s_1} + x_{t_2s_1}^* - x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right)\epsilon_j\Big|\mathcal{F}_{s_1}\right] \\ = & \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_2-j} \iiint U_1(d_{t_2s_1}x)V(z_{s_1})L_1\left(\frac{d_{t_2s_1}x + x_{t_2s_1}^* - x_{s_2}}{h_1}\right) L_2\left(\frac{z - z_{s_2}}{h_2}\right) u \\ & \times p_{t_2s_1}(x, z, u)dxdzdu \\ = & h_1h_2^d d_{t_2s_1}^{-1} \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_2-j} \iiint U_1(h_1x - x_{t_2s_1}^* + x_{s_2})V(z_{s_1})L_1(x)L_2(z)u \\ & \times p_{t_2s_1}\left(\frac{h_1x - x_{t_2s_1}^* + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}, u\right) dxdzdu \\ = & h_1h_2^d d_{t_2s_1}^{-1} V(z_{s_1}) \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_2-j} \iiint dxdzdu U_1(h_1x - x_{t_2s_1}^* + x_{s_2})L_1(x)L_2(z)u \\ & \times \left[p_{t_2s_1}\left(\frac{h_1x - x_{t_2s_1}^* + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}, u\right) - f_{t_2s_1(m_0)}\left(\frac{h_1x - x_{t_2s_1}^* + x_{s_2}}{d_{t_2s_1}}\right)\kappa_j(h_2z + z_{s_2}, u)\right]\end{aligned}$$

$$
\begin{aligned}
&+ h_1 h_2^d d_{t_2 s_1}^{-1} V(z_{s_1}) V(z_{s_2}) \int U_1(h_1 x - x_{t_2 s_1}^* + x_{s_2}) L_1(x) f_{t_2 s_1 (m_0)} \left(\frac{h_1 x - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}} \right) dx \\
\equiv & V(z_{s_1}) T_{11a}^* + V(z_{s_1}) V(z_{s_2}) T_{11b}^*.
\end{aligned}
$$

Here, $|T_{11a}^*| \leq C h_1^2 h_2^{2d} d_{t_1 t_2}^{-1} d_{t_2 s_1}^{-2}$, and it can be written as $T_{11a}^* = T_{11a}^*(x_{t_2 s_1}^* - x_{s_2}, z_{s_2})$ for the calculation of expectation in next stage. For T_{11b}^*, denote $T_{11b}^* = T_{11b}^*(x_{t_2 s_1}^* - x_{s_2})$ where

$$
\begin{aligned}
T_{11b}^*(x_{t_2 s_1}^* - x_{s_2}) \equiv & h_1 h_2^d d_{t_2 s_1}^{-1} \int U_1(h_1 x - x_{t_2 s_1}^* + x_{s_2}) L_1(x) f_{t_2 s_1 (m_0)} \left(\frac{h_1 x - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}} \right) dx \\
V(z_{s_2}) \equiv & \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_2-j} \iint L_2(z) u \ \kappa_j(h_2 z + z_{s_2}, u) dz du.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
T_{12}^* = & \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_2-j} E \left[U_1(\tilde{x}_{t_2 s_1} + \bar{x}_{t_1 s_1}) V(z_{s_1}) L_1 \left(\frac{x_{t_2 s_1} + x_{t_2 s_1}^* - x_{s_2}}{h_1} \right) L_2 \left(\frac{z_{t_2} - z_{s_2}}{h_2} \right) \epsilon_j \middle| \mathcal{F}_{s_1} \right] \\
= & V(z_{s_1}) \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_2-j} \iiint U_1(d_{t_2 s_1} x + \tilde{b}_{t_2,j} u) L_1 \left(\frac{d_{t_2 s_1} x + b_{t_2,j} u + x_{t_2 s_1}^* - x_{s_2}}{h_1} \right) \\
& \times L_2 \left(\frac{z - z_{s_2}}{h_2} \right) u \ p_{t_2 s_1 / j}(x, z) f_\epsilon(u) dx dz du \\
= & h_1 h_2^d d_{t_2 s_1}^{-1} V(z_{s_1}) \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_2-j} \iiint U_1(h_1 x + (\tilde{b}_{t_2,j} - b_{t_2,j}) u - x_{t_2 s_1}^* + x_{s_2}) L_1(x) \\
& \times L_2(z) u \ p_{t_2 s_1 / j} \left(\frac{h_1 x - b_{t_2,j} u - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2} \right) f_\epsilon(u) dx dz du \\
= & (1 + o(1)) h_1 h_2^d d_{t_2 s_1}^{-1} V(z_{s_1}) \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_2-j} \iiint U_1(h_1 x - x_{t_2 s_1}^* + x_{s_2}) L_1(x) L_2(z) u f_\epsilon(u) \\
& \times \left[p_{t_2 s_1 / j} \left(\frac{h_1 x - b_{t_2,j} u - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2} \right) \right. \\
& \left. - p_{t_2 s_1 / j} \left(\frac{h_1 x - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2} \right) \right] dx dz du \equiv V(z_{s_1}) \ T_{12a}^*,
\end{aligned}
$$

where, similarly to T_2, by lemmas A.5 and A.6, $|T_{12a}^*| \leq C h_1^2 h_2^{2d} d_{t_1 t_2}^{-1} d_{t_2 s_1}^{-2}$, a.s. Also, denote $T_{12a}^* = T_{12a}^*(x_{t_2 s_1}^* - x_{s_2}, z_{s_2})$.

Furthermore,

$$
\begin{aligned}
T_{13}^* = & \sum_{j=-\infty}^{s_1} \rho_{t_2-j} \epsilon_j E \left[U_1(\tilde{x}_{t_2 s_1} + \bar{x}_{t_1 s_1}) V(z_{s_1}) L_1 \left(\frac{x_{t_2 s_1} + x_{t_2 s_1}^* - x_{s_2}}{h_1} \right) L_2 \left(\frac{z_{t_2} - z_{s_2}}{h_2} \right) \middle| \mathcal{F}_{s_1} \right] \\
= & V(z_{s_1}) \sum_{j=-\infty}^{s_1} \rho_{t_2-j} \epsilon_j \iint U_1(d_{t_2 s_1} x) L_1 \left(\frac{d_{t_2 s_1} x + x_{t_2 s_1}^* - x_{s_2}}{h_1} \right) L_2 \left(\frac{z - z_{s_2}}{h_2} \right) p_{t_2 s_1}(x, z) dx dz \\
= & h_1 h_2^d d_{t_2 s_1}^{-1} V(z_{s_1}) \sum_{j=-\infty}^{s_1} \rho_{t_2-j} \epsilon_j \iint U_1(h_1 x - x_{t_2 s_1}^* + x_{s_2}) L_1(x) L_2(z) \\
& \times p_{t_2 s_1} \left(\frac{h_1 x - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2} \right) dx dz \equiv V(z_{s_1}) \ T_{13a}^* \sum_{j=-\infty}^{s_1} \rho_{t_2-j} \epsilon_j,
\end{aligned}
$$

where $|T_{13a}^*| \leq C h_1^2 h_2^{2d} d_{t_1 t_2}^{-1} d_{t_2 s_1}^{-1}$ a.s. and for later use denote $T_{13a}^* = T_{13a}^*(x_{t_2 s_1}^* - x_{s_2}, z_{s_2})$ where

$$
T_{13a}^*(x_{t_2 s_1}^* - x_{s_2}, z_{s_2})
$$

$$= \iint U_1(h_1x - x^*_{t_2s_1} + x_{s_2}) L_1(x) L_2(z) p_{t_2s_1}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}\right) dxdz. \tag{D.15}$$

For the third term of (D.12), it can be derived the same as the first term and it has the same order, so we omit it hereafter.

The fourth term of (D.12) is different from the other in that T_3 is a product of a linear process $\sum_{j=-\infty}^{t_2} \rho_{t_1-j}\epsilon_j$ and T_3'. The linear process is $o_P((t_1-t_2)^{-1})$ and contains the same information as e_{t_2}. It follows that

$$\begin{aligned}
& E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) e_{t_2} \sum_{j=-\infty}^{t_2} \rho_{t_1-j}\epsilon_j \Big| \mathcal{F}_{s_1}\right] \\
&= \sum_{j=-\infty}^{t_2} \rho_{t_1-j}\rho_{t_2-j} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \epsilon_j^2 \Big| \mathcal{F}_{s_1}\right] \\
&\quad + \sum_{j_1=-\infty}^{t_2} \sum_{j_2=-\infty,\neq j_1}^{t_2} \rho_{t_1-j_1}\rho_{t_2-j_2} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \epsilon_{j_1}\epsilon_{j_2} \Big| \mathcal{F}_{s_1}\right] \\
&= \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_1-j}\rho_{t_2-j} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \epsilon_j^2 \Big| \mathcal{F}_{s_1}\right] \\
&\quad + \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_1-j}\rho_{t_2-j} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \epsilon_j^2 \Big| \mathcal{F}_{s_1}\right] \\
&\quad + \sum_{j=-\infty}^{s_1} \rho_{t_1-j}\rho_{t_2-j}\epsilon_j^2 E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \Big| \mathcal{F}_{s_1}\right] \\
&\quad + \sum_{j_1=t_2-m_0+2}^{t_2} \sum_{j_2=t_2-m_0+1}^{j_1-1} \rho_{t_1-j_1}\rho_{t_2-j_2} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \epsilon_{j_1}\epsilon_{j_2} \Big| \mathcal{F}_{s_1}\right] \\
&\quad + \sum_{j_1=t_2-m_0+1}^{t_2} \sum_{j_2=s_1+1}^{t_2-m_0} \rho_{t_1-j_1}\rho_{t_2-j_2} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \epsilon_{j_1}\epsilon_{j_2} \Big| \mathcal{F}_{s_1}\right] \\
&\quad + \sum_{j_1=t_2-m_0+1}^{t_2} \sum_{j_2=-\infty}^{s_1} \rho_{t_1-j_1}\rho_{t_2-j_2}\epsilon_{j_2} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \epsilon_{j_1} \Big| \mathcal{F}_{s_1}\right] \\
&\quad + \sum_{j_1=s_1+2}^{t_2-m_0} \sum_{j_2=s_1+1}^{j_1-1} \rho_{t_1-j_1}\rho_{t_2-j_2} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \epsilon_{j_1}\epsilon_{j_2} \Big| \mathcal{F}_{s_1}\right] \\
&\quad + \sum_{j_1=s_1+1}^{t_2-m_0} \sum_{j_2=-\infty}^{s_1} \rho_{t_1-j_1}\rho_{t_2-j_2}\epsilon_{j_2} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \epsilon_{j_1} \Big| \mathcal{F}_{s_1}\right] \\
&\quad + \sum_{j_1=-\infty}^{s_1} \sum_{j_2=-\infty}^{j_1-1} \rho_{t_1-j_1}\rho_{t_2-j_2}\epsilon_{j_1}\epsilon_{j_2} E\left[T_3'\, L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) \Big| \mathcal{F}_{s_1}\right] \\
&:= \sum_{k=1}^{9} T_{3k}, \quad \text{say.}
\end{aligned}$$

The first three terms T_{31}, T_{32} and T_{33} can be calculated using the decomposition $x_{t_2} = x_{t_2s_1} + x^*_{t_2s_1}$ where $x^*_{t_2s_1}$ is adapted with $\mathcal{F}_{s_1}$, and the density $p_{t_2s_1}(x,z,u)$ for $(d^{-1}_{t_2s_1}x_{t_2s_1}, z_{t_2}, \epsilon_j)$ where $t_2 - d + 1 \leq j \leq t_2$, the density $q_{t_2s_1}(x,z)$ for $(d^{-1}_{t_2s_1}x_{t_2s_1}, z_{t_2})$ as well as the density $f_\epsilon(u)$ for ϵ_j. Also note that $T_3' = T_3'(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1}, z_{s_1})$, and $d^{-1}_{t_2s_1}\tilde{x}_{t_2s_1}$ and $d^{-1}_{t_2s_1}x_{t_2s_1}$ have asymptotically the same density due to the reason mentioned before. Indeed,

$$
\begin{aligned}
T_{31} = & \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_1-j}\rho_{t_2-j} E\Big[T_3'(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1}, z_{s_1}) L_1\left(\frac{x_{t_2s_1} + x^*_{t_2s_1} - x_{s_2}}{h_1}\right) \\
& \times L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right) \epsilon_j^2 \Big| \mathcal{F}_{s_1}\Big] \\
= & \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_1-j}\rho_{t_2-j} \iiint T_3'(d_{t_2s_1}x + \bar{x}_{t_1s_1}, z_{s_1}) L_1\left(\frac{d_{t_2s_1}x + x^*_{t_2s_1} - x_{s_2}}{h_1}\right) \\
& \times L_2\left(\frac{z - z_{s_2}}{h_2}\right) u^2 p_{t_2s_1}(x, z, u) dxdzdu \\
=& h_1 h_2^d d_{t_2s_1}^{-1} \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_1-j}\rho_{t_2-j} \iiint T_3'(h_1x - x^*_{t_2s_1} + x_{s_2} + \bar{x}_{t_1s_1}, z_{s_1}) L_1(x) L_2(z) u^2 \\
& \times p_{t_2s_1}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}, u\right) dxdzdu \\
=& h_1 h_2^d d_{t_2s_1}^{-1} \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_1-j}\rho_{t_2-j} \iiint T_3'(h_1x - x^*_{t_2s_1} + x_{s_2} + \bar{x}_{t_1s_1}, z_{s_1}) L_1(x) L_2(z) u^2 \\
& \times \Big[p_{t_2s_1}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}, h_2z + z_{s_2}, u\right) \\
& - f_{t_2s_1(m_0)}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}\right) \kappa_j(h_2z + z_{s_2}, u)\Big] dxdzdu \\
& + h_1 h_2^d d_{t_2s_1}^{-1} \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_1-j}\rho_{t_2-j} \iiint T_3'(h_1x - x^*_{t_2s_1} + x_{s_2} + \bar{x}_{t_1s_1}, z_{s_1}) L_1(x) L_2(z) u^2 \\
& \times f_{t_2s_1(m_0)}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}\right) \kappa_j(h_2z + z_{s_2}, u) dxdzdu,
\end{aligned}
$$

where $\kappa_j(z,u)$ is the joint density of (z_t, ϵ_j) for $t - m_0 + 1 \le j \le t$, which, by virtue of the bound for T_3', the convergence of ρ_j and Lemma A.6, implies that the first term in absolute value is bounded by $Ch_1^2 h_2^{2d} d_{t_1t_2}^{-1} d_{t_2s_1}^{-2} \rho_{t_1-t_2}$ a.s. that is, this term has double super rates ($\rho_{t_1-t_2}$ decays much faster than $d_{t_1t_2}^{-1}$); the second term, denoted by $T^*_{31} = T^*_{31}(x^*_{t_2s_1} - x_{s_2}, z_{s_1}, z_{s_2})$,

$$
\begin{aligned}
T^*_{31} \equiv & h_1 h_2^d d_{t_2s_1}^{-1} \sum_{j=t_2-m_0+1}^{t_2} \rho_{t_1-j}\rho_{t_2-j} \iiint T_3'(h_1x - x^*_{t_2s_1} + x_{s_2} + \bar{x}_{t_1s_1}, z_{s_1}) L_1(x) L_2(z) u^2 \\
& \times f_{t_2s_1(m_0)}\left(\frac{h_1x - x^*_{t_2s_1} + x_{s_2}}{d_{t_2s_1}}\right) \kappa_j(h_2z + z_{s_2}, u) dxdzdu
\end{aligned}
$$

for the computation in next step.

Similarly,

$$
\begin{aligned}
T_{32} = & \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_1-j}\rho_{t_2-j} E\left[T_3'(\tilde{x}_{t_2s_1} + \bar{x}_{t_1s_1}, z_{s_1}) L_1\left(\frac{x_{t_2} - x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right) \epsilon_j^2 \Big| \mathcal{F}_{s_1}\right] \\
= & \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_1-j}\rho_{t_2-j} E\Big[T_3'(\tilde{x}_{t_2s_1/j} + \tilde{b}_{t_2,j}\epsilon_j + \bar{x}_{t_1s_1}, z_{s_1}) L_1\left(\frac{x_{t_2s_1/j} + b_{t_2,j}\epsilon_j + x^*_{t_2s_1} - x_{s_2}}{h_1}\right) \\
& \times L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right) \epsilon_j^2 \Big| \mathcal{F}_{s_1}\Big] \\
= & \sum_{j=s_1+1}^{t_2-m_0} \rho_{t_1-j}\rho_{t_2-j} \iiint T_3'(d_{t_2s_1}x + \tilde{b}_{t_2,j}u, z_{s_1}) L_1\left(\frac{d_{t_2s_1}x + b_{t_2,j}u + x^*_{t_2s_1} - x_{s_2}}{h_1}\right)
\end{aligned}
$$

$$\times L_2\left(\frac{z-z_{s_2}}{h_2}\right)u^2 q_{t_2 s_1/j}(x,z)dxdzf_\epsilon(u)du$$

$$=h_1 h_2^d d_{t_2 s_1}^{-1}\sum_{j=s_1+1}^{t_2-m_0}\rho_{t_1-j}\rho_{t_2-j}\iiint T_3'(h_1 x - x_{t_2 s_1}^* + x_{s_2} + (\tilde{b}_{t_2,j}-b_{t_2,j})u, z_{s_1})L_1(x)L_2(z)u^2$$

$$\times q_{t_2 s_1/j}\left(\frac{h_1 x - b_{t_2,j}u - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2}\right)dxdzf_\epsilon(u)du$$

$$=(1+o(1))h_1 h_2^d d_{t_2 s_1}^{-1}\sum_{j=s_1+1}^{t_2-m_0}\rho_{t_1-j}\rho_{t_2-j}\iiint T_3'(h_1 x - x_{t_2 s_1}^* + x_{s_2}, z_{s_1})L_1(x)L_2(z)u^2$$

$$\times\Bigg[q_{t_2 s_1/j}\left(\frac{h_1 x - b_{t_2,j}u - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2}\right)$$

$$- f_{t_2 s_1(m_0)/j}\left(\frac{h_1 x - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}\right)p(h_2 z + z_{s_2})\Bigg]dxdzf_\epsilon(u)du$$

$$+ h_1 h_2^d d_{t_2 s_1}^{-1}\sum_{j=s_1+1}^{t_2-m_0}\rho_{t_1-j}\rho_{t_2-j}\iiint T_3'(h_1 x - x_{t_2 s_1}^* + x_{s_2}, z_{s_1})L_1(x)L_2(z)u^2$$

$$\times f_{t_2 s_1(m_0)/j}\left(\frac{h_1 x - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}\right)p(h_2 z + z_{s_2})dxdzf_\epsilon(u)du$$

where $p(\cdot)$ is the density of z_t, and clearly the first term has double super rates due to Lemma A.6 and the convergence of ρ_j; we thus only leave the second term, denoted by $T_{32}^* = T_{32}^*(x_{t_2 s_1}^* - x_{s_2}, z_{s_1}, z_{s_2})$, for later computation,

$$T_{32}^* \equiv h_1 h_2^d d_{t_2 s_1}^{-1}\sum_{j=s_1+1}^{t_2-m_0}\rho_{t_1-j}\rho_{t_2-j}\iiint T_3'(h_1 x - x_{t_2 s_1}^* + x_{s_2}, z_{s_1})L_1(x)L_2(z)u^2$$

$$\times f_{t_2 s_1(m_0)/j}\left(\frac{h_1 x - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}\right)p(h_2 z + z_{s_2})dxdzf_\epsilon(u)du.$$

Moreover, we have

$$T_{33} = \sum_{j=-\infty}^{s_1}\rho_{t_1-j}\rho_{t_2-j}\epsilon_j^2 E\left[T_3'(\tilde{x}_{t_2 s_1} + \bar{x}_{t_1 s_1}, z_{s_1})L_1\left(\frac{x_{t_2 s_1} + x_{t_2 s_1}^* - x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2} - z_{s_2}}{h_2}\right)\Big|\mathcal{F}_{s_1}\right]$$

$$= \sum_{j=-\infty}^{s_1}\rho_{t_1-j}\rho_{t_2-j}\epsilon_j^2\iint T_3'(d_{t_2 s_1}x, z_{s_1})L_1\left(\frac{d_{t_2 s_1}x + x_{t_2 s_1}^* - x_{s_2}}{h_1}\right)L_2\left(\frac{z - z_{s_2}}{h_2}\right)$$

$$\times q_{t_2 s_1}(x,z)dxdz$$

$$=h_1 h_2^d d_{t_2 s_1}^{-1}\sum_{j=-\infty}^{s_1}\rho_{t_1-j}\rho_{t_2-j}\epsilon_j^2\iint T_3'(h_1 x - x_{t_2 s_1}^* + x_{s_2}, z_{s_1})L_1(x)L_2(z)$$

$$\times q_{t_2 s_1}\left(\frac{h_1 x - x_{t_2 s_1}^* + x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2}\right)dxdz \equiv T_{33}'\sum_{j=-\infty}^{s_1}\rho_{t_1-j}\rho_{t_2-j}\epsilon_j^2,$$

implying that T_{33} has double super rates, derived from ρ_{t_1-j} and ρ_{t_2-j} with $j \le s_1$ respectively, and hence is omitted hereafter.

On the other hand, for T_{3k} where $4 \le k \le 9$, we can use the density $p_{t_2 s_1}(x,z,u,v)$ in Lemma A.6 for $(d_{t_2 s_1}^{-1}x_{t_2 s_1}, z_{t_2}, \epsilon_{j_1}, \epsilon_{j_2})$ where $t_2 - m_0 + 1 \le j_1 \ne j_2 \le t_2$, the density $p_{t_2 s_1}(x,z,u)$ for $(d_{t_2 s_1}^{-1}x_{t_2 s_1}, z_{t_2}, \epsilon_{j_1})$ where $t_2 - m_0 + 1 \le j_1 \le t_2$, the density $q_{t_2 s_1}(x,z)$ for $(d_{t_2 s_1}^{-1}x_{t_2 s_1}, z_{t_2})$ as well as the density $f_\epsilon(u)$ for ϵ_j. Thus,

$$T_{34} = \sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}$$

$$
\begin{aligned}
&\times E\left[T_3'(\tilde{x}_{t_2 s_1}+\bar{x}_{t_1 s_1}, z_{s_1}) L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\epsilon_{j_1}\epsilon_{j_2}\Big|\mathcal{F}_{s_1}\right]\\
=&\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\int\cdots\int T_3'(d_{t_2 s_1}x, z_{s_1}) L_1\left(\frac{d_{t_2 s_1}x+x^*_{t_2 s_1}-x_{s_2}}{h_1}\right)\\
&\times L_2\left(\frac{z-z_{s_2}}{h_2}\right) uv\ p_{t_2 s_1}(x,z,u,v)dxdzdudv\\
=&h_1 h_2^d d_{t_2 s_1}^{-1}\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\int\cdots\int T_3'(h_1 x - x^*_{t_2 s_1}+x_{s_2}, z_{s_1}) L_1(x)\\
&\times L_2(z)uv\ p_{t_2 s_1}\left(\frac{h_1 x - x^*_{t_2 s_1}+x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2}, u, v\right) dxdzdudv\\
=&h_1 h_2^d d_{t_2 s_1}^{-1}\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\int\cdots\int T_3'(h_1 x - x^*_{t_2 s_1}+x_{s_2}, z_{s_1}) L_1(x)\\
&\times L_2(z)uv\ \Big[p_{t_2 s_1}\left(\frac{h_1 x - x^*_{t_2 s_1}+x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2}, u, v\right)\\
&\quad - f_{t_2 s_1(m_0)}\left(\frac{h_1 x - x^*_{t_2 s_1}+x_{s_2}}{d_{t_2 s_1}}\right)\kappa_{j_1 j_2}(h_2 z + z_{s_2}, u, v)\Big] dxdzdudv\\
&+ h_1 h_2^d d_{t_2 s_1}^{-1}\int T_3'(h_1 x - x^*_{t_2 s_1}+x_{s_2}, z_{s_1}) L_1(x) f_{t_2 s_1(m_0)}\left(\frac{h_1 x - x^*_{t_2 s_1}+x_{s_2}}{d_{t_2 s_1}}\right) dx\\
&\quad\times\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\iiint L_2(z)uv\ \kappa_{j_1 j_2}(h_2 z + z_{s_2}, u, v)dzdudv,
\end{aligned}
$$

where $\kappa_{j_1,j_2}(z,u,v)$ is the joint density of $(z_t, \epsilon_{j_1}, \epsilon_{j_2})$ for $t-m_0+1\le j_1\ne j_2\le t$, and similarly the first term has double super rates (one from $\rho_{t_1-j_1}$ for $j_1\le t_2$, one from Lemma A.6 for the difference of the densities), while the seconde one, denoted by $T^*_{34}=T^*_{34}(x^*_{t_2 s_1}-x_{s_2}, z_{s_2}, z_{s_1})$,

$$
\begin{aligned}
T^*_{34}\equiv&h_1 h_2^d d_{t_2 s_1}^{-1}\int T_3'(h_1 x - x^*_{t_2 s_1}+x_{s_2}, z_{s_1}) L_1(x) f_{t_2 s_1(m_0)}\left(\frac{h_1 x - x^*_{t_2 s_1}+x_{s_2}}{d_{t_2 s_1}}\right) dx\\
&\times\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\iiint L_2(z)uv\ \kappa_{j_1 j_2}(h_2 z + z_{s_2}, u, v)dzdudv,
\end{aligned}
$$

is remained to the next step. Also, observe that

$$
\begin{aligned}
T_{35}=&\sum_{j_1=t_2-m_0+1}^{t_2}\sum_{j_2=s_1+1}^{t_2-m_0}\rho_{t_1-j_1}\rho_{t_2-j_2}E\Big[T_3'(\tilde{x}_{t_2 s_1/j_2}+\tilde{b}_{t_2 j_2}\epsilon_{j_2}+\bar{x}_{t_1 s_1}, z_{s_1})\\
&\times L_1\left(\frac{x_{t_2 s_1/j_2}+b_{t_2 j_2}\epsilon_{j_2}+x^*_{t_2 s_1}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\epsilon_{j_1}\epsilon_{j_2}\Big|\mathcal{F}_{s_1}\Big]\\
=&\sum_{j_1=t_2-m_0+1}^{t_2}\sum_{j_2=s_1+1}^{t_2-m_0}\rho_{t_1-j_1}\rho_{t_2-j_2}\int\cdots\int T_3'(d_{t_2 s_1}x+\tilde{b}_{t_2,j_2}v, z_{s_1})\\
&\times L_1\left(\frac{d_{t_2 s_1}x+b_{t_2,j_2}v+x^*_{t_2 s_1}-x_{s_2}}{h_1}\right) L_2\left(\frac{z-z_{s_2}}{h_2}\right) uv\ p_{t_2 s_1}(x,z,u) f_\epsilon(v)dxdzdudv\\
=&h_1 h_2^d d_{t_2 s_1}^{-1}\sum_{j_1=t_2-m_0+1}^{t_2}\sum_{j_2=s_1+1}^{t_2-m_0}\rho_{t_1-j_1}\rho_{t_2-j_2}\int\cdots\int T_3'(h_1 x - x^*_{t_2 s_1}+x_{s_2}+(\tilde{b}_{t_2 j_2}-b_{t_2 j_2})v, z_{s_1})\\
&\times L_1(x)L_2(z)uv\ p_{t_2 s_1}\left(\frac{h_1 x - b_{t_2 j_2}v - x^*_{t_2 s_1}+x_{s_2}}{d_{t_2 s_1}}, h_2 z + z_{s_2}, u\right) f_\epsilon(v)dxdzdudv
\end{aligned}
$$

$$
\begin{aligned}
=&(1+o(1))h_1h_2^d d_{t_2s_1}^{-1}\sum_{j_1=t_2-m_0+1}^{t_2}\sum_{j_2=s_1+1}^{t_2-m_0}\rho_{t_1-j_1}\rho_{t_2-j_2}\int\cdots\int T_3'(h_1x-x_{t_2s_1}^*+x_{s_2},z_{s_1})\\
&\times L_1(x)L_2(z)uv\; p_{t_2s_1}\left(\frac{h_1x-b_{t_2j_2}v-x_{t_2s_1}^*+x_{s_2}}{d_{t_2s_1}},h_2z+z_{s_2},u\right)f_\epsilon(v)dxdzdudv\\
=&h_1h_2^d d_{t_2s_1}^{-1}\sum_{j_1=t_2-m_0+1}^{t_2}\sum_{j_2=s_1+1}^{t_2-m_0}\rho_{t_1-j_1}\rho_{t_2-j_2}\int\cdots\int T_3'(h_1x-x_{t_2s_1}^*+x_{s_2},z_{s_1})\\
&\times L_1(x)L_2(z)uv\;\Bigg[p_{t_2s_1}\left(\frac{h_1x-b_{t_2j_2}v-x_{t_2s_1}^*+x_{s_2}}{d_{t_2s_1}},h_2z+z_{s_2},u\right)\\
&\quad -p_{t_2s_1}\left(\frac{h_1x-x_{t_2s_1}^*+x_{s_2}}{d_{t_2s_1}},h_2z+z_{s_2},u\right)\Bigg]f_\epsilon(v)dxdzdudv
\end{aligned}
$$

implying that $|T_{35}|\le Ch_1^2h_2^{2d}d_{t_1t_2}^{-1}d_{t_2s_1}^{-2}(t_1-t_2)^{-1}$ a.s. where we have used the convergence of ρ_j, the bound of T_3', Assumption 2.1 and Lemma A.6.

$$
\begin{aligned}
T_{36}=&\sum_{j_2=-\infty}^{s_1}\rho_{t_2-j_2}\epsilon_{j_2}\sum_{j_1=t_2-m_0+1}^{t_2}\rho_{t_1-j_1}E\left[T_3'L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\epsilon_{j_1}\Big|\mathcal{F}_{s_1}\right]\\
=&\sum_{j_2=-\infty}^{s_1}\rho_{t_2-j_2}\epsilon_{j_2}\sum_{j_1=t_2-m_0+1}^{t_2}\rho_{t_1-j_1}\iiint T_3'(d_{t_2s_1}x+\tilde b_{t_2,j_1}u,z_{s_1})\\
&\times L_1\left(\frac{d_{t_2s_1}x+b_{t_2,j_1}u+x_{t_2s_1}^*-x_{s_2}}{h_1}\right)L_2\left(\frac{z-z_{s_2}}{h_2}\right)u\;p_{t_2s_1/j_1}(x,z,u)dxdzdu\\
=&h_1h_2^d d_{t_2s_1}^{-1}\sum_{j_2=-\infty}^{s_1}\rho_{t_2-j_2}\epsilon_{j_2}\sum_{j_1=t_2-m_0+1}^{t_2}\rho_{t_1-j_1}\iiint T_3'(h_1x+(\tilde b_{t_2,j_1}-b_{t_2,j_1})u-x_{t_2s_1}^*+x_{s_2},z_{s_1})\\
&\times L_1(x)L_2\left(\frac{z-z_{s_2}}{h_2}\right)u\;p_{t_2s_1/j_1}\left(\frac{h_1x-b_{t_2,j_1}u-x_{t_2s_1}^*+x_{s_2}}{d_{t_2s_1}},h_2z+z_{s_2},u\right)dxdzdu\\
\equiv&T_{36}'\sum_{j_2=-\infty}^{s_1}\rho_{t_2-j_2}\epsilon_{j_2},
\end{aligned}
$$

where $|T_{36}'|\le Ch_1^2h_2^{2d}d_{t_1t_2}^{-1}(t_1-t_2)^{-1}d_{t_2s_1}^{-1}$ a.s. by Lemma A.6. Notice also that from the residue $\sum_{j_2=-\infty}^{s_1}\rho_{t_2-j_2}\epsilon_{j_2}$, we can derive another super rate:

$$
T_{37}=\sum_{j_1=s_1+2}^{t_2-m_0}\sum_{j_2=s_1+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}E\left[T_3'L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\epsilon_{j_1}\epsilon_{j_2}\Big|\mathcal{F}_{s_1}\right],
$$

which, by a similar calculation as T_{35}, satisfies that $|T_{37}|\le Ch_1^2h_2^{2d}d_{t_1t_2}^{-1}d_{t_2s_1}^{-2}(t_1-t_2)^{-1}$ a.s.

$$
\begin{aligned}
T_{38}=&\sum_{j_1=s_1+1}^{t_2-m_0}\sum_{j_2=-\infty}^{s_1}\rho_{t_1-j_1}\rho_{t_2-j_2}\epsilon_{j_2}E\left[T_3'L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\epsilon_{j_1}\Big|\mathcal{F}_{s_1}\right]\\
\equiv&T_{38}'\sum_{j_2=-\infty}^{s_1}\rho_{t_2-j_2}\epsilon_{j_2},
\end{aligned}
$$

which gives that $|T_{38}'|\le Ch_1^2h_2^{2d}d_{t_1t_2}^{-1}d_{t_2s_1}^{-1}(t_1-t_2)^{-1}$ a.s., similarly,

$$
T_{39}=\sum_{j_1=-\infty}^{s_1}\sum_{j_2=-\infty}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\epsilon_{j_1}\epsilon_{j_2}E\left[T_3'L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)\Big|\mathcal{F}_{s_1}\right]
$$

$$\equiv T'_{39} \sum_{j_1=-\infty}^{s_1} \sum_{j_2=-\infty}^{j_1-1} \rho_{t_1-j_1}\rho_{t_2-j_2}\epsilon_{j_1}\epsilon_{j_2}$$

which yields $|T'_{39}| \le Ch_1^2 h_2^{2d} d_{t_1t_2}^{-1} d_{t_2s_1}^{-1}$ a.s..

The terms T_{3k}, $k = 5, \cdots, 9$ all have double super rates and then they all are neglected for further calculation. Now, the first two conditional expectations in (D.7) have calculated, giving that

$$E[B_{n15}] = 2\frac{d_n^2}{n^2} \sum_{t_1=4}^{n} \sum_{t_2=3}^{t_1-1} \sum_{s_1=2}^{t_1-1} \sum_{s_2=1}^{s_1-1} E\Big[\pi_1(x_{s_1})\pi_2(z_{s_1}) L_1\left(\frac{x_{t_1}-x_{s_1}}{h_1}\right) L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)$$
$$\times \pi_1(x_{s_2})\pi_2(z_{s_2}) L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) e_{t_1}e_{t_2}e_{s_1}e_{s_2}\Big]$$
$$=2\frac{d_n^2}{n^2} \sum_{t_1=4}^{n} \sum_{t_2=3}^{t_1-1} \sum_{s_1=2}^{t_1-1} \sum_{s_2=1}^{s_1-1} E\Big\{E\Big[(T_1+T_1^*+T_2+T_3)\pi_1(x_{s_1})\pi_2(z_{s_1})$$
$$\times \pi_1(x_{s_2})\pi_2(z_{s_2}) L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right) L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right) e_{t_2}e_{s_1}e_{s_2}\Big|\mathcal{F}_{s_1}\Big]\Big\}$$
$$=2\frac{d_n^2}{n^2} \sum_{t_1=4}^{n} \sum_{t_2=3}^{t_1-1} \sum_{s_1=2}^{t_1-1} \sum_{s_2=1}^{s_1-1} E\Big\{(T_{11}+T_{11}^{**}+T_{12}+T_{13})\pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}\Big\} \tag{D.16}$$
$$+2\frac{d_n^2}{n^2} \sum_{t_1=4}^{n} \sum_{t_2=3}^{t_1-1} \sum_{s_1=2}^{t_1-1} \sum_{s_2=1}^{s_1-1} E\Big\{(T_{11}^*+T_{12}^*+T_{13}^*)\pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}\Big\} \tag{D.17}$$
$$+2\frac{d_n^2}{n^2} \sum_{t_1=4}^{n} \sum_{t_2=3}^{t_1-1} \sum_{s_1=2}^{t_1-1} \sum_{s_2=1}^{s_1-1} E\Big\{(T_{21}+T_{21}^{**}+T_{22}+T_{23})\pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}\Big\} \tag{D.18}$$
$$+2\frac{d_n^2}{n^2} \sum_{t_1=4}^{n} \sum_{t_2=3}^{t_1-1} \sum_{s_1=2}^{t_1-1} \sum_{s_2=1}^{s_1-1} E\Big\{\sum_{k=1}^{9} T_{3k}\ \pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}\Big\}. \tag{D.19}$$

Step III. Consider (D.16) first.

As we have shown $|T_{11}| \le Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2}$ and $|T_{12}| \le Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2}$, these two terms have double super rates. We are to show here such terms are negligible for the further calculation, using $E\{T_{11}\pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}\}$ as an example to provide the reader with the reasoning. In fact, noting that $\pi_2(\cdot)$ is a probability density which is continuous implied by the Lipschitz condition in Assumption 2.3 and thus is bounded,

$$|E[T_{11}\pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}]|$$
$$\le Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2} E\{E[\pi_1(x_{s_1})|e_{s_1}||\mathcal{F}_{s_2}]\pi_1(x_{s_2})|e_{s_2}|\}$$
$$\le Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2} \sum_{j=s_2+1}^{s_1} |\rho_{s_1-j}| E\{E[\pi_1(x_{s_1s_2/j} + b_{s_1,j}\epsilon_j + x^*_{s_1s_2})|\epsilon_j||\mathcal{F}_{s_2}]\pi_1(x_{s_2})|e_{s_2}|\}$$
$$+ Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2} E\{E[\pi_1(x_{s_1s_2} + x^*_{s_1s_2})|\mathcal{F}_{s_2}] \sum_{j=-\infty}^{s_2} |\rho_{s_1-j}\epsilon_j|\pi_1(x_{s_2})|e_{s_2}|\}$$
$$=Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2} \sum_{j=s_2+1}^{s_1} |\rho_{s_1-j}|$$
$$\times E\{\iint \pi_1(d_{s_1s_2}x + b_{s_1,j}u + x^*_{s_1s_2})|u| f_{s_1s_2/j}(x)dx f_\epsilon(u)du\, \pi_1(x_{s_2})|e_{s_2}|\}$$
$$+ Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-2} E\{\int \pi_1(d_{s_1s_2}x + x^*_{s_1s_2}) f_{s_1s_2}(x)dx \sum_{j=-\infty}^{s_2} |\rho_{s_1-j}\epsilon_j|\pi_1(x_{s_2})|e_{s_2}|\}$$

$$
\begin{aligned}
=&Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}\sum_{j=s_2+1}^{s_1}|\rho_{s_1-j}|\\
&\times E\{\iint \pi_1(x)|u|f_{s_1s_2/j}\left(\frac{x-b_{s_1,j}u-x^*_{s_1s_2}}{d_{s_1s_2}}\right)dxf_\epsilon(u)du\pi_1(x_{s_2})|e_{s_2}|\}\\
&+Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}E\{\int \pi_1(x)f_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)dx\sum_{j=-\infty}^{s_2}|\rho_{s_1-j}\epsilon_j|\pi_1(x_{s_2})|e_{s_2}|\}\\
\le&Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}E[\pi_1(x_{s_2})|e_{s_2}|]\\
&+Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}E[\sum_{j=-\infty}^{s_2}|\rho_{s_1-j}\epsilon_j|\pi_1(x_{s_2})|e_{s_2}|]\\
\le&Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}\sum_{j=-\infty}^{s_2}|\rho_{s_2-j}|E[\pi_1(x_{s_2})|\epsilon_j|]\\
&+Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}\sum_{j=-\infty}^{s_2}|\rho_{s_1-j}\rho_{s_2-j}|E[\pi_1(x_{s_2})|\epsilon_j|^2]\\
&+Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}\sum_{j_1=-\infty}^{s_2}\sum_{j_2=-\infty,\neq j_1}^{s_2}|\rho_{s_1-j}\rho_{s_2-j}|E[\pi_1(x_{s_2})|\epsilon_{j_1}\epsilon_{j_2}|]\\
=&Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}d_{s_2}^{-1}(1+o(1)),
\end{aligned}
$$

due to the fact that $\sum_{j=-\infty}^{s_2}|\rho_{s_1-j}|=o((s_1-s_2)^{-1})$. Therefore, one of the terms in $E[B_{n15}]$ is $o(1)$.

For T_{13} in (D.16), noting that $T_{13}=T_{13}'\sum_{j=-\infty}^{s_1}\rho_{t_2-j}\epsilon_j$ and $|T_{13}'|\le Ch_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-1}$, it has double super rates as well, in view of the convergence of ρ_{t_2-j} for $j\le s_1$, hence is neglected.

Now, we focus on the calculation of $E[T_{11}^{**}\ \pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}]=E\Big\{E[T_{11}^{**}\ \pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}]\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_2}\Big\}$. Notice that $T_{11}^{**}=U_2(x^*_{t_2s_1}-x_{s_2},z_{s_1})V(z_{s_2})$, similarly to (D.11), $x^*_{t_2s_1}-x_{s_2}=x_{s_1}-x_{s_2}+\bar{x}_{t_2s_1}=x_{s_1s_2}+\bar{x}_{s_1s_2}+\bar{x}_{t_2s_1}=\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2}$, and $x_{s_1}=x_{s_1s_2}+x^*_{s_1s_2}$.

$$
\begin{aligned}
&E[T_{11}^{**}\pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}]\\
=&V(z_{s_2})\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}E[U_2(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2},z_{s_1})\pi_1(x_{s_1s_2}+x^*_{s_1s_2})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}]\\
&+V(z_{s_2})\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}E[U_2(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2},z_{s_1})\pi_1(x_{s_1s_2}+x^*_{s_1s_2})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}]\\
&+V(z_{s_2})\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_jE[U_2(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2},z_{s_1})\pi_1(x_{s_1s_2}+x^*_{s_1s_2})\pi_2(z_{s_1})|\mathcal{F}_{s_2}]\\
=&V(z_{s_2})\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint U_2(d_{s_1s_2}x,z)\pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)up_{s_1s_2}(x,z,u)dxdzdu\\
&+V(z_{s_2})\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint U_2(d_{s_1s_2}x+\tilde{b}_{s_1,j}u,z)\pi_1(d_{s_1s_2}x+b_{s_1,j}u+x^*_{s_1s_2})\pi_2(z)u\\
&\times p_{s_1s_2/j}(x,z)f_\epsilon(u)dxdzdu\\
&+V(z_{s_2})\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j\iint U_2(d_{s_1s_2}x,z)\pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)p_{s_1s_2}(x,z)dxdz\\
=&d_{s_1s_2}^{-1}V(z_{s_2})\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint U_2(x-x^*_{s_1s_2},z)\pi_1(x)\pi_2(z)up_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)dxdzdu
\end{aligned}
$$

$$
\begin{aligned}
&+ d_{s_1s_2}^{-1}V(z_{s_2})\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint U_2(x+(\tilde{b}_{s_1,j}-b_{s_1,j})u-x^*_{s_1s_2},z)\pi_1(x)\pi_2(z)u\\
&\quad\times p_{s_1s_2/j}\left(\frac{x-b_{s_1,j}u-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)f_\epsilon(u)dxdzdu\\
&+ d_{s_1s_2}^{-1}V(z_{s_2})\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j\iint U_2(d_{s_1s_2}x,z)\pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)dxdz\\
=&d_{s_1s_2}^{-1}V(z_{s_2})\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint U_2(x-x^*_{s_1s_2},z)\pi_1(x)\pi_2(z)u\\
&\left[p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)-f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z,u)\right]dxdzdu\\
&+ d_{s_1s_2}^{-1}V(z_{s_2})\iiint U_2(x-x^*_{s_1s_2},z)\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\\
&\quad\times\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\pi_2(z)u\kappa_j(z,u)dxdzdu\\
&+(1+o(1))d_{s_1s_2}^{-1}V(z_{s_2})\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint U_2(x-x^*_{s_1s_2},z)\pi_1(x)\pi_2(z)u\\
&\quad\times\left[p_{s_1s_2/j}\left(\frac{x-b_{s_1,j}u-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)-p_{s_1s_2/j}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)\right]f_\epsilon(u)dxdzdu\\
&+ d_{s_1s_2}^{-1}V(z_{s_2})\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j\iint U_2(d_{s_1s_2}x,z)\pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)dxdz\\
\equiv&V(z_{s_2})[T^{**}_{11a}+T^{**}_{11b}+T^{**}_{11c}+T^{**}_{11d}],\quad\text{say.}
\end{aligned}
$$

Here, $|T^{**}_{11a}|\le h_1^2h_2^{2d}d_{t_1t_2}^{-2}d_{t_2s_1}^{-1}d_{s_1s_2}^{-2}$ by the bound of U' and Lemma A.6, T^{**}_{11c} has a similar order while T^{**}_{11d} may have a rate $o((s_1-s_2)^{-1})$ derived from the sum $\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j$. All these terms have double super rates and then we neglect them hereafter. We denote $T^{**}_{11b}=T^{**}_{11b}(x^*_{s_1s_2})$ with

$$
\begin{aligned}
T^{**}_{11b}(x^*_{s_1s_2})\equiv&d_{s_1s_2}^{-1}\iiint U_2(x-x^*_{s_1s_2},z)\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\\
&\times\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\pi_2(z)u\kappa_j(z,u)dxdzdu,
\end{aligned}
$$

and leave $V(z_{s_2})T^{**}_{11b}$ to the next step calculation.

Now, we turn to (D.17). Notice that $T^*_{11}=V(z_{s_1})T^*_{11a}+V(z_{s_1})V(z_{s_2})T^*_{11b}$ and T^*_{11a} and T^*_{11b} are functions of $(x^*_{t_2s_1}-x_{s_2},z_{s_2})$ and $x^*_{t_2s_1}-x_{s_2}$, respectively. Also, $x^*_{t_2s_1}-x_{s_2}=x_{s_1}-x_{s_2}+\bar{x}_{t_2s_1}=x_{s_1s_2}+\bar{x}_{s_1s_2}+\bar{x}_{t_2s_1}=\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2}$. Hence,

$$
\begin{aligned}
&E[T^*_{11}\pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}]=E[V(z_{s_1})T^*_{11a}(x^*_{t_2s_1}-x_{s_2},z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}]\\
&+V(z_{s_2})E[V(z_{s_1})T^*_{11b}(x^*_{t_2s_1}-x_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}]\\
=&\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}E[V(z_{s_1})T^*_{11a}(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2},z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}]\\
&+\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}E[V(z_{s_1})T^*_{11a}(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2},z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}]\\
&+\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_jE[V(z_{s_1})T^*_{11a}(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2},z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})|\mathcal{F}_{s_2}]
\end{aligned}
$$

$$
\begin{aligned}
&+V(z_{s_2})\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}E[V(z_{s_1})T^*_{11b}(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}]\\
&+V(z_{s_2})\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}E[V(z_{s_1})T^*_{11b}(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}]\\
&+V(z_{s_2})\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_jE[V(z_{s_1})T^*_{11b}(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})|\mathcal{F}_{s_2}]\\
=&\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint V(z)T^*_{11a}(d_{s_1s_2}x,z_{s_2})\pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)up_{s_1s_2}(x,z,u)dxdzdu\\
&+\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint V(z)T^*_{11a}(d_{s_1s_2}x+\tilde{b}_{s_1,j}u,z_{s_2})\\
&\qquad\times\pi_1(d_{s_1s_2}x+b_{s_1,j}u+x^*_{s_1s_2})\pi_2(z)up_{s_1s_2}(x,z)f_\epsilon(u)dudxdz\\
&+\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j\iint V(z)T^*_{11a}(d_{s_1s_2}x,z_{s_2})\pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)p_{s_1s_2}(x,z)dxdz\\
&+V(z_{s_2})\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint V(z)T^*_{11b}(d_{s_1s_2}x)\pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)up_{s_1s_2}(x,z,u)dxdzdu\\
&+V(z_{s_2})\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint V(z)T^*_{11b}(d_{s_1s_2}x+\tilde{b}_{s_1,j}u)\\
&\qquad\times\pi_1(d_{s_1s_2}x+b_{s_1,j}u+x^*_{s_1s_2})\pi_2(z)u\;p_{s_1s_2}(x,z)f_\epsilon(u)dudxdz\\
&+V(z_{s_2})\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j\iint V(z)T^*_{11b}(d_{s_1s_2}x)\pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)p_{s_1s_2}(x,z)dxdz\\
=&d_{s_1s_2}^{-1}\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint V(z)T^*_{11a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)up_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)dxdzdu\\
&+d_{s_1s_2}^{-1}\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint V(z)T^*_{11a}(x+(\tilde{b}_{s_1,j}-b_{s_1,j})u-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)u\\
&\qquad\times p_{s_1s_2}\left(\frac{x-b_{s_1,j}u-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)f_\epsilon(u)dudxdz\\
&+d_{s_1s_2}^{-1}\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j\iint V(z)T^*_{11a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)dxdz\\
&+d_{s_1s_2}^{-1}V(z_{s_2})\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint V(z)T^*_{11b}(x-x^*_{s_1s_2})\pi_1(x)\pi_2(z)up_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)dxdzdu\\
&+d_{s_1s_2}^{-1}V(z_{s_2})\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint V(z)T^*_{11b}(x+(\tilde{b}_{s_1,j}-b_{s_1,j})u-x^*_{s_1s_2})\pi_1(x)\pi_2(z)u\\
&\qquad\times\;p_{s_1s_2}\left(\frac{x-b_{s_1,j}u-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)f_\epsilon(u)dudxdz\\
&+d_{s_1s_2}^{-1}V(z_{s_2})\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j\iint V(z)T^*_{11b}(x-x^*_{s_1s_2})\pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)dxdz\\
=&d_{s_1s_2}^{-1}\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint V(z)T^*_{11a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)u
\end{aligned}
$$

$$\times \left[p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}, z, u\right) - f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z,u)\right] dxdzdu$$

$$+ d^{-1}_{s_1s_2} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z) T^*_{11a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)u$$

$$\times f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z,u) dxdzdu$$

$$+ (1+o(1)) d^{-1}_{s_1s_2} \sum_{j=s_2+1}^{s_1-m_0} \rho_{s_1-j} \iiint V(z) T^*_{11a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)u f_\epsilon(u)$$

$$\times \left[p_{s_1s_2}\left(\frac{x-b_{s_1,j}u - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right) - p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}, z\right)\right] dudxdz$$

$$+ d^{-1}_{s_1s_2} \sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j \iint V(z) T^*_{11a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z) p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}, z\right) dxdz$$

$$+ d^{-1}_{s_1s_2} V(z_{s_2}) \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z) T^*_{11b}(x - x^*_{s_1s_2})\pi_1(x)\pi_2(z)u$$

$$\times \left[p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}, z, u\right) - f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z,u)\right] dxdzdu$$

$$+ d^{-1}_{s_1s_2} V(z_{s_2}) \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z) T^*_{11b}(x - x^*_{s_1s_2})\pi_1(x)\pi_2(z)u$$

$$\times f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z,u) dxdzdu$$

$$+ (1+o(1)) d^{-1}_{s_1s_2} V(z_{s_2}) \sum_{j=s_2+1}^{s_1-m_0} \rho_{s_1-j} \iiint V(z) T^*_{11b}(x - x^*_{s_1s_2})\pi_1(x)\pi_2(z)u f_\epsilon(u)$$

$$\times \left[p_{s_1s_2}\left(\frac{x-b_{s_1,j}u - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right) - p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}, z\right)\right] dudxdz$$

$$+ d^{-1}_{s_1s_2} V(z_{s_2}) \sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j \iint V(z) T^*_{11b}(x - x^*_{s_1s_2})\pi_1(x)\pi_2(z) p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}, z\right) dxdz$$

$$\equiv \sum_{k=1}^{8} T^*_{11k}, \quad \text{say.}$$

All the terms except T^*_{112} and T^*_{116} have double super rates, derived either from Lemma A.6 or the convergence of ρ_j, which are therefore neglected hereafter. Denote $T^*_{112} = T^*_{112}(x^*_{s_1s_2}, z_{s_2})$ and $T^*_{116} = V(z_{s_2})T^*_{116}(x^*_{s_1s_2})$ for later computation, where

$$T^*_{112} \equiv d^{-1}_{s_1s_2} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z) T^*_{11a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)u \times f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z,u) dxdzdu \tag{D.20}$$

$$T^*_{116}(x^*_{s_1s_2}) \equiv d^{-1}_{s_1s_2} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z) T^*_{11b}(x - x^*_{s_1s_2})\pi_1(x)\pi_2(z)u \times f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z,u) dxdzdu. \tag{D.21}$$

For $T^*_{12} = V(z_{s_1})T^*_{12a}(x^*_{t_2s_1} - x_{s_2}, z_{s_2})$, noting again $x^*_{t_2s_1} - x_{s_2} = \tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}$,

$$E[T^*_{12}\pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}]$$

$$
\begin{aligned}
=&\sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} E[V(z_{s_1})T^*_{12a}(x^*_{t_2 s_1} - x_{s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}] \\
&+ \sum_{j=s_2+1}^{s_1-m_0} \rho_{s_1-j} E[V(z_{s_1})T^*_{12a}(x^*_{t_2 s_1} - x_{s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}] \\
&+ \sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j E[V(z_{s_1})T^*_{12a}(x^*_{t_2 s_1} - x_{s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})|\mathcal{F}_{s_2}] \\
=&\sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z)T^*_{12a}(d_{s_1s_2}x, z_{s_2})\pi_1(d_{s_1s_2}x + x^*_{s_1s_2})\pi_2(z)u p_{s_1s_2}(x,z,u)dxdzdu \\
&+ \sum_{j=s_2+1}^{s_1-m_0} \rho_{s_1-j} \iiint V(z)T^*_{12a}(d_{s_1s_2}x + \tilde{b}_{s_1,j}u, z_{s_2})\pi_1(d_{s_1s_2}x + b_{s_1,j}u + x^*_{s_1s_2}) \\
&\quad \times \pi_2(z)u\ p_{s_1s_2/j}(x,z)f_\epsilon(u)dxdzdu \\
&+ \sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j \iint V(z)T^*_{12a}(d_{s_1s_2}x, z_{s_2})\pi_1(d_{s_1s_2}x + x^*_{s_1s_2})\pi_2(z)p_{s_1s_2}(x,z)dxdz \\
=&d^{-1}_{s_1s_2}\sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z)T^*_{12a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\quad \times p_{s_1s_2}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}, z, u\right)dxdzdu \\
&+ d^{-1}_{s_1s_2}\sum_{j=s_2+1}^{s_1-m_0} \rho_{s_1-j} \iiint V(z)T^*_{12a}(x + (\tilde{b}_{s_1,j} - b_{s_1,j})u - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\quad \times\ p_{s_1s_2/j}\left(\frac{x - b_{s_1,j}u - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right)f_\epsilon(u)dxdzdu \\
&+ d^{-1}_{s_1s_2}\sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j \iint V(z)T^*_{12a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right)dxdz \\
=&d^{-1}_{s_1s_2}\sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z)T^*_{12a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\quad \times \left[p_{s_1s_2}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}, z, u\right) - f_{s_1s_2(m_0)}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z,u)\right]dxdzdu \\
&+ d^{-1}_{s_1s_2}\sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z)T^*_{12a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\quad \times f_{s_1s_2(m_0)}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z,u)dxdzdu \\
&+ (1+o(1))d^{-1}_{s_1s_2}\sum_{j=s_2+1}^{s_1-m_0} \rho_{s_1-j} \iiint V(z)T^*_{12a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\quad \times \left[p_{s_1s_2/j}\left(\frac{x - b_{s_1,j}u - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right) - p_{s_1s_2/j}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right)\right]f_\epsilon(u)dxdzdu \\
&+ d^{-1}_{s_1s_2}\sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j \iint V(z)T^*_{12a}(x - x^*_{s_1s_2}, z_{s_2})\pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right)dxdz.
\end{aligned}
$$

All terms except the second one have double super rates since T^*_{12a} already has one super rate and, by Lemma A.6 and the convergence of $\sum_{j=-\infty}^{s_2}\rho_{s_1-j} = o((s_1-s_2)^{-1})$, we may derive another super rate for them. Hence these are neglected and denote the second one as

$$T^*_{12b}(x^*_{s_1s_2}, z_{s_2}) \equiv d^{-1}_{s_1s_2} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z) T^*_{12a}(x - x^*_{s_1s_2}, z_{s_2}) \pi_1(x)\pi_2(z) u \times f_{s_1s_2(m_0)}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}\right) \kappa_j(z,u) dxdzdu \tag{D.22}$$

for later calculation.

For $T^*_{13} = V(z_{s_1})\ T^*_{13a}(x^*_{t_2s_1} - x_{s_2}, z_{s_2}) \sum_{j=-\infty}^{s_1} \rho_{t_2-j}\epsilon_j$, observe that

$$\begin{aligned}
&E[T^*_{13}\pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}] \\
=&E[V(z_{s_1})\ T^*_{13a}\pi_1(x_{s_1})\pi_2(z_{s_1}) \sum_{j=-\infty}^{s_1} \rho_{t_2-j}\epsilon_j\ e_{s_1}|\mathcal{F}_{s_2}] \\
=&\sum_{j=-\infty}^{s_1} \rho_{t_2-j}\rho_{s_1-j} E[V(z_{s_1})\ T^*_{13a}\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_j^2|\mathcal{F}_{s_2}] \\
&+2\sum_{j_1=-\infty}^{s_1}\sum_{j_2=-\infty}^{j_1-1} \rho_{t_2-j_1}\rho_{s_1-j_2} E[V(z_{s_1})\ T^*_{13a}\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_{j_1}\epsilon_{j_2}|\mathcal{F}_{s_2}] \\
=&\sum_{j=s_1-m_0+1}^{s_1} \rho_{t_2-j}\rho_{s_1-j} E[V(z_{s_1})\ T^*_{13a}(\tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_j^2|\mathcal{F}_{s_2}] \\
&+\sum_{j=s_2+1}^{s_1-m_0} \rho_{t_2-j}\rho_{s_1-j} E[V(z_{s_1})\ T^*_{13a}(\tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_j^2|\mathcal{F}_{s_2}] \\
&+\sum_{j=-\infty}^{s_2} \rho_{t_2-j}\rho_{s_1-j}\epsilon_j^2\ E[V(z_{s_1})T^*_{13a}(\tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})|\mathcal{F}_{s_2}] \\
&+2\sum_{j_1=s_1-m_0+2}^{s_1}\sum_{j_2=s_1-m_0+1}^{j_1-1} \rho_{t_2-j_1}\rho_{s_1-j_2} E[V(z_{s_1})T^*_{13a}(\tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_{j_1}\epsilon_{j_2}|\mathcal{F}_{s_2}] \\
&+2\sum_{j_1=s_1-m_0+1}^{s_1}\sum_{j_2=s_2+1}^{s_1-m_0} \rho_{t_2-j_1}\rho_{s_1-j_2} E[V(z_{s_1})T^*_{13a}(\tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_{j_1}\epsilon_{j_2}|\mathcal{F}_{s_2}] \\
&+2\sum_{j_1=s_1-m_0+1}^{s_1}\sum_{j_2=-\infty}^{s_2} \rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_2} E[V(z_{s_1})T^*_{13a}(\tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_{j_1}|\mathcal{F}_{s_2}] \\
&+2\sum_{j_1=s_2+2}^{s_1-m_0}\sum_{j_2=s_2+1}^{j_1-1} \rho_{t_2-j_1}\rho_{s_1-j_2} E[V(z_{s_1})T^*_{13a}(\tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_{j_1}\epsilon_{j_2}|\mathcal{F}_{s_2}] \\
&+2\sum_{j_1=s_2+1}^{s_1-m_0}\sum_{j_2=-\infty}^{s_2} \rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_2} E[V(z_{s_1})T^*_{13a}(\tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})\epsilon_{j_1}|\mathcal{F}_{s_2}] \\
&+2\sum_{j_1=-\infty}^{s_2}\sum_{j_2=-\infty}^{j_1-1} \rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_1}\epsilon_{j_2} E[V(z_{s_1})T^*_{13a}(\tilde{x}_{s_1s_2} + \bar{x}_{t_2s_2}, z_{s_2})\pi_1(x_{s_1})\pi_2(z_{s_1})|\mathcal{F}_{s_2}] \\
=&\sum_{j=s_1-m_0+1}^{s_1} \rho_{t_2-j}\rho_{s_1-j} \iiint V(z)T^*_{13a}(d_{s_1s_2}x, z_{s_2})\pi_1(d_{s_1s_2}x + x^*_{s_1s_2}) \\
&\times \pi_2(z)u^2 p_{s_1s_2}(x,z,u)dxdzdu \\
&+\sum_{j=s_2+1}^{s_1-m_0} \rho_{t_2-j}\rho_{s_1-j} \iiint V(z)T^*_{13a}(d_{s_1s_2}x + \tilde{b}_{s_1,j}u, z_{s_2})\pi_1(d_{s_1s_2}x + b_{s_1,j}u + x^*_{s_1s_2})
\end{aligned}$$

$$
\begin{aligned}
&\times \pi_2(z)u^2 p_{s_1s_2/j}(x,z)f_\epsilon(u)dxdzdu \\
&+\sum_{j=-\infty}^{s_2}\rho_{t_2-j}\rho_{s_1-j}\epsilon_j^2\iint V(z)T^*_{13a}(d_{s_1s_2}x,z_{s_2})\pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)p_{s_1s_2}(x,z)dxdz \\
&+2\sum_{j_1=s_1-m_0+2}^{s_1}\sum_{j_2=s_1-m_0+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T^*_{13a}(d_{s_1s_2}x,z_{s_2})\pi_1(d_{s_1s_2}x+x^*_{s_1s_2}) \\
&\times \pi_2(z)uvp_{s_1s_2}(x,z,u,v)dxdzdudv \\
&+2\sum_{j_1=s_1-m_0+1}^{s_1}\sum_{j_2=s_2+1}^{s_1-m_0}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T^*_{13a}(d_{s_1s_2}x+\tilde{b}_{s_1,j_2}v,z_{s_2}) \\
&\times \pi_1(d_{s_1s_2}x+b_{s_1,j_2}v+x^*_{s_1s_2})\pi_2(z)uvp_{s_1s_2/j_2}(x,z,u)f_\epsilon(v)dxdzdudv \\
&+2\sum_{j_1=s_1-m_0+1}^{s_1}\sum_{j_2=-\infty}^{s_2}\rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_2}\iiint V(z)T^*_{13a}(d_{s_1s_2}x,z_{s_2})\pi_1(d_{s_1s_2}x+x^*_{s_1s_2}) \\
&\times \pi_2(z)up_{s_1s_2}(x,z,u)dxdzdu \\
&+2\sum_{j_1=s_2+2}^{s_1-m_0}\sum_{j_2=s_2+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T^*_{13a}(d_{s_1s_2}x+\tilde{b}_{s_1,j_1}u+\tilde{b}_{s_1,j_2}v,z_{s_2}) \\
&\times \pi_1(d_{s_1s_2}x+b_{s_1,j_1}u+b_{s_1,j_2}v+x^*_{s_1s_2})\pi_2(z)uvp_{s_1s_2/j_1j_2}(x,z)f_\epsilon(u)f_\epsilon(v)dxdzdudv \\
&+2\sum_{j_1=s_2+1}^{s_1-m_0}\sum_{j_2=-\infty}^{s_2}\rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_2}\iiint V(z)T^*_{13a}(d_{s_1s_2}x+\tilde{b}_{s_1,j_1}u,z_{s_2}) \\
&\times \pi_1(d_{s_1s_2}x+b_{s_1,j_1}u+x^*_{s_1s_2})\pi_2(z)up_{s_1s_2/j_1}(x,z)f_\epsilon(u)dxdzdu \\
&+2\sum_{j_1=-\infty}^{s_2}\sum_{j_2=-\infty}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_1}\epsilon_{j_2}\iint V(z)T^*_{13a}(d_{s_1s_2}x,z_{s_2}) \\
&\times \pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)p_{s_1s_2}(x,z)dxdz \\
=&d^{-1}_{s_1s_2}\sum_{j=s_1-m_0+1}^{s_1}\rho_{t_2-j}\rho_{s_1-j}\iiint V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x) \\
&\times \pi_2(z)u^2p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)dxdzdu \\
&+d^{-1}_{s_1s_2}\sum_{j=s_2+1}^{s_1-m_0}\rho_{t_2-j}\rho_{s_1-j}\iiint V(z)T^*_{13a}(x-(\tilde{b}_{s_1,j}-b_{s_1,j})u-x^*_{s_1s_2},z_{s_2})\pi_1(x) \\
&\times \pi_2(z)u^2p_{s_1s_2/j}\left(\frac{x-b_{s_1,j}u-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)f_\epsilon(u)dxdzdu \\
&+d^{-1}_{s_1s_2}\sum_{j=-\infty}^{s_2}\rho_{t_2-j}\rho_{s_1-j}\epsilon_j^2\iint V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)dxdz \\
&+2d^{-1}_{s_1s_2}\sum_{j_1=s_1-m_0+2}^{s_1}\sum_{j_2=s_1-m_0+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x) \\
&\times \pi_2(z)uvp_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u,v\right)dxdzdudv \\
&+2d^{-1}_{s_1s_2}\sum_{j_1=s_1-m_0+1}^{s_1}\sum_{j_2=s_2+1}^{s_1-m_0}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T^*_{13a}(x+(\tilde{b}_{s_1,j_2}-b_{s_1,j_2})v-x^*_{s_1s_2},z_{s_2})
\end{aligned}
$$

$$
\begin{aligned}
&\times \pi_1(x)\pi_2(z)uvp_{s_1s_2/j_2}\left(\frac{x-b_{s_1,j_2}v-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)f_\epsilon(v)dxdzdudv\\
&+2d^{-1}_{s_1s_2}\sum_{j_1=s_1-m_0+1}^{s_1}\sum_{j_2=-\infty}^{s_2}\rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_2}\iiint V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\\
&\times \pi_2(z)up_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)dxdzdu\\
&+2d^{-1}_{s_1s_2}\sum_{j_1=s_2+2}^{s_1-m_0}\sum_{j_2=s_2+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)\\
&\times T^*_{13a}(x+(\tilde{b}_{s_1,j_1}-b_{s_1,j_1})u+(\tilde{b}_{s_1,j_2}-b_{s_1,j_2})v-x^*_{s_1s_2},z_{s_2})\\
&\times \pi_1(x)\pi_2(z)uvp_{s_1s_2/j_1j_2}\left(\frac{x-b_{s_1,j_1}u-b_{s_1,j_2}v-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)f_\epsilon(u)f_\epsilon(v)dxdzdudv\\
&+2d^{-1}_{s_1s_2}\sum_{j_1=s_2+1}^{s_1-m_0}\sum_{j_2=-\infty}^{s_2}\rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_2}\iiint V(z)T^*_{13a}(x+(\tilde{b}_{s_1,j_1}-b_{s_1,j_1})u-x^*_{s_1s_2},z_{s_2})\\
&\times \pi_1(x)\pi_2(z)up_{s_1s_2/j_1}\left(\frac{x-b_{s_1,j_1}u-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)f_\epsilon(u)dxdzdu\\
&+2d^{-1}_{s_1s_2}\sum_{j_1=-\infty}^{s_2}\sum_{j_2=-\infty}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_1}\epsilon_{j_2}\iint V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\\
&\times \pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)dxdz\\
=&d^{-1}_{s_1s_2}\sum_{j=s_1-m_0+1}^{s_1}\rho_{t_2-j}\rho_{s_1-j}\iiint V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)u^2\\
&\times\left[p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)-f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(u,z)\right]dxdzdu\\
&+d^{-1}_{s_1s_2}\sum_{j=s_1-m_0+1}^{s_1}\rho_{t_2-j}\rho_{s_1-j}\iiint V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)u^2\\
&\times f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(u,z)dxdzdu\\
&+(1+o(1))d^{-1}_{s_1s_2}\sum_{j=s_2+1}^{s_1-m_0}\rho_{t_2-j}\rho_{s_1-j}\iiint V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)u^2\\
&\times\left[p_{s_1s_2/j}\left(\frac{x-b_{s_1,j}u-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)-f_{s_1s_2(m_0)/j}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)p(z)\right]f_\epsilon(u)dxdzdu\\
&+d^{-1}_{s_1s_2}\sum_{j=s_2+1}^{s_1-m_0}\rho_{t_2-j}\rho_{s_1-j}\iiint V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)u^2\\
&\times f_{s_1s_2(m_0)/j}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)p(z)f_\epsilon(u)dxdzdu\\
&+d^{-1}_{s_1s_2}\sum_{j=-\infty}^{s_2}\rho_{t_2-j}\rho_{s_1-j}\epsilon_j^2\iint V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)dxdz\\
&+2d^{-1}_{s_1s_2}\sum_{j_1=s_1-m_0+2}^{s_1}\sum_{j_2=s_1-m_0+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T^*_{13a}(x-x^*_{s_1s_2},z_{s_2})\pi_1(x)\pi_2(z)uv\\
&\times\left[p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u,v\right)-f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_{j_1j_2}(z,u,v)\right]dxdzdudv
\end{aligned}
$$

$$+ 2d_{s_1s_2}^{-1}\sum_{j_1=s_1-m_0+2}^{s_1}\sum_{j_2=s_1-m_0+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T_{13a}^*(x-x_{s_1s_2}^*,z_{s_2})\pi_1(x)\pi_2(z)uv$$
$$\times f_{s_1s_2(m_0)}\left(\frac{x-x_{s_1s_2}^*}{d_{s_1s_2}}\right)\kappa_{j_1j_2}(z,u,v)dxdzdudv$$
$$+ 2(1+o(1))d_{s_1s_2}^{-1}\sum_{j_1=s_1-m_0+1}^{s_1}\sum_{j_2=s_2+1}^{s_1-m_0}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T_{13a}^*(x-x_{s_1s_2}^*,z_{s_2})\pi_1(x)\pi_2(z)uv$$
$$\times\Big[p_{s_1s_2/j_2}\left(\frac{x-b_{s_1,j_2}v-x_{s_1s_2}^*}{d_{s_1s_2}},z,u\right)-f_{s_1s_2(m_0)/j_2}\left(\frac{x-x_{s_1s_2}^*}{d_{s_1s_2}}\right)\kappa_{j_1}(z,u)\Big]f_\epsilon(v)dxdzdudv$$
$$+ 2d_{s_1s_2}^{-1}\sum_{j_1=s_1-m_0+1}^{s_1}\sum_{j_2=-\infty}^{s_2}\rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_2}\iiint V(z)T_{13a}^*(x-x_{s_1s_2}^*,z_{s_2})\pi_1(x)\pi_2(z)u$$
$$\times p_{s_1s_2}\left(\frac{x-x_{s_1s_2}^*}{d_{s_1s_2}},z,u\right)dxdzdu$$
$$+ 2(1+o(1))d_{s_1s_2}^{-1}\sum_{j_1=s_2+2}^{s_1-m_0}\sum_{j_2=s_2+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)$$
$$\times T_{13a}^*(x-x_{s_1s_2}^*,z_{s_2})\pi_1(x)\pi_2(z)uv$$
$$\times\Big[p_{s_1s_2/j_1j_2}\left(\frac{x-b_{s_1,j_1}u-b_{s_1,j_2}v-x_{s_1s_2}^*}{d_{s_1s_2}},z\right)-f_{s_1s_2(m_0)/j_1j_2}\left(\frac{x-x_{s_1s_2}^*}{d_{s_1s_2}}\right)p(z)\Big]$$
$$\times f_\epsilon(u)f_\epsilon(v)dxdzdudv$$
$$+ 2(1+o(1))d_{s_1s_2}^{-1}\sum_{j_1=s_2+1}^{s_1-m_0}\sum_{j_2=-\infty}^{s_2}\rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_2}\iiint V(z)T_{13a}^*(x-x_{s_1s_2}^*,z_{s_2})\pi_1(x)\pi_2(z)u$$
$$\times\Big[p_{s_1s_2/j_1}\left(\frac{x-b_{s_1,j_1}u-x_{s_1s_2}^*}{d_{s_1s_2}},z\right)-f_{s_1s_2(m_0)/j_1}\left(\frac{x-x_{s_1s_2}^*}{d_{s_1s_2}}\right)p(z)\Big]f_\epsilon(u)dxdzdu$$
$$+ 2d_{s_1s_2}^{-1}\sum_{j_1=-\infty}^{s_2}\sum_{j_2=-\infty}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\epsilon_{j_1}\epsilon_{j_2}\iint V(z)T_{13a}^*(x-x_{s_1s_2}^*,z_{s_2})$$
$$\times\pi_1(x)\pi_2(z)p_{s_1s_2}\left(\frac{x-x_{s_1s_2}^*}{d_{s_1s_2}},z\right)dxdz\equiv\sum_{k=1}^{12}T_{13k}^*,$$

where $\kappa_{j_1j_2}(z,u,v)$ is the joint density of $(z_t,\epsilon_{j_1},\epsilon_{j_2})$ for $t-m_0+1\le j_1\ne j_2\le t$.

All the terms above except T_{13k}^* $(k=2,4,7)$ have double super rates, derived either from both Lemma A.6 and the related coefficients ρ_j or from the coefficients ρ_j only. In T_{139}^*, $\sum_{j_1=s_1-m_0+1}^{s_1}\sum_{j_2=-\infty}^{s_2}\rho_{t_2-j_1}\rho_{s_1-}$ $o((t_2-s_1)^{-1}(s_1-s_2)^{-1})$. Thus, we only consider T_{13k}^* $(k=2,4,7)$ in the next stage, specified by

$$T_{132}^*(x_{s_1s_2}^*,z_{s_2})\equiv d_{s_1s_2}^{-1}\sum_{j=s_1-m_0+1}^{s_1}\rho_{t_2-j}\rho_{s_1-j}\iiint V(z)T_{13a}^*(x-x_{s_1s_2}^*,z_{s_2})\pi_1(x)\pi_2(z)u^2$$
$$\times f_{s_1s_2(m_0)}\left(\frac{x-x_{s_1s_2}^*}{d_{s_1s_2}}\right)\kappa_j(u,z)dxdzdu, \tag{D.23}$$
$$T_{134}^*(x_{s_1s_2}^*,z_{s_2})\equiv d_{s_1s_2}^{-1}\sum_{j=s_2+1}^{s_1-m_0}\rho_{t_2-j}\rho_{s_1-j}\iiint V(z)T_{13a}^*(x-x_{s_1s_2}^*,z_{s_2})\pi_1(x)\pi_2(z)u^2$$
$$\times f_{s_1s_2(m_0)/j}\left(\frac{x-x_{s_1s_2}^*}{d_{s_1s_2}}\right)p(z)f_\epsilon(u)dxdzdu, \tag{D.24}$$
$$T_{137}^*(x_{s_1s_2}^*,z_{s_2})\equiv d_{s_1s_2}^{-1}\sum_{j_1=s_1-m_0+2}^{s_1}\sum_{j_2=s_1-m_0+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T_{13a}^*(x-x_{s_1s_2}^*,z_{s_2})$$

$$\times \pi_1(x)\pi_2(z)uvf_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_{j_1j_2}(z,u,v)dxdzdudv. \tag{D.25}$$

Moreover, each term in $E\{(T_{21}+T^{**}_{21}+T_{22}+T_{23})\pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}\}$ of (D.18) can be dealt with in the same way as for those involving $T_{11}, T^{**}_{11}, T_{12}$ and T_{13}, respectively.

As claimed earlier, for $E\{\sum_{k=1}^{9} T_{3k}\ \pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}\}$ of (D.19), we only need to consider the terms T^*_{3k}, $k=1,2,4$, defined before.

Notice that $T^*_{31}=T^*_{31}(x^*_{t_2s_1}-x_{s_2},z_{s_1},z_{s_2})$ and $x^*_{t_2s_1}-x_{s_2}=\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2}$ where $\bar{x}_{t_2s_2}=O_P(1)$ and $d^{-1}_{s_1s_2}\tilde{x}_{s_1s_2}$ and $d^{-1}_{s_1s_2}x_{s_1s_2}$ have asymptotically the same distribution due to the same reason as before. Hence, by Lemma A.6,

$$\begin{aligned}
&E[T^*_{31}\pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}]\\
=&\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}E[T^*_{31}(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2},z_{s_1},z_{s_2})\pi_1(x_{s_1s_2}+x^*_{s_1s_2})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}]\\
&+\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}E[T^*_{31}(\tilde{x}_{s_1s_2/j}+\tilde{b}_{s_1,j}\epsilon_j+\bar{x}_{t_2s_2},z_{s_1},z_{s_2})\pi_1(x_{s_1s_2/j}+b_{s_1,j}\epsilon_j+x^*_{s_1s_2})\pi_2(z_{s_1})\epsilon_j|\mathcal{F}_{s_2}]\\
&+\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_jE[T^*_{31}(\tilde{x}_{s_1s_2}+\bar{x}_{t_2s_2},z_{s_1},z_{s_2})\pi_1(x_{s_1s_2}+x^*_{s_1s_2})\pi_2(z_{s_1})|\mathcal{F}_{s_2}]\\
=&\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint T^*_{31}(d_{s_1s_2}x,z,z_{s_2})\pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)up_{s_1s_2}(x,z,u)dxdzdu\\
&+\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint T^*_{31}(d_{s_1s_2}x+\tilde{b}_{s_1,j}u,z,z_{s_2})\pi_1(d_{s_1s_2}x+b_{s_1,j}u+x^*_{s_1s_2})\pi_2(z)u\\
&\times q_{s_1s_2/j}(x,z)f_\epsilon(u)dxdzdu\\
&+\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j\iint T^*_{31}(d_{s_1s_2}x,z,z_{s_2})\pi_1(d_{s_1s_2}x+x^*_{s_1s_2})\pi_2(z)q_{s_1s_2}(x,z)dxdz\\
=&d^{-1}_{s_1s_2}\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint T^*_{31}(x-x^*_{s_1s_2},z,z_{s_2})\pi_1(x)\pi_2(z)up_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)dxdzdu\\
&+d^{-1}_{s_1s_2}\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint T^*_{31}(x-x^*_{s_1s_2}+(\tilde{b}_{s_1,j}-b_{s_1,j})u,z,z_{s_2})\pi_1(x)\pi_2(z)u\\
&\times q_{s_1s_2/j}\left(\frac{x-b_{s_1,j}u-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)f_\epsilon(u)dxdzdu\\
&+d^{-1}_{s_1s_2}\sum_{j=-\infty}^{s_2}\rho_{s_1-j}\epsilon_j\iint T^*_{31}(d_{s_1s_2}x,z,z_{s_2})\pi_1(x)\pi_2(z)q_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)dxdz\\
=&d^{-1}_{s_1s_2}\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint T^*_{31}(x-x^*_{s_1s_2},z,z_{s_2})\pi_1(x)\pi_2(z)u\\
&\times\left[p_{s_1s_2}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}},z,u\right)-f_{s_1s_2(m_0)}\left(\frac{x-x^*_{s_1s_2}}{d_{s_1s_2}}\right)q(z,u)\right]dxdzdu\\
&+(1+o(1))d^{-1}_{s_1s_2}\sum_{j=s_2+1}^{s_1-m_0}\rho_{s_1-j}\iiint T^*_{31}(x-x^*_{s_1s_2},z,z_{s_2})\pi_1(x)\pi_2(z)u\\
&\times q_{s_1s_2/j}\left(\frac{x-b_{s_1,j}u-x^*_{s_1s_2}}{d_{s_1s_2}},z\right)f_\epsilon(u)dxdzdu
\end{aligned}$$

$$
+ d_{s_1s_2}^{-1} \sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j \iint T_{31}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)q_{s_1s_2}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}, z\right) dxdz
$$

$$
\begin{aligned}
=& d_{s_1s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T_{31}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times \left[p_{s_1s_2}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}, z, u\right) - f_{s_1s_2(m_0)}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}\right)\kappa_j(z,u)\right] dxdzdu \\
&+ d_{s_1s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T_{31}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times f_{s_1s_2(m_0)}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}\right)\kappa_j(z,u)dxdzdu \\
&+ (1+o(1))d_{s_1s_2}^{-1} \sum_{j=s_2+1}^{s_1-m_0} \rho_{s_1-j} \iiint T_{31}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times \left[q_{s_1s_2/j}\left(\frac{x - b_{s_1,j}u - x_{s_1s_2}^*}{d_{s_1s_2}}, z\right) - q_{s_1s_2/j}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}, z\right)\right] f_\epsilon(u)dxdzdu \\
&+ d_{s_1s_2}^{-1} \sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j \iint T_{31}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)q_{s_1s_2}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}, z\right) dxdz.
\end{aligned}
$$

Since T_{31}^* already has one super rate, all the terms except the second one have double super rates and therefore are neglected. The second term is denoted by T_{31a}^*,

$$
\begin{aligned}
T_{31a}^*(x_{s_1s_2}^*, z_{s_2}) \equiv & d_{s_1s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T_{31}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times f_{s_1s_2(m_0)}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}\right)\kappa_j(z,u)dxdzdu, \qquad \text{(D.26)}
\end{aligned}
$$

for the later calculation.

Similarly, $T_{32}^* = T_{32}^*(x_{t_2s_1}^* - x_{s_2}, z_{s_1}, z_{s_2})$ and

$$
\begin{aligned}
& E[T_{32}^*\pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}] \\
=& d_{s_1s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T_{32}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times \left[p_{s_1s_2}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}, z, u\right) - f_{s_1s_2(m_0)}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}\right)\kappa_j(z,u)\right] dxdzdu \\
&+ d_{s_1s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T_{32}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times f_{s_1s_2(m_0)}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}\right)\kappa_j(z,u)dxdzdu \\
&+ (1+o(1))d_{s_1s_2}^{-1} \sum_{j=s_2+1}^{s_1-m_0} \rho_{s_1-j} \iiint T_{32}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times \left[q_{s_1s_2/j}\left(\frac{x - b_{s_1,j}u - x_{s_1s_2}^*}{d_{s_1s_2}}, z\right) - q_{s_1s_2/j}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}, z\right)\right] f_\epsilon(u)dxdzdu \\
&+ d_{s_1s_2}^{-1} \sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j \iint T_{32}^*(x - x_{s_1s_2}^*, z, z_{s_2})\pi_1(x)\pi_2(z)q_{s_1s_2}\left(\frac{x - x_{s_1s_2}^*}{d_{s_1s_2}}, z\right) dxdz.
\end{aligned}
$$

Due to the same reason, we denote the second term by T^*_{32a},

$$T^*_{32a}(x^*_{s_1s_2}, z_{s_2}) \equiv d^{-1}_{s_1s_2} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T^*_{32}(x - x^*_{s_1s_2}, z, z_{s_2})\pi_1(x)\pi_2(z)u \\ \times f_{s_1s_2(m_0)}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z, u)dxdzdu, \tag{D.27}$$

for the later calculation. All the other terms are omitted.

Meanwhile, the same holds for $T^*_{34} = T^*_{34}(x^*_{t_2s_1} - x_{s_2}, z_{s_1}, z_{s_2})$, that is,

$$\begin{aligned}
&E[T^*_{34}\pi_1(x_{s_1})\pi_2(z_{s_1})e_{s_1}|\mathcal{F}_{s_2}] \\
=&d^{-1}_{s_1s_2} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T^*_{34}(x - x^*_{s_1s_2}, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times \left[p_{s_1s_2}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}, z, u\right) - f_{s_1s_2(m_0)}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z, u)\right] dxdzdu \\
&+ d^{-1}_{s_1s_2} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T^*_{34}(x - x^*_{s_1s_2}, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times f_{s_1s_2(m_0)}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z, u)dxdzdu \\
&+ (1 + o(1))d^{-1}_{s_1s_2} \sum_{j=s_2+1}^{s_1-m_0} \rho_{s_1-j} \iiint T^*_{34}(x - x^*_{s_1s_2}, z, z_{s_2})\pi_1(x)\pi_2(z)u \\
&\times \left[q_{s_1s_2/j}\left(\frac{x - b_{s_1,j}u - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right) - q_{s_1s_2/j}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right)\right] f_\epsilon(u)dxdzdu \\
&+ d^{-1}_{s_1s_2} \sum_{j=-\infty}^{s_2} \rho_{s_1-j}\epsilon_j \iint T^*_{34}(x - x^*_{s_1s_2}, z, z_{s_2})\pi_1(x)\pi_2(z)q_{s_1s_2}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}, z\right) dxdz,
\end{aligned}$$

and denote the second term by T^*_{34a},

$$T^*_{34a}(x^*_{s_1s_2}, z_{s_2}) \equiv d^{-1}_{s_1s_2} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T^*_{34}(x - x^*_{s_1s_2}, z, z_{s_2})\pi_1(x)\pi_2(z)u \\ \times f_{s_1s_2(m_0)}\left(\frac{x - x^*_{s_1s_2}}{d_{s_1s_2}}\right)\kappa_j(z, u)dxdzdu, \tag{D.28}$$

for the later calculation.

Step IV. The last step in (D.7). All the remaining terms are addressed in this step, which are (1) $V(z_{s_2})T^{**}_{11b}(x^*_{s_1s_2})$ [from T^{**}_{11}]; (2) $T^*_{112}(x^*_{s_1s_2}, z_{s_2})$ and $V(z_{s_2})T^*_{116}(x^*_{s_1s_2})$ [from T^*_{11}]; (3) $T^*_{12b}(x^*_{s_1s_2}, z_{s_2})$ [from T^*_{12a} that is derived by T^*_{12}]; (4) $T^*_{132}(x^*_{s_1s_2}, z_{s_2})$, $T^*_{134}(x^*_{s_1s_2}, z_{s_2})$ and $T^*_{137}(x^*_{s_1s_2}, z_{s_2})$ [from T^*_{13}]; (5) $T^*_{31a}(x^*_{s_1s_2}, z_{s_2})$, $T^*_{32a}(x^*_{s_1s_2}, z_{s_2})$ and $T^*_{34a}(x^*_{s_1s_2}, z_{s_2})$ [from T^*_{31}, T^*_{32} and T^*_{34}, respectively]. We shall deal with them term by term in the following sub–steps.

Since all terms have $(x^*_{s_1s_2}, z_{s_2})$ as their argument the calculations are almost the same. Once the first one $V(z_{s_2})T^{**}_{11b}(x^*_{s_1s_2})$ is computed, we would be able to write down the results for the rest directly.

To begin with, notice that $x^*_{s_1s_2} = x_{s_2} + \bar{x}_{s_1s_2}$, and thus $d^{-1}_{s_2}x^*_{s_1s_2}$ and $d^{-1}_{s_2}x_{s_2}$ have asymptotically the same distribution.

Sub-step (1):

$$E[V(z_{s_2})T^{**}_{11b}(x^*_{s_1s_2})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_2}]$$

$$
\begin{aligned}
=& \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} E[V(z_{s_2}) T_{11b}^{**}(x_{s_1 s_2}^{*}) \pi_1(x_{s_2}) \pi_2(z_{s_2}) \epsilon_j] \\
&+ \sum_{j=-\infty}^{s_2-m_0} \rho_{s_2-j} E[V(z_{s_2}) T_{11b}^{**}(x_{s_1 s_2/j}^{*} + \tilde{b}_{s_2,j}\epsilon_j) \pi_1(x_{s_2/j} + b_{s_2,j}\epsilon_j) \pi_2(z_{s_2}) \epsilon_j] \\
=& \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iiint V(z) T_{11b}^{**}(d_{s_2}x) \pi_1(d_{s_2}x) \pi_2(z) u p_{s_2}(x,z,u) dxdzdu \\
&+ \sum_{j=-\infty}^{s_2-m_0} \rho_{s_2-j} \iiint V(z) T_{11b}^{**}(d_{s_2}x + \tilde{b}_{s_2,j}u) \pi_1(d_{s_2}x + b_{s_2,j}u) \pi_2(z) u p_{s_2}(x,z) f_\epsilon(u) dxdzdu \\
=& d_{s_2}^{-1} \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iiint V(z) T_{11b}^{**}(x) \pi_1(x) \pi_2(z) u p_{s_2}(d_{s_2}^{-1}x, z, u) dxdzdu \\
&+ d_{s_2}^{-1} \sum_{j=-\infty}^{s_2-m_0} \rho_{s_2-j} \iiint V(z) T_{11b}^{**}(x + (\tilde{b}_{s_2,j} - b_{s_2,j})u) \pi_1(x) \pi_2(z) u \\
&\quad \times p_{s_2}\left(\frac{x - b_{s_2,j}u}{d_{s_2}}, z\right) f_\epsilon(u) dxdzdu \\
=& d_{s_2}^{-1} \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iiint V(z) T_{11b}^{**}(x) \pi_1(x) \pi_2(z) u \\
&\quad \times \left[p_{s_2}(d_{s_2}^{-1}x, z, u) - f_{s_2(m_0)}(d_{s_2}^{-1}x) \kappa_j(z,u)\right] dxdzdu \\
&+ d_{s_2}^{-1} \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \int T_{11b}^{**}(x) \pi_1(x) f_{s_2(m_0)}(d_{s_2}^{-1}x) dx \iint V(z) \pi_2(z) u \kappa_j(z,u) dzdu \\
&+ (1+o(1)) d_{s_2}^{-1} \sum_{j=-\infty}^{s_2-m_0} \rho_{s_2-j} \iiint V(z) T_{11b}^{**}(x) \pi_1(x) \pi_2(z) u \\
&\quad \times \left[p_{s_2}\left(\frac{x - b_{s_2,j}u}{d_{s_2}}, z\right) - p_{s_2}\left(\frac{x}{d_{s_2}}, z\right)\right] f_\epsilon(u) dxdzdu.
\end{aligned}
$$

Since T_1 is included in T_{11b}^{**} according to (D.13) and T_1 has a super rate, it is evident by Lemma A.6 that the first term and the third term have double super rates. We only need to address the second term. Note that

$$
\begin{aligned}
& d_{s_2}^{-1} \int T_{11b}^{**}(x) \pi_1(x) f_{s_2(m_0)}(d_{s_2}^{-1}x) dx \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iint V(z) \pi_2(z) u \kappa_j(z,u) dzdu \\
=& d_{s_2}^{-1} \int T_{11b}^{**}(x) \pi_1(x) f_{s_2(m_0)}(d_{s_2}^{-1}x) dx \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} E[V(z_{s_2}) \pi_2(z_{s_2}) \epsilon_j],
\end{aligned}
$$

and by the definition in (D.14),

$$
\begin{aligned}
V(z_{s_2}) =& \sum_{k=t_2-m_0+1}^{t_2} \rho_{t_2-k} \iint L_2(z) u \kappa_k(h_2 z + z_{s_2}, u) dudz \\
=& \int L_2(z) dz \sum_{k=t_2-m_0+1}^{t_2} \rho_{t_2-k} \int u \kappa_k(z_{s_2}, u) du \\
&+ \sum_{k=t_2-m_0+1}^{t_2} \rho_{t_2-k} \iint L_2(z) u [\kappa_k(h_2 z + z_{s_2}, u) - \kappa_k(z_{s_2}, u)] dudz \\
=& V_1(z_{s_2}) + V_2(z_{s_2}), \quad \text{say.}
\end{aligned}
$$

Observe that, for $s_2 - m_0 + 1 \le j \le s_2$,

$$\begin{aligned}
E[V_1(z_{s_2})\pi_2(z_{s_2})\epsilon_j] = & \iint V_1(z)\pi_2(z)w\kappa_j(z,w)dwdz \\
=C & \sum_{k=t_2-m_0+1}^{t_2} \rho_{t_2-k} \iiint u\kappa_k(z,u)du\pi_2(z)w\kappa_j(z,w)dwdz \\
=C & \sum_{k=t_2-m_0+1}^{t_2} \rho_{t_2-k} \int \pi_2(z)p^2(z)E[\epsilon_k|z_{s_2}=z]E[\epsilon_j|z_{s_2}=z]dz = 0
\end{aligned}$$

since $E[\epsilon_k|z_{s_2}=z] = E[\epsilon_k] = 0$ by the independence of ϵ_k and z_{s_2}, where $C = \int L_2(z)dz$. On the other hand, it follows from the differentiability of $\kappa_k(z,u)$ that

$$\begin{aligned}
|V_2(z_{s_2})| \le & \sum_{k=t_2-m_0+1}^{t_2} |\rho_{t_2-k}| \iint L_2(z)|u|h_2\|z\| \left\| \frac{\partial}{\partial z}\kappa_k(z^*,u) \right\| dudz \\
\le h_2 \int \|z\|L_2(z)dz & \sum_{k=t_2-m_0+1}^{t_2} |\rho_{t_2-k}| \int |u| \sup_z \left\| \frac{\partial}{\partial z}\kappa_k(z,u) \right\| du = Ch_2, \quad a.s.,
\end{aligned}$$

where $\int \|z\|L_2(z)dz = \iint \|z\|K_2(v+z)K_2(v)dvdz \le \iint (\|z\|+\|v\|)K_2(v)K_2(v)dvdz < \infty$ by Assumption 2.3. These imply that the second term in absolute value is bounded by

$$\begin{aligned}
& Cd_{s_2}^{-1} \int |T_{11b}^{**}(x)|\pi_1(x)dx \sum_{j=s_2-m_0+1}^{s_2} |\rho_{s_2-j}||E[V(z_{s_2})\pi_2(z_{s_2})\epsilon_j]| \\
=& Cd_{s_2}^{-1} \int |T_{11b}^{**}(x)|\pi_1(x)dx \sum_{j=s_2-m_0+1}^{s_2} |\rho_{s_2-j}||E[V_2(z_{s_2})\pi_2(z_{s_2})\epsilon_j]| \\
\le& Ch_1^2 h_2^{2d+1} d_{t_1t_2}^{-2} d_{t_2s_1}^{-1} d_{s_1s_2}^{-1} d_{s_2}^{-1},
\end{aligned}$$

by recalling the notation of $T_{11b}^{**}(\cdot)$, for any $\xi \in \mathbb{R}$,

$$\begin{aligned}
T_{11b}^{**}(\xi) =& d_{s_1s_2}^{-1} \iint U_2(x-\xi,z)\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)dx \\
& \times \left(\pi_2(z) \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \int u\kappa_j(z,u)du\right)dz,
\end{aligned}$$

giving that

$$\begin{aligned}
|T_{11b}^{**}(\xi)| \le& Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-1} d_{s_1s_2}^{-1} \int \left| \pi_2(z) \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \int u\kappa_j(z,u)du \right| dz \\
=& Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-1} d_{s_1s_2}^{-1} \int \pi_2(z) \left| \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \int u\kappa_j(z,u)du \right| dz \\
\le& Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-1} d_{s_1s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} |\rho_{s_1-j}| \iint |u|\pi_2(z)\kappa_j(z,u)dudz \\
=& Ch_1^2 h_2^{2d} d_{t_1t_2}^{-2} d_{t_2s_1}^{-1} d_{s_1s_2}^{-1},
\end{aligned}$$

consulting the bound of U_2 in (D.13).

In conclusion, for the calculation of $E[V(z_{s_2})T_{11b}^{**}(x_{s_1s_2}^*)\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_2}]$, both the first and the third terms have double super rates, while the second term has one super rate and one extra h_2.

Sub-step (2):

$$
\begin{aligned}
&E[T^*_{112}(x^*_{s_1s_2}, z_{s_2})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_2}]\\
=&d_{s_2}^{-1}\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}\iiint T^*_{112}(x,z)\pi_1(x)\pi_2(z)u\Big[p_{s_2}(d_{s_2}^{-1}x,z,u)-f_{s_2(m_0)}(d_{s_2}^{-1}x)\kappa_j(z,u)\Big]dxdzdu\\
&+d_{s_2}^{-1}\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}\iiint T^*_{112}(x,z)\pi_1(x)f_{s_2(m_0)}(d_{s_2}^{-1}x)\pi_2(z)u\kappa_j(z,u)dxdzdu\\
&+(1+o(1))d_{s_2}^{-1}\sum_{j=-\infty}^{s_2-m_0}\rho_{s_2-j}\iiint T^*_{112}(x,z)\pi_1(x)\pi_2(z)u\\
&\quad\times\left[p_{s_2}\left(\frac{x-b_{s_2,j}u}{d_{s_2}},z\right)-p_{s_2}\left(\frac{x}{d_{s_2}},z\right)\right]f_\epsilon(u)dxdzdu.
\end{aligned}
$$

As before, we only need to deal with the second term, since the other two clearly have double super rates. From (D.20), it can be seen

$$
\begin{aligned}
T^*_{112}(\xi,\eta)\equiv&d_{s_1s_2}^{-1}\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint V(z)T^*_{11a}(x-\xi,\eta)\pi_1(x)\pi_2(z)u\\
&\times f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)\kappa_j(z,u)dxdzdu\\
=&d_{s_1s_2}^{-1}\int T^*_{11a}(x-\xi,\eta)\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)dx\\
&\times\int V(z)\pi_2(z)\left(\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\int u\kappa_j(z,u)du\right)dz\\
=&d_{s_1s_2}^{-1}\int T^*_{11a}(x-\xi,\eta)\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)dx\\
&\times\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\int\int u\kappa_j(z,u)V(z)\pi_2(z)dudz\\
=&d_{s_1s_2}^{-1}\int T^*_{11a}(x-\xi,\eta)\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)dx\\
&\times\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}E[V(z_{s_1})\pi_2(z_{s_1})\epsilon_j],
\end{aligned}
$$

where the definition of $V(z_{s_1})$ is given by (D.10), and similarly to (1), $V(z_{s_1})=V_1(z_{s_1})+V_2(z_{s_1})$. A similar derivation will give the same conclusion as for $E[T^*_{112}(x^*_{s_1s_2}, z_{s_2})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_2}]$.

In addition,

$$
\begin{aligned}
&E[V(z_{s_2})T^*_{116}(x^*_{s_1s_2})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_2}]\\
=&d_{s_2}^{-1}\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}\iiint V(z)T^*_{116}(x)\pi_1(x)\pi_2(z)u\\
&\times\Big[p_{s_2}(d_{s_2}^{-1}x,z,u)-f_{s_2(m_0)}(d_{s_2}^{-1}x)\kappa_j(z,u)\Big]dxdzdu\\
&+d_{s_2}^{-1}\int T^*_{116}(x)\pi_1(x)f_{s_2(m_0)}(d_{s_2}^{-1}x)dx\iint V(z)\pi_2(z)u\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}\kappa_j(z,u)dzdu\\
&+(1+o(1))d_{s_2}^{-1}\sum_{j=-\infty}^{s_2-m_0}\rho_{s_2-j}\iiint V(z)T^*_{116}(x)\pi_1(x)\pi_2(z)u
\end{aligned}
$$

$$\times \left[p_{s_2}\left(\frac{x-b_{s_2,j}u}{d_{s_2}}, z\right) - p_{s_2}\left(\frac{x}{d_{s_2}}, z\right)\right] f_\epsilon(u) dx dz du.$$

Here, we only need to deal with the second term, which is

$$\begin{aligned}
& d_{s_2}^{-1} \int T_{116}^*(x)\pi_1(x) f_{s_2(m_0)}(d_{s_2}^{-1}x) dx \iint V(z)\pi_2(z) u \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j}\kappa_j(z,u) dz du \\
=& d_{s_2}^{-1} \int T_{116}^*(x)\pi_1(x) f_{s_2(m_0)}(d_{s_2}^{-1}x) dx \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} E[V(z_{s_2})\pi_2(z_{s_2})\epsilon_j].
\end{aligned}$$

Notice by (D.21) that,

$$\begin{aligned}
T_{116}^*(\xi) \equiv& d_{s_1 s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z) T_{11b}^*(x-\xi)\pi_1(x)\pi_2(z) u \\
& \times f_{s_1 s_2(m_0)}\left(\frac{x-\xi}{d_{s_1 s_2}}\right) \kappa_j(z,u) dx dz du \\
=& d_{s_1 s_2}^{-1} \int T_{11b}^*(x-\xi)\pi_1(x) f_{s_1 s_2(m_0)}\left(\frac{x-\xi}{d_{s_1 s_2}}\right) dx \\
& \times \int V(z)\pi_2(z) \left(\sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \int u\kappa_j(z,u) du\right) dz \\
=& d_{s_1 s_2}^{-1} \int T_{11b}^*(x-\xi)\pi_1(x) f_{s_1 s_2(m_0)}\left(\frac{x-\xi}{d_{s_1 s_2}}\right) dx \\
& \times \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} E[V(z_{s_1})\pi_2(z_{s_1})\epsilon_j].
\end{aligned}$$

Hence, this term has two extra h_2. This is because in the function form $V(z_{s_2})T_{116}^*(x_{s_1 s_2}^*)$, the variables are separable.

Sub-step (3): Observe that

$$\begin{aligned}
& E[T_{12b}^*(x_{s_1 s_2}^*, z_{s_2})\pi_1(x_{s_2})\pi_2(z_{s_2}) e_{s_2}] \\
=& d_{s_2}^{-1} \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iiint T_{12b}^*(x,z)\pi_1(x)\pi_2(z) u \\
& \times \left[p_{s_2}(d_{s_2}^{-1}x, z, u) - f_{s_2(m_0)}(d_{s_2}^{-1}x)\kappa_j(z,u)\right] dx dz du \\
& + d_{s_2}^{-1} \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iiint T_{12b}^*(x,z)\pi_1(x) f_{s_2(m_0)}(d_{s_2}^{-1}x)\pi_2(z) u\kappa_j(z,u) dx dz du \\
& + (1+o(1)) d_{s_2}^{-1} \sum_{j=-\infty}^{s_2-m_0} \rho_{s_2-j} \iiint T_{12b}^*(x,z)\pi_1(x)\pi_2(z) u \\
& \times \left[p_{s_2}\left(\frac{x-b_{s_2,j}u}{d_{s_2}}, z\right) - p_{s_2}\left(\frac{x}{d_{s_2}}, z\right)\right] f_\epsilon(u) dx dz du.
\end{aligned}$$

Only the second term is necessarily to be addressed. Note by (D.22) that

$$\begin{aligned}
T_{12b}^*(\xi,\eta) \equiv& d_{s_1 s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint V(z) T_{12a}^*(x-\xi,\eta)\pi_1(x)\pi_2(z) u \\
& \times f_{s_1 s_2(m_0)}\left(\frac{x-\xi}{d_{s_1 s_2}}\right) \kappa_j(z,u) dx dz du
\end{aligned}$$

$$
\begin{aligned}
=&d_{s_1s_2}^{-1}\int T_{12a}^*(x-\xi,\eta)\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)dx\\
&\times\int V(z)\pi_2(z)\left(\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\int u\kappa_j(z,u)du\right)dz\\
=&d_{s_1s_2}^{-1}\int T_{12a}^*(x-\xi,\eta)\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)dx\\
&\times\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}E[V(z_{s_1})\pi_2(z_{s_1})\epsilon_j].
\end{aligned}
$$

Hence, this term is the same as the one in (2).

Sub-step (4): For the sake of exposition, $T_{132}^*(x_{s_1s_2}^*,z_{s_2})$ and $T_{137}^*(x_{s_1s_2}^*,z_{s_2})$ are considered jointly.

$$
\begin{aligned}
&E\{[T_{132}^*(x_{s_1s_2}^*,z_{s_2})+T_{132}^*(x_{s_1s_2}^*,z_{s_2})]\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_2}\}\\
=&d_{s_2}^{-1}\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}\iiint[T_{132}^*(x,z)+T_{137}^*(x,z)]\pi_1(x)\pi_2(z)u\\
&\times\left[p_{s_2}(d_{s_2}^{-1}x,z,u)-f_{s_2(m_0)}(d_{s_2}^{-1}x)\kappa_j(z,u)\right]dxdzdu\\
&+d_{s_2}^{-1}\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}\iiint[T_{132}^*(x,z)+T_{137}^*(x,z)]\\
&\times\pi_1(x)f_{s_2(m_0)}(d_{s_2}^{-1}x)\pi_2(z)u\kappa_j(z,u)dxdzdu\\
&+(1+o(1))d_{s_2}^{-1}\sum_{j=-\infty}^{s_2-m_0}\rho_{s_2-j}\iiint[T_{132}^*(x,z)+T_{137}^*(x,z)]\pi_1(x)\pi_2(z)u\\
&\times\left[p_{s_2}\left(\frac{x-b_{s_2,j}u}{d_{s_2}},z\right)-p_{s_2}\left(\frac{x}{d_{s_2}},z\right)\right]f_\epsilon(u)dxdzdu.
\end{aligned}
\tag{D.29}
$$

Before addressing the second term as usual, let us have a look at the sum $T_{132}^*(\xi,\eta)+T_{137}^*(\xi,\eta)$ from (D.23) and (D.25),

$$
\begin{aligned}
&T_{132}^*(\xi,\eta)+T_{137}^*(\xi,\eta)\\
=&d_{s_1s_2}^{-1}\sum_{j=s_1-m_0+1}^{s_1}\rho_{t_2-j}\rho_{s_1-j}\iiint V(z)T_{13a}^*(x-\xi,\eta)\pi_1(x)\pi_2(z)u^2\\
&\times f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)\kappa_j(u,z)dxdzdu\\
&+d_{s_1s_2}^{-1}\sum_{j_1=s_1-m_0+2}^{s_1}\sum_{j_2=s_1-m_0+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\int\cdots\int V(z)T_{13a}^*(x-\xi,\eta)\\
&\times\pi_1(x)\pi_2(z)uvf_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)\kappa_{j_1j_2}(z,u,v)dxdzdudv\\
=&d_{s_1s_2}^{-1}\iiint T_{13a}^*(x-\xi,\eta)\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)dx\\
&\times\Big[\sum_{j=s_1-m_0+1}^{s_1}\rho_{t_2-j}\rho_{s_1-j}\iint V(z)\pi_2(z)u^2\kappa_j(u,z)dzdu\\
&\quad+\sum_{j_1=s_1-m_0+2}^{s_1}\sum_{j_2=s_1-m_0+1}^{j_1-1}\rho_{t_2-j_1}\rho_{s_1-j_2}\iiint V(z)\pi_2(z)uv\kappa_{j_1j_2}(z,u,v)dzdudv\Big]
\end{aligned}
$$

$$
\begin{aligned}
=& d_{s_1 s_2}^{-1} \iiint T_{13a}^*(x-\xi,\eta)\pi_1(x) f_{s_1 s_2 (m_0)}\left(\frac{x-\xi}{d_{s_1 s_2}}\right) dx \\
& \times E[V(z_{s_1})\pi_2(z_{s_1})\alpha_{t_2}(s_1, s_1-m_0+1)\alpha_{s_1}(s_1, s_1-m_0+1)].
\end{aligned}
$$

where $\alpha_t(a,b)$ with integers $a \geq b$ stands for the period of $e_t = \sum_{j=-\infty}^{t} \rho_{t-j}\epsilon_j$ including all ϵ_j where $b \leq j \leq a$, i.e. $\alpha_t(a,b) = \rho_{t-a}\epsilon_a + \cdots + \rho_{t-b}\epsilon_b$.

Because of Lemma A.6, in (D.15) we can write

$$
\begin{aligned}
& T_{13a}^*(\tau_1,\tau_2) \\
=& \int U_1(h_1 x - \tau_1) L_1(x) f_{t_2 s_1 (m_0)}\left(\frac{h_1 x - \tau_1}{d_{t_2 s_1}}\right) dx \int L_2(z) p(h_2 z + \tau_2) dz \\
=& \int U_1(h_1 x - \tau_1) L_1(x) f_{t_2 s_1 (m_0)}\left(\frac{h_1 x - \tau_1}{d_{t_2 s_1}}\right) dx \int L_2(z) dz \; p(\tau_2) \\
& + \int U_1(h_1 x - \tau_1) L_1(x) f_{t_2 s_1 (m_0)}\left(\frac{h_1 x - \tau_1}{d_{t_2 s_1}}\right) dx \int L_2(z)[p(h_2 z + \tau_2) - p(\tau_2)] dz \\
=& A_1(\tau_1,\tau_2) + A_2(\tau_1,\tau_2), \quad \text{say,}
\end{aligned}
$$

and these two terms play a similar role to $V_1(\cdot)$ and $V_2(\cdot)$ in (1), respectively. Accordingly, the term (D.29) can be written as a sum of two quantities in terms of $A_1(\tau_1,\tau_2)$ and $A_2(\tau_1,\tau_2)$, so to speak, $\widetilde{A_1}$ and $\widetilde{A_2}$. We give the derivation for the first one only, given that the second term will be bounded by $O(h_2)$ by the mean value theorem for $p(\cdot)$. Notice that in $A_1(\tau_1,\tau_2)$, τ_1 and τ_2 are separated in different functions, which implies that in the integral in $\widetilde{A_1}$, x and z are separable. Thus,

$$
\begin{aligned}
\widetilde{A_1} =& C d_{s_2}^{-1} \int \pi_1(x) \int U_1(h_1\zeta - x) L_1(\zeta) f_{t_2 s_1 (m_0)}\left(\frac{h_1 \zeta - x}{d_{t_2 s_1}}\right) d\zeta dx \\
& \times E[V(z_{s_1})\pi_2(z_{s_1})\alpha_{t_2}(s_1, s_1-m_0+1)\alpha_{s_1}(s_1, s_1-m_0+1)] \\
& \times \int p(z)\pi_2(z) \int \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} u \kappa_j(z,u) du dz \\
=& C d_{s_2}^{-1} \int \pi_1(x) \int U_1(h_1\zeta - x) L_1(\zeta) f_{t_2 s_1 (m_0)}\left(\frac{h_1 \zeta - x}{d_{t_2 s_1}}\right) d\zeta dx \\
& \times E[V(z_{s_1})\pi_2(z_{s_1})\alpha_{t_2}(s_1, s_1-m_0+1)\alpha_{s_1}(s_1, s_1-m_0+1)] \\
& \times \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} E[p(z_{s_2})\pi_2(z_{s_2})\epsilon_j] = 0,
\end{aligned}
$$

by Assumption 2.1.

Notice that there is a super rate included in $|E[V(z_{s_1})\pi_2(z_{s_1})\alpha_{t_2}(s_1, s_1-m_0+1)\alpha_{s_1}(s_1, s_1-m_0+1)]|$ implicitly, since $t_2 - s_1$ ia sufficiently large. This implies that the other two terms have double super rates.

Now, we turn to $T_{134}^*(x_{s_1 s_2}^*, z_{s_2})$.

$$
\begin{aligned}
& E[T_{134}^*(x_{s_1 s_2}^*, z_{s_2})\pi_1(x_{s_2})\pi_2(z_{s_2}) e_{s_2}] \\
=& d_{s_2}^{-1} \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iiint T_{134}^*(x,z)\pi_1(x)\pi_2(z) u \Big[p_{s_2}(d_{s_2}^{-1}x, z, u) - f_{s_2(m_0)}(d_{s_2}^{-1}x)\kappa_j(z,u)\Big] dx dz du \\
& + d_{s_2}^{-1} \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iiint T_{134}^*(x,z)\pi_1(x) f_{s_2(m_0)}(d_{s_2}^{-1}x)\pi_2(z) u \kappa_j(z,u) dx dz du
\end{aligned}
$$

$$+ (1+o(1))d_{s_2}^{-1} \sum_{j=-\infty}^{s_2-m_0} \rho_{s_2-j} \iiint T_{134}^*(x,z)\pi_1(x)\pi_2(z)u$$

$$\times \left[p_{s_2}\left(\frac{x-b_{s_2,j}u}{d_{s_2}}, z\right) - p_{s_2}\left(\frac{x}{d_{s_2}}, z\right) \right] f_\epsilon(u)dxdzdu.$$

Similarly, by virtue of (D.24) and (D.15), we have

$$T_{134}^*(\xi,\eta) = d_{s_1s_2}^{-1} \sum_{j=s_2+1}^{s_1-m_0} \rho_{t_2-j}\rho_{s_1-j} \iiint V(z)T_{13a}^*(x-\xi,\eta)\pi_1(x)\pi_2(z)u^2$$
$$\times f_{s_1s_2(m_0)/j}\left(\frac{x-\xi}{d_{s_1s_2}}\right) p(z)f_\epsilon(u)dxdzdu$$

and similar to the above we may derive an $O(h_2)$ from $T_{13a}^*(x-\xi,\eta)$.

Sub-step (5): For concise expression, two terms, $T_{31a}^*(x_{s_1s_2}^*, z_{s_2})$ and $T_{34a}^*(x_{s_1s_2}^*, z_{s_2})$, are treated together. Observe that

$$E\{[T_{31a}^*(x_{s_1s_2}^*, z_{s_2}) + T_{34a}^*(x_{s_1s_2}^*, z_{s_2})]\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_2}\} \tag{D.30}$$
$$= d_{s_2}^{-1} \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iiint [T_{31a}^*(x,z) + T_{34a}^*(x,z)]\pi_1(x)\pi_2(z)u$$
$$\times \left[p_{s_2}(d_{s_2}^{-1}x, z, u) - f_{s_2(m_0)}(d_{s_2}^{-1}x)\kappa_j(z,u) \right] dxdzdu$$
$$+ d_{s_2}^{-1} \sum_{j=s_2-m_0+1}^{s_2} \rho_{s_2-j} \iiint [T_{31a}^*(x,z) + T_{34a}^*(x,z)]\pi_1(x)f_{s_2(m_0)}(d_{s_2}^{-1}x)\pi_2(z)u\kappa_j(z,u)dxdzdu$$
$$+ (1+o(1))d_{s_2}^{-1} \sum_{j=-\infty}^{s_2-m_0} \rho_{s_2-j} \iiint [T_{31a}^*(x,z) + T_{34a}^*(x,z)]\pi_1(x)\pi_2(z)u$$
$$\times \left[p_{s_2}\left(\frac{x-b_{s_2,j}u}{d_{s_2}}, z\right) - p_{s_2}\left(\frac{x}{d_{s_2}}, z\right) \right] f_\epsilon(u)dxdzdu.$$

Before the second term is dealt with, we have a look at $T_{31a}^*(x,z)+T_{34a}^*(x,z)$ by (D.26) and (D.28),

$$T_{31a}^*(\xi,\eta) + T_{34a}^*(\xi,\eta)$$
$$= d_{s_1s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T_{31}^*(x-\xi,z,\eta)\pi_1(x)\pi_2(z)u$$
$$\times f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)\kappa_j(z,u)dxdzdu$$
$$+ d_{s_1s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint T_{34}^*(x-\xi,z,\eta)\pi_1(x)\pi_2(z)u$$
$$\times f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)\kappa_j(z,u)dxdzdu$$
$$= d_{s_1s_2}^{-1} \sum_{j=s_1-m_0+1}^{s_1} \rho_{s_1-j} \iiint [T_{31}^*(x-\xi,z,\eta) + T_{34}^*(x-\xi,z,\eta)]$$
$$\times \pi_1(x)\pi_2(z)uf_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)\kappa_j(z,u)dxdzdu$$
$$= d_{s_1s_2}^{-1} \iint [T_{31}^*(x-\xi,z,\eta) + T_{34}^*(x-\xi,z,\eta)]$$

$$\times \pi_1(x)\pi_2(z)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)\left(\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\int u\kappa_j(z,u)du\right)dxdz,$$

and

$$\begin{aligned}
&T_{31}^*(\tau_1,\tau_2,\tau_3)+T_{34}^*(\tau_1,\tau_2,\tau_3)\\
=&h_1h_2^d d_{t_2s_1}^{-1}\int T_3'(h_1x-\tau_1,\tau_2)L_1(x)f_{t_2s_1(m_0)}\left(\frac{h_1x-\tau_1}{d_{t_2s_1}}\right)dx\\
&\times\iint\sum_{j=t_2-m_0+1}^{t_2}\rho_{t_1-j}\rho_{t_2-j}L_2(z)u^2\kappa_j(h_2z+\tau_3,u)dzdu\\
&+2h_1h_2^d d_{t_2s_1}^{-1}\int T_3'(h_1x-\tau_1,\tau_2)L_1(x)f_{t_2s_1(m_0)}\left(\frac{h_1x-\tau_1}{d_{t_2s_1}}\right)dx\\
&\times\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\iiint L_2(z)uv\ \kappa_{j_1j_2}(h_2z+\tau_3,u,v)dzdudv\\
=&h_1h_2^d d_{t_2s_1}^{-1}\int T_3'(h_1x-\tau_1,\tau_2)L_1(x)f_{t_2s_1(m_0)}\left(\frac{h_1x-\tau_1}{d_{t_2s_1}}\right)dx\int L_2(z)dz\\
&\times\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\iint uv\ \kappa_{j_1j_2}(\tau_3,u,v)dudv\\
&+h_1h_2^d d_{t_2s_1}^{-1}\int T_3'(h_1x-\tau_1,\tau_2)L_1(x)f_{t_2s_1(m_0)}\left(\frac{h_1x-\tau_1}{d_{t_2s_1}}\right)dx\int L_2(z)dz\\
&\times\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\iint uv[\kappa_{j_1j_2}(h_2z+\tau_3,u,v)-\kappa_{j_1j_2}(\tau_3,u,v)]dudv,
\end{aligned}$$

where the last term has an extra $O(h_2)$, except for all other rates the same as the first term. We then concentrate on the first one.

Noting from (D.8), because of Lemma A.6,

$$\begin{aligned}
&T_3'(\zeta_1,\zeta_2)\\
\equiv&\iint L_1(x)L_2(z)q_{t_1t_2}\left(\frac{h_1x-\zeta_1}{d_{t_1t_2}},h_2z+\zeta_2\right)dxdz\\
=&\iint L_1(x)L_2(z)f_{t_1t_2(m_0)}\left(\frac{h_1x-\zeta_1}{d_{t_1t_2}}\right)p(h_2z+\zeta_2)dxdz(1+O(d_{t_1t_2}^{-1}))\\
=&\int L_1(x)f_{t_1t_2(m_0)}\left(\frac{h_1x-\zeta_1}{d_{t_1t_2}}\right)dx\int L_2(z)dzp(\zeta_2)(1+O(h_2)+O(d_{t_1t_2}^{-1})),
\end{aligned}$$

that is, $T_3'(\zeta_1,\zeta_2)$ has three terms by two approximations, derived by Lemma A.6 for the first approximation and similarly to the first part of Sub-step (4) for the second approximation. Accordingly, $T_{31a}^*(\xi,\eta)+T_{34a}^*(\xi,\eta)$ can be written as a sum of three quantities. Given the order for the last two, we only consider the first term, which implies that the corresponding term in $T_{31a}^*(\xi,\eta)+T_{34a}^*(\xi,\eta)$ is

$$\begin{aligned}
&Cd_{s_1s_2}^{-1}\int\pi_1(x)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)dx\\
&\times\int L_2(z)dz\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\iint uv\ \kappa_{j_1j_2}(\eta,u,v)dudv\\
&\times\int p(z)\pi_2(z)\left(\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\int u\kappa_j(z,u)du\right)dz
\end{aligned}$$

$$
\begin{aligned}
=&Cd_{s_1s_2}^{-1}\sum_{j_1=t_2-m_0+2}^{t_2}\sum_{j_2=t_2-m_0+1}^{j_1-1}\rho_{t_1-j_1}\rho_{t_2-j_2}\iint uv\ \kappa_{j_1j_2}(\eta,u,v)dudv\\
&\times\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}E[p(z_{s_1})\pi_2(z_{s_1})\epsilon_j]=0.
\end{aligned}
$$

Given that there is a super rate from the coefficients $\rho_{t_1-j_1}$, this term has the same order as the others, i.e. one super rate and one extra $O(h_2)$.

Additionally, for $T^*_{32a}(x^*_{s_1s_2},z_{s_2})$, observe that

$$
\begin{aligned}
&E[T^*_{32a}(x^*_{s_1s_2},z_{s_2})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_2}]\\
=&d_{s_2}^{-1}\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}\iiint T^*_{32a}(x,z)\pi_1(x)\pi_2(z)u\\
&\times\left[p_{s_2}(d_{s_2}^{-1}x,z,u)-f_{s_2(m_0)}(d_{s_2}^{-1}x)\kappa_j(z,u)\right]dxdzdu\\
&+d_{s_2}^{-1}\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}\iiint T^*_{32a}(x,z)\pi_1(x)f_{s_2(m_0)}(d_{s_2}^{-1}x)\pi_2(z)u\kappa_j(z,u)dxdzdu\\
&+(1+o(1))d_{s_2}^{-1}\sum_{j=-\infty}^{s_2-m_0}\rho_{s_2-j}\iiint T^*_{32a}(x,z)\pi_1(x)\pi_2(z)u\\
&\times\left[p_{s_2}\left(\frac{x-b_{s_2,j}u}{d_{s_2}},z\right)-p_{s_2}\left(\frac{x}{d_{s_2}},z\right)\right]f_\epsilon(u)dxdzdu.
\end{aligned}
$$

As usual we only need to deal with the second term. For any (ξ,η), it follows from (D.27) that

$$
\begin{aligned}
T^*_{32a}(\xi,\eta)=&d_{s_1s_2}^{-1}\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\iiint T^*_{32}(x-\xi,z,\eta)\pi_1(x)\pi_2(z)u\\
&\times f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)\kappa_j(z,u)dxdzdu\\
=&d_{s_1s_2}^{-1}\iint T^*_{32}(x-\xi,z,\eta)\pi_1(x)\pi_2(z)f_{s_1s_2(m_0)}\left(\frac{x-\xi}{d_{s_1s_2}}\right)\\
&\times\left(\sum_{j=s_1-m_0+1}^{s_1}\rho_{s_1-j}\int u\kappa_j(z,u)du\right)dxdz.
\end{aligned}
$$

Notice also that for any (τ_1,τ_2,τ_3),

$$
\begin{aligned}
&T^*_{32}(\tau_1,\tau_2,\tau_3)\\
=&h_1h_2^dd_{t_2s_1}^{-1}\int T_3'(h_1x-\tau_1,\tau_2)L_1(x)f_{t_2s_1(m_0)}\left(\frac{h_1x-\tau_1}{d_{t_2s_1}}\right)dx\\
&\times\int L_2(z)p(h_2z+\tau_3)dz\sum_{j=t_2-m_0+1}^{t_2}\rho_{t_1-j}\rho_{t_2-j}\int u^2f_\epsilon(u)du\\
=&h_1h_2^dd_{t_2s_1}^{-1}\int T_3'(h_1x-\tau_1,\tau_2)L_1(x)f_{t_2s_1(m_0)}\left(\frac{h_1x-\tau_1}{d_{t_2s_1}}\right)dx\int L_2(z)dz\\
&\times p(\tau_3)\sum_{j=t_2-m_0+1}^{t_2}\rho_{t_1-j}\rho_{t_2-j}\int u^2f_\epsilon(u)du(1+O(h_2)),
\end{aligned}
$$

derived similarly for $p(h_2z+\tau_3)=p(\tau_3)+[p(h_2z+\tau_3)-p(\tau_3)]$ as before. Accordingly, $T^*_{32a}(\xi,\eta)$ can be rephrased as a sum of two quantities. In view of the order for the second one, we only consider the first here, which is equal to

$$
\begin{aligned}
&Ch_1h_2^d d_{t_2s_1}^{-1}d_{s_2}^{-1}\int L_1(x)dx\int L_2(z)dz\sum_{j=t_2-m_0+1}^{t_2}\rho_{t_1-j}\rho_{t_2-j}\int u^2f_\epsilon(u)du\\
&\times\int p(z)\pi_2(z)\left(\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}u\kappa_j(z,u)du\right)dz\\
=&Ch_1h_2^d d_{t_2s_1}^{-1}d_{s_2}^{-1}\int L_1(x)dx\int L_2(z)dz\sum_{j=t_2-m_0+1}^{t_2}\rho_{t_1-j}\rho_{t_2-j}\int u^2f_\epsilon(u)du\\
&\times\sum_{j=s_2-m_0+1}^{s_2}\rho_{s_2-j}E[p(z_{s_2})\pi_2(z_{s_2})\epsilon_j]=0,
\end{aligned}
$$

because of the same reason.

Therefore, under the assumptions all terms in $E[B_{n15}]$ either have double super rates or have one super rate plus an extra h_2, which, along with (D.16)-(D.19), imply that

$$
\begin{aligned}
&|E[B_{n15}]|\le 2\frac{d_n^2}{n^2}\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_1-1}\sum_{s_2=1}^{s_1-1}|E\pi_1(x_{s_1})\pi_2(z_{s_1})L_1\left(\frac{x_{t_1}-x_{s_1}}{h_1}\right)L_2\left(\frac{z_{t_1}-z_{s_1}}{h_2}\right)\\
&\times\pi_1(x_{s_2})\pi_2(z_{s_2})L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{s_1}e_{t_1}e_{s_2}e_{t_2}|\\
=&2\frac{d_n^2}{n^2}\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_1-1}\sum_{s_2=1}^{s_1-1}|E(T_1+T_2+T_3)\pi_1(x_{s_1})\pi_2(z_{s_1})\\
&\times\pi_1(x_{s_2})\pi_2(z_{s_2})L_1\left(\frac{x_{t_2}-x_{s_2}}{h_1}\right)L_2\left(\frac{z_{t_2}-z_{s_2}}{h_2}\right)e_{s_1}e_{s_2}e_{t_2}|\\
\le&O(1)\frac{1}{n}\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_1-1}\sum_{s_2=1}^{s_1-1}|E[(T_{11}+T_{12}+T_{13})\pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}]|\\
&+O(1)\frac{1}{n}\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_1-1}\sum_{s_2=1}^{s_1-1}|E[(T_{21}+T_{22}+T_{23})\pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{s_1}e_{s_2}]|\\
&+O(1)\frac{1}{n}\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_1-1}\sum_{s_2=1}^{s_1-1}|E[\sum_{k=1}^{9}T_{3k}\ \pi_1(x_{s_1})\pi_2(z_{s_1})\pi_1(x_{s_2})\pi_2(z_{s_2})e_{t_2}e_{s_1}e_{s_2}\sum_{j=-\infty}^{t_2}\rho_{t_1-j}\epsilon_j]|\\
\le&O(1)h_1^2h_2^{2d}\frac{1}{n}\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_1-1}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-1}\rho_{t_1-t_2}d_{t_2s_1}^{-1}d_{s_1s_2}^{-2}d_{s_2}^{-1}\\
&+O(1)h_1^2h_2^{2d+1}\frac{1}{n}\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_1-1}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-1}d_{t_2s_1}^{-1}d_{s_1s_2}^{-2}d_{s_2}^{-1}\\
=&O(1)h_1^2h_2^{2d}\frac{1}{n}n\ln(n)+O(1)h_1^2h_2^{2d+1}\frac{1}{n}n^{3/2}\\
=&O(1)h_1^2h_2^{2d}\ln(n)+O(1)h_1^2h_2^{2d+1}\sqrt{n}=o(1),
\end{aligned}
$$

due to Assumption 2.2, where in the last steps we only calculate the leading term. This finishes the proof of $B_{n1}=o_P(1)$.

Now we move on to the proof of $B_{ni}=o_P(1)$, $i=2,3,4$. Notice that their difference from B_{n1} is only the function form involved. In particular, $B_i(t,s)$, $i=2,3,4$, in (B.4) are still functions of $(x_t-x_s)/h_1$, $(z_t-z_s)/h_2$, x_s and z_s all or partially involved in integrals. However, this does not affect the calculation of expectation at all, in view of the integrability of K_1 and K_2 as well as Assumption 2.3. Thus, we can conclude that $B_{ni} = o_P(1)$, $i = 2, 3, 4$, follow in the same fashion as the proof of $B_{n1} = o_P(1)$. The proof of Theorem 2.1 is finally completed. □

Proof of Corollary 2.1. As mentioned in the paper on page 8, d_n can be substituted by $\widehat{d}_n = \widehat{\phi}\sqrt{n}$ and $\widehat{d}_n/d_n \to_P 1$ where $\widehat{\phi}$, the estimate of ϕ, is estimated in the same way under some mild conditions given in Assumption 2.4 for the linear process, as what has been done in Lemma 3.1 of Dong and Gao (2014) where the similar quantity is estimated for a fractional integrated process. For the sake of simplification, we omit to repeat the procedure. Hence, in order to prove Corollary 2.1, in view of Theorem 2.1, it suffices to show that $\frac{1}{n}\sum_{t=1}^n \left(y_t - g(x_t, z_t; \widehat{\theta})\right)^2 \pi_2(z_t) \to_P E\left[e_1^2\pi_2(z_1)\right]$. Note that, under H_0, $\widehat{e}_t = y_t - g(x_t, z_t; \widehat{\theta}) = e_t + \widehat{r}_n(x_t, z_t; \theta_0)$, where $\widehat{r}_n(x_t, z_t; \theta_0) \equiv g(x_t, z_t; \theta_0) - g(x_t, z_t; \widehat{\theta})$. Thus, we have

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^n \left(y_t - g(x_t, z_t; \widehat{\theta})\right)^2 \pi_2(z_t) =& \frac{1}{n}\sum_{t=1}^n e_t^2\pi_2(z_t) + \frac{1}{n}\sum_{t=1}^n \widehat{r}_n^2(x_t, z_t; \theta_0)\pi_2(z_t) \\ &+ \frac{2}{n}\sum_{t=1}^n \widehat{r}_n(x_t, z_t; \theta_0)e_t\pi_2(z_t).\end{aligned} \tag{D.31}$$

We first prove that $\frac{1}{n}\sum_{t=1}^n e_t^2\pi_2(z_t) \to_P E[e_1^2\pi_2(z_1)]$.

Notice that $e_t^2 = \sum_{j=-\infty}^t \rho_{t-j}^2\epsilon_j^2 + 2\sum_{j_1=-\infty}^t\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\epsilon_{j_1}\rho_{t-j_2}\epsilon_{j_2}$, and $z_t = \psi(\epsilon_t, \cdots, \epsilon_{t-m_0+1})$. It follows that

$$\begin{aligned}E[e_t^2\pi_2(z_t)] =& \sum_{j=-\infty}^t \rho_{t-j}^2 E[\epsilon_j^2\pi_2(z_t)] + 2\sum_{j_1=-\infty}^t\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}E[\epsilon_{j_1}\epsilon_{j_2}\pi_2(z_t)] \\ =& \sum_{j=t-m_0+1}^t \rho_{t-j}^2 E[\epsilon_j^2\pi_2(z_t)] + E[\pi_2(z_t)]\sum_{j=-\infty}^{t-m_0}\rho_{t-j}^2E[\epsilon_j^2] \\ &+ 2\sum_{j_1=t-m_0+2}^t\sum_{j_2=t-m_0+1}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}E[\epsilon_{j_1}\epsilon_{j_2}\pi_2(z_t)]\end{aligned} \tag{D.32}$$

and

$$\begin{aligned}&E\left(\frac{1}{n}\sum_{t=1}^n e_t^2\pi_2(z_t) - E[e_1^2\pi_2(z_1)]\right)^2 = \frac{1}{n^2}E\left(\sum_{t=1}^n(e_t^2\pi_2(z_t) - E[e_t^2\pi_2(z_t)])\right)^2 \\ =&\frac{1}{n^2}\sum_{t=1}^n E(e_t^2\pi_2(z_t) - E[e_t^2\pi_2(z_t)])^2 \\ &+ 2\frac{1}{n^2}\sum_{t=2}^n\sum_{s=1}^{t-1}E[(e_t^2\pi_2(z_t) - E[e_t^2\pi_2(z_t)])(e_s^2\pi_2(z_s) - E[e_s^2\pi_2(z_s)])] \\ =&o(1) + 2\frac{1}{n^2}\sum_{t=2}^n\sum_{s=1}^{t-1}\{E[e_t^2\pi_2(z_t)e_s^2\pi_2(z_s)] - E[e_t^2\pi_2(z_t)]E[e_s^2\pi_2(z_s)]\}.\end{aligned}$$

It is clear that we only need to consider both $t - s$ and s are large, say, $t - s > m_n$ and $s > m_n$ where $m_n \to \infty$ and $m_n^4/n \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned}&E[e_t^2\pi_2(z_t)e_s^2\pi_2(z_s)] \\ =&E\pi_2(z_t)\pi_2(z_s)\left(\sum_{j=-\infty}^t \rho_{t-j}^2\epsilon_j^2 + 2\sum_{j_1=-\infty}^t\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\epsilon_{j_1}\rho_{t-j_2}\epsilon_{j_2}\right)\end{aligned}$$

$$
\begin{aligned}
&\times\left(\sum_{\ell=-\infty}^{s}\rho_{s-\ell}^2\epsilon_\ell^2+2\sum_{\ell_1=-\infty}^{s}\sum_{\ell_2=-\infty}^{\ell_1-1}\rho_{s-\ell_1}\epsilon_{\ell_1}\rho_{s-\ell_2}\epsilon_{\ell_2}\right)\\
=&E\pi_2(z_t)\pi_2(z_s)\sum_{j=-\infty}^{t}\rho_{t-j}^2\epsilon_j^2\sum_{\ell=-\infty}^{s}\rho_{s-\ell}^2\epsilon_\ell^2\\
&+2E\pi_2(z_t)\pi_2(z_s)\sum_{j=-\infty}^{t}\rho_{t-j}^2\epsilon_j^2\sum_{\ell_1=-\infty}^{s}\sum_{\ell_2=-\infty}^{\ell_1-1}\rho_{s-\ell_1}\epsilon_{\ell_1}\rho_{s-\ell_2}\epsilon_{\ell_2}\\
&+2E\pi_2(z_t)\pi_2(z_s)\sum_{\ell=-\infty}^{s}\rho_{s-\ell}^2\epsilon_\ell^2\sum_{j_1=-\infty}^{t}\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\epsilon_{j_1}\rho_{t-j_2}\epsilon_{j_2}\\
&+4E\pi_2(z_t)\pi_2(z_s)\sum_{j_1=-\infty}^{t}\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\epsilon_{j_1}\rho_{t-j_2}\epsilon_{j_2}\sum_{\ell_1=-\infty}^{s}\sum_{\ell_2=-\infty}^{\ell_1-1}\rho_{s-\ell_1}\epsilon_{\ell_1}\rho_{s-\ell_2}\epsilon_{\ell_2}\\
:=&I_1+2I_2+2I_3+4I_4,\quad \text{say.}
\end{aligned}
$$

Observe further that

$$
\begin{aligned}
I_1=&E\pi_2(z_t)\pi_2(z_s)\sum_{j=-\infty}^{t}\rho_{t-j}^2\epsilon_j^2\sum_{\ell=-\infty}^{s}\rho_{s-\ell}^2\epsilon_\ell^2\\
=&E[\pi_2(z_t)]\sum_{j=-\infty}^{t-m_0}\sum_{\ell=-\infty}^{s-m_0}\rho_{s-\ell}^2\rho_{t-j}^2E[\epsilon_j^2\pi_2(z_s)\epsilon_\ell^2]\\
&+E[\pi_2(z_s)]\sum_{j=t-m_0+1}^{t}\rho_{t-j}^2E[\pi_2(z_t)\epsilon_j^2]\sum_{\ell=-\infty}^{s-m_0}\rho_{s-\ell}^2E[\epsilon_\ell^2]\\
&+E[\pi_2(z_t)]\sum_{j=-\infty}^{t-m_0}\sum_{\ell=s-m_0+1}^{s}\rho_{t-j}^2\rho_{s-\ell}^2E[\epsilon_j^2\pi_2(z_s)\epsilon_\ell^2]\\
&+\sum_{j=t-m_0+1}^{t}\sum_{\ell=s-m_0+1}^{s}\rho_{t-j}^2\rho_{s-\ell}^2E[\pi_2(z_t)\epsilon_j^2]E[\pi_2(z_s)\epsilon_\ell^2]\\
=&E[\pi_2(z_t)]\sum_{j=s+1}^{t-m_0}\sum_{\ell=-\infty}^{s-m_0}\rho_{s-\ell}^2\rho_{t-j}^2E[\epsilon_j^2]E[\pi_2(z_s)]E[\epsilon_\ell^2]\\
&+E[\pi_2(z_t)]\sum_{j=s-m_0+1}^{s}\sum_{\ell=-\infty}^{s-m_0}\rho_{s-\ell}^2\rho_{t-j}^2E[\epsilon_j^2\pi_2(z_s)]E[\epsilon_\ell^2]\\
&+E[\pi_2(z_t)]E[\pi_2(z_s)]\sum_{j=-\infty}^{s-m_0}\sum_{\ell=-\infty}^{s-m_0}\rho_{s-\ell}^2\rho_{t-j}^2E[\epsilon_j^2\epsilon_\ell^2]\\
&+E[\pi_2(z_s)]\sum_{j=t-m_0+1}^{t}\rho_{t-j}^2E[\pi_2(z_t)\epsilon_j^2]\sum_{\ell=-\infty}^{s-m_0}\rho_{s-\ell}^2E[\epsilon_\ell^2]\\
&+E[\pi_2(z_t)]\sum_{j=-\infty}^{t-d}\sum_{\ell=s-m_0+1}^{s}\rho_{t-j}^2\rho_{s-\ell}^2E[\epsilon_j^2\pi_2(z_s)\epsilon_\ell^2]\\
&+\sum_{j=t-m_0+1}^{t}\sum_{\ell=s-m_0+1}^{s}\rho_{t-j}^2\rho_{s-\ell}^2E[\pi_2(z_t)\epsilon_j^2]E[\pi_2(z_s)\epsilon_\ell^2].
\end{aligned}
$$

Also, notice that

$$\begin{aligned}
I_2 =& E\pi_2(z_t)\pi_2(z_s)\sum_{j=-\infty}^{t}\rho_{t-j}^2\epsilon_j^2\sum_{\ell_1=-\infty}^{s}\sum_{\ell_2=-\infty}^{\ell_1-1}\rho_{s-\ell_1}\epsilon_{\ell_1}\rho_{s-\ell_2}\epsilon_{\ell_2}\\
=&\sum_{j=t-m_0+1}^{t}\rho_{t-j}^2E[\pi_2(z_t)\epsilon_j^2]\sum_{\ell_1=-\infty}^{s}\sum_{\ell_2=-\infty}^{\ell_1-1}\rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+\sum_{j=s+1}^{t-m_0}\rho_{t-j}^2E[\pi_2(z_t)]E[\epsilon_j^2]\sum_{\ell_1=-\infty}^{s}\sum_{\ell_2=-\infty}^{\ell_1-1}\rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+E[\pi_2(z_t)]\sum_{j=-\infty}^{s}\rho_{t-j}^2\sum_{\ell_1=-\infty}^{s}\sum_{\ell_2=-\infty}^{\ell_1-1}\rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{\ell_1}\epsilon_{\ell_2}\epsilon_j^2]\\
=&\sum_{j=t-m_0+1}^{t}\rho_{t-j}^2E[\pi_2(z_t)\epsilon_j^2]\sum_{\ell_1=s-m_0+2}^{s}\sum_{\ell_2=s-m_0+1}^{\ell_1-1}\rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+\sum_{j=s+1}^{t-m_0}\rho_{t-j}^2E[\pi_2(z_t)]E[\epsilon_j^2]\sum_{\ell_1=s-m_0+2}^{s}\sum_{\ell_2=s-m_0+1}^{\ell_1-1}\rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+E[\pi_2(z_t)]\sum_{j=-\infty}^{s}\rho_{t-j}^2\sum_{\ell_1=-\infty}^{s}\sum_{\ell_2=-\infty}^{\ell_1-1}\rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{\ell_1}\epsilon_{\ell_2}\epsilon_j^2],
\end{aligned}$$

which yields that $|I_2|\le O(1)\sum_{j=-\infty}^{s}\rho_{t-j}^2\le C|\rho_{t-s}|(t-s)^{-1}$ by Assumption 2.1.

Similarly,

$$\begin{aligned}
I_3 =& E\pi_2(z_t)\pi_2(z_s)\sum_{\ell=-\infty}^{s}\rho_{s-\ell}^2\epsilon_\ell^2\sum_{j_1=-\infty}^{t}\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\epsilon_{j_1}\rho_{t-j_2}\epsilon_{j_2}\\
=& E[\pi_2(z_t)]\sum_{j_1=-\infty}^{s}\sum_{j_2=-\infty}^{j_1-1}\sum_{\ell=-\infty}^{s}\rho_{s-\ell}^2\rho_{t-j_1}\rho_{t-j_2}E[\pi_2(z_s)\epsilon_\ell^2\epsilon_{j_1}\epsilon_{j_2}],
\end{aligned}$$

implying that $|I_3|\le O(1)\sum_{j_1=-\infty}^{s}\sum_{j_2=-\infty}^{j_1-1}|\rho_{t-j_1}\rho_{t-j_2}|\le(t-s)^{-2}$ by Assumption 2.1 and $\sum_j j|\rho_j|<\infty$.

Next, we calculate I_4.

$$\begin{aligned}
I_4 =& E\pi_2(z_t)\pi_2(z_s)\sum_{j_1=-\infty}^{t}\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\epsilon_{j_1}\rho_{t-j_2}\epsilon_{j_2}\sum_{\ell_1=-\infty}^{s}\sum_{\ell_2=-\infty}^{\ell_1-1}\rho_{s-\ell_1}\epsilon_{\ell_1}\rho_{s-\ell_2}\epsilon_{\ell_2}\\
=&\sum_{j_1=t-m_0+2}^{t}\sum_{j_2=t-m_0+1}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}E[\pi_2(z_t)\epsilon_{j_1}\epsilon_{j_2}]\sum_{\ell_1=s-m_0+2}^{s}\sum_{\ell_2=s-m_0+1}^{\ell_1-1}\rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+E[\pi_2(z_t)]\sum_{j_1=s-m_0+2}^{s}\sum_{j_2=s-m_0+1}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}\sum_{\ell_1=s-m_0+2}^{s}\sum_{\ell_2=s-m_0+1}^{\ell_1-1}\rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{j_1}\epsilon_{j_2}\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+E[\pi_2(z_t)]E[\pi_2(z_s)]\sum_{j_1=-\infty}^{s-m_0}\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}\rho_{s-j_1}\rho_{s-j_2}E[\epsilon_{j_1}^2]E[\epsilon_{j_2}^2]\\
=& E[\pi_2(z_t)]\sum_{j_1=s-m_0+2}^{s}\sum_{j_2=s-m_0+1}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}\sum_{\ell_1=s-m_0+2}^{s}\sum_{\ell_2=s-m_0+1}^{\ell_1-1}\rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{j_1}\epsilon_{j_2}\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+E[\pi_2(z_t)]E[\pi_2(z_s)]\sum_{j_1=-\infty}^{s-m_0}\sum_{j_2=-\infty}^{j_1-1}\rho_{t-j_1}\rho_{t-j_2}\rho_{s-j_1}\rho_{s-j_2}E[\epsilon_{j_1}^2]E[\epsilon_{j_2}^2].
\end{aligned}$$

It then follows from (D.32) that

$$
\begin{aligned}
&E[e_t^2\pi_2(z_t)e_s^2\pi_2(z_s)] - E[e_t^2\pi_2(z_t)]E[e_s^2\pi_2(z_s)]\\
=&I_1 + 2I_2 + 2I_3 + 4I_4 - E[e_t^2\pi_2(z_t)]E[e_s^2\pi_2(z_s)]\\
=&\nu_4 \ E[\pi_2(z_t)]E[\pi_2(z_s)] \sum_{j=-\infty}^{s-m_0}\sum_{\ell=-\infty}^{s-m_0} \rho_{s-\ell}^2\rho_{t-j}^2 + 2I_2 + 2I_3\\
&+ E[\pi_2(z_t)] \sum_{j_1=s-m_0+2}^{s}\sum_{j_2=s-m_0+1}^{j_1-1} \rho_{t-j_1}\rho_{t-j_2} \sum_{\ell_1=s-m_0+2}^{s}\sum_{\ell_2=s-m_0+1}^{\ell_1-1} \rho_{s-\ell_1}\rho_{s-\ell_2}E[\pi_2(z_s)\epsilon_{j_1}\epsilon_{j_2}\epsilon_{\ell_1}\epsilon_{\ell_2}]
\end{aligned}
$$

where $\nu_4 = E\epsilon_1^4$, which implies that $|E[e_t^2\pi_2(z_t)e_s^2\pi_2(z_s)] - E[e_t^2\pi_2(z_t)]E[e_s^2\pi_2(z_s)]| \leq C(|\rho_{t-s}|(t-s)^{-1} + (t-s)^{-2})$. Consequently,

$$
\begin{aligned}
&E\left(\frac{1}{n}\sum_{t=1}^{n} e_t^2\pi_2(z_t) - E[e_1^2\pi_2(z_1)]\right)^2\\
=&o(1) + 2\frac{1}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}\{E[e_t^2\pi_2(z_t)e_s^2\pi_2(z_s)] - E[e_t^2\pi_2(z_t)]E[e_s^2\pi_2(z_s)]\}\\
=&o(1) + O(1)\frac{1}{n^2}\sum_{t=m_n+1}^{n}\sum_{s=m_n}^{t-m_n}(|\rho_{t-s}|(t-s)^{-1} + (t-s)^{-2})\\
=&o(1) + C\frac{1}{n^2}n = o(1).
\end{aligned}
$$

This finishes the proof of $\frac{1}{n}\sum_{t=1}^{n} e_t^2\pi_2(z_t) \to_P E[e_1^2\pi_2(z_1)]$.

Second, since $\widehat{\theta} \to_P \theta_0$, we need only to consider the case where $\|\widehat{\theta} - \theta_0\| < \epsilon$ holds in probability for some $\epsilon > 0$ in what follows. By Assumption 2.3(iii), $|\widehat{r}_n(x_t, z_t; \theta_0)| \leq |G_1(x_t, z_t; \theta_0)^\tau(\widehat{\theta} - \theta_0)| + G_2(x_t, z_t; \theta_0)\|\widehat{\theta} - \theta_0\|^2$. Whence,

$$
\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\widehat{r}_n^2(x_t, z_t; \theta_0)\pi_2(z_t)\\
\leq&\frac{1}{n}\sum_{t=1}^{n}|G_1(x_t, z_t; \theta_0)^\tau(\widehat{\theta} - \theta_0)|^2\pi_2(z_t) + \ \frac{1}{n}\sum_{t=1}^{n}G_2^2(x_t, z_t; \theta_0)\pi_2(z_t)\|\widehat{\theta} - \theta_0\|^4\\
&+ \ \frac{2}{n}\sum_{t=1}^{n}\pi_2(z_t)|G_1(x_t, z_t; \theta_0)^\tau(\widehat{\theta} - \theta_0)|G_2(x_t, z_t; \theta_0)\|\widehat{\theta} - \theta_0\|^2\\
\leq& \ \|\widehat{\theta} - \theta_0\|^2 \cdot \frac{1}{n}\sum_{t=1}^{n}\|G_1(x_t, z_t; \theta_0)\|^2\pi_2(z_t) + \|\widehat{\theta} - \theta_0)\|^4 \cdot \frac{1}{n}\sum_{t=1}^{n}G_2^2(x_t, z_t; \theta_0) \ \pi_2(z_t)\\
&+ \|\widehat{\theta} - \theta_0\|^2\frac{2}{n}\sum_{t=1}^{n}\pi_2(z_t)|G_1(x_t, z_t; \theta_0)^\tau(\widehat{\theta} - \theta_0)|G_2(x_t, z_t; \theta_0)\\
\equiv&A_1 + A_2 + A_3.
\end{aligned}
$$

Making use of the density $q_t(x, z)$ of $(d_t^{-1}x_t, z_t)$ in Lemma A.6, it follows immediately that $A_1 = o_P(1)$ and $A_2 = o_P(1)$. Since $A_1 = o_P(1)$ and $A_2 = o_P(1)$ imply that $A_3 = o_P(1)$, and hence $\frac{1}{n}\sum_{t=1}^{n}\widehat{r}_n(x_t, z_t; \theta_0)e_t\pi_2(z_t) = o_P(1)$ by Cauchy-Schwartz inequality. This finishes the proof. □

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