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# **Bayesian Assessment of Lorenz and Stochastic Dominance**

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# Bayesian Assessment of Lorenz and Stochastic Dominance

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## Summary

We introduce a Bayesian approach for assessing Lorenz and stochastic dominance. For two income distributions, say  $X$  and  $Y$ , estimated via Markov chain Monte Carlo, we describe how to compute posterior probabilities for (i)  $X$  dominates  $Y$ , (ii)  $Y$  dominates  $X$ , and (iii) neither  $Y$  nor  $X$  is dominant. The proposed approach is applied to Indonesian income distributions using mixtures of gamma densities that ensure flexible modelling. Probability curves depicting the probability of dominance at each population proportion are used to explain changes in dominance probabilities over restricted ranges relevant for poverty orderings. They also explain some seemingly contradictory outcomes from the  $p$ -values of some sampling theory tests.

*Keywords:* Dominance probabilities, poverty comparisons, MCMC, gamma mixture.

*JEL classification:* C11, C12, D31, I32

## 1. INTRODUCTION

Statistical tests of dominance regularly appear in the economics and finance literature. They employ statistical methods to compare the distributions of two random variables (or one random variable at two points in time) in such a way as to determine if one ‘dominates’ another. These tests have been used to compare income distributions in welfare analysis, distributions of asset returns in portfolio analysis, and risk analyses in actuarial science. We focus on Lorenz, stochastic and poverty dominance in income distributions and, in contrast to the existing literature, we propose a Bayesian approach to assessing dominance. Applications of stochastic dominance to asset returns in portfolio analysis can be found in Wong et al. (2008), Sriboonchitta et al. (2010) and Bai et al. (2011); for applications in actuarial sciences and risk analysis, see Kaas et al. (1994) and Denuit et al. (2005). Maasoumi and Millimet (2005) use stochastic dominance tests to examine trends in environmental quality.

The theoretical foundations for using Lorenz and stochastic dominance to analyse social welfare were laid by Atkinson (1970) and Shorrocks (1983). Among other things, it was shown that, for the class of monotonically increasing concave, but otherwise arbitrary, income utility functions, generalized Lorenz dominance (which is equivalent to second-order stochastic dominance) implies, and is implied by, a social welfare ordering. Robustness of stochastic dominance criteria to changes in the functional form of the social welfare function allows practitioners to avoid specifying a functional form for the utility or poverty function, and provides a compelling reason for its use in analyses of social welfare. Other inequality measures, such as the Gini coefficient and Atkinson’s inequality index are useful when income distributions cannot be ordered according to stochastic dominance criteria, but they involve placing more restrictive assumptions on the functional form of the social welfare function or poverty index. Details of these various concepts, and the relationships between them, can be found in Atkinson and

Bourguignon (1982), Chakravarty (2009), Davidson and Duclos (2000), Le Breton and Peluso (2009), Lambert (2001), and Maasoumi (1997).

Empirical studies of stochastic, Lorenz and poverty dominance have typically been undertaken in a sampling-theory framework using nonparametric hypothesis tests. Stochastic dominance tests have been proposed, developed, extended and/or applied by Bishop et al. (1989), McFadden (1989), Kaur et al. (1994), Bishop et al (1995), Anderson (1996), Davidson and Duclos (2000, 2013), Maasoumi and Heshmati (2000, 2008), Barrett and Donald (2003), Linton et al. (2005), Horváth et al. (2006), Linton et al. (2010), Berrendero and Cárcamo (2011), Bennett (2013) and Donald and Hsu (2016). Many of these papers also examine Lorenz dominance and poverty orderings. Additional papers concerned with Lorenz dominance include Beach and Davidson (1983), Bishop et al. (1991), Dardanoni and Forcina (1999), Schluter and Trede (2002) and Barrett et al (2014). Tests specifically concerned with poverty dominance include those proposed by Tabri (2015) and Barrett et al. (2016). Extensions to multivariate scenarios include Duclos et al (2006), McCaig and Yatchew (2007), and Bennett and Mitra (2013).

We propose a novel Bayesian approach to assessing dominance, and illustrate how it can be applied to income distribution data. There are a number of differences between existing sampling theory approaches and our proposed Bayesian approach. Sampling theory approaches begin with specification of null and alternative hypotheses, with some tests specifying  $X$  dominates  $Y$  (say) as the null hypothesis, and others using  $X$  does not dominate  $Y$  as the null. To exhaust all possibilities, these hypotheses are often also reversed, specifying that  $Y$  does or does not dominate  $X$ . A test statistic and its limiting distribution are derived and results are reported in terms of  $p$ -values. In our Bayesian approach we begin with flexible parametric specifications for both income distributions and compute posterior probabilities of dominance as the probability that the parameters of the income distributions lie in the constrained region implied by dominance inequalities. We compute the probability that  $X$  dominates  $Y$ , the probability that  $Y$  dominates  $X$ ,

and the probability that neither distribution is dominant. This general approach – computing posterior probabilities of parameters lying within constrained regions – has been used by Geweke (1988) to assess explosive and oscillatory behavior of real per capita GDP for 19 OECD countries, and by Geweke (1986) in a variety of other contexts. A possible criticism of our proposal for computing posterior probabilities of dominance is its dependencies on how the income distribution is modelled through the likelihood function, and the prior information that is placed on the unknown parameters. These dependencies can be minimized, however. Subjectivity from specification of prior information can be limited by using relatively uninformative priors and is unlikely to have a major impact given the relatively large sample sizes that are typically used to estimate income distributions. The choice of likelihood function is likely to be more critical. In preliminary work, two of us (Chotikapanich and Griffiths, 2006) found results that were sensitive to a choice between the Dagum and Singh-Maddala income distributions, and concluded that a relatively flexible likelihood function is necessary for robust results. To minimize the impact of the likelihood function, in this paper we choose a more flexible likelihood function: a mixture of gamma densities. The added flexibility achieved by a mixture makes the analysis more complex, however. We are confronted with the problem of choosing an arbitrary number of points to assess dominance across the  $(0, \infty)$  interval, or having to employ numerical methods to invert the distribution function of the mixture so that the quantile function can be used and the analysis restricted to the  $(0, 1)$  interval. We propose numerical methods to resolve this issue, and introduce several refinements to earlier work. The techniques are illustrated using Indonesian household income distributions for the years 1999, 2002, 2005 and 2008.

The remainder of the paper is structured as follows. Conditions for dominance are stated in Section 2 and the proposed method for Bayesian assessment is described in Section 3. In Section 4 we introduce a gamma mixture model and describe how parameter draws from the joint posterior density of its parameters can be used to find corresponding values for the quantile

function, the generalized Lorenz curve and the Lorenz curve for the mixture of gamma densities. A Markov chain Monte Carlo (MCMC) algorithm for drawing observations from the joint posterior density for the parameters is summarized in Appendix A. Appendix B describes a numerical algorithm for inverting the distribution function of the gamma mixture to obtain its quantiles. Data and estimation results for some Indonesian income distributions are considered in Section 5. Dominance results and a comparison with some sampling theory results are presented in Section 6; some concluding remarks are offered in Section 7.

## 2. DOMINANCE CONDITIONS

To introduce the dominance conditions, we consider an income distribution  $X$  that is described by density and distribution functions  $p_X(x)$  and  $F_X(x)$ , respectively, with finite mean income  $\mu_X = E(X)$ . A number of expressions have been used in the literature for the Lorenz curve that gives the proportion of total income earned by the poorest proportion  $u$  of the population. The one that is useful for noting the equivalence of generalized Lorenz dominance with second-order stochastic dominance (see below) is

$$L_X(u) = \frac{1}{\mu_X} \int_0^u F_X^{-1}(t) dt \quad 0 \leq u \leq 1 \quad (1)$$

For computing values of the Lorenz curve to use for Bayesian assessment of dominance, the most convenient expression is

$$L_X(u) = F_X^{(1)}[F_X^{-1}(u)] \quad (2)$$

where  $F_X^{(1)}(x) = (1/\mu_X) \int_0^x t p_X(t) dt$  is the first moment distribution function for  $X$ . We say that an income distribution for  $X$  Lorenz dominates (LD) a distribution for  $Y$  (say), expressed as  $X \geq_{LD} Y$ , if and only if

$$L_X(u) \geq L_Y(u) \quad \text{for all } 0 \leq u \leq 1 \quad \text{and} \quad L_X(u) > L_Y(u) \quad \text{for some } 0 < u < 1 \quad (3)$$

While this definition is the typical one used in the economics literature (see, for example, Lambert (2001) and Barrett et al. (2014)), the definition used in much of the statistics literature follows the opposite convention, with  $L_Y(u) \geq L_X(u)$  being the condition for  $X \geq_{LD} Y$ . See, for example, Kleiber and Kotz (2003). Since  $L_X(u) \geq L_Y(u)$  implies higher welfare for distribution  $X$  in the sense that, other thing equal, less inequality is preferred to more inequality, we refer to this condition as one where  $X$  dominates  $Y$ . This is the case when the Lorenz curve of  $X$  lies nowhere below that of  $Y$  for all population proportions  $u$ . For two income distributions with the same mean income, Lorenz dominance implies greater utility with respect to all strictly increasing and concave social welfare functions.

Because Lorenz dominance considers only the degree of inequality and not the level of income, generalized Lorenz dominance (GLD) was introduced to recognize that higher levels of income are associated with higher levels of welfare. We say that  $X$  generalized-Lorenz dominates  $Y$ , written as  $X \geq_{GLD} Y$  if and only if

$$\mu_X L_X(u) \geq \mu_Y L_Y(u) \text{ for all } 0 \leq u \leq 1 \quad (4)$$

with strict inequality holding for some  $0 < u < 1$ . For all strictly increasing and concave social welfare functions, GLD provides an unambiguous ranking of distributions when the generalized Lorenz curves do not intersect. Given the expression for the Lorenz curve in equation (1), the condition in (4) can also be expressed as

$$\int_0^u F_X^{-1}(t) dt \geq \int_0^u F_Y^{-1}(t) dt \text{ for all } 0 \leq u \leq 1 \quad (5)$$

with strict inequality holding for some  $0 < u < 1$ . Condition (5) can be viewed as the sum of the incomes of the bottom  $u$  proportion in  $X$  being at least as great as the corresponding sum for  $Y$  for any population proportion  $u$ . Writing the relation for GLD in this way demonstrates its equivalence to second-order stochastic dominance (SSD). See, for example, Maasoumi (1997) or Kleiber and Kotz (2003, p.25). An equivalent condition for SSD is

$$\int_0^x F_X(t) dt \leq \int_0^x F_Y(t) dt \quad \text{for all } 0 < x < \infty \quad (6)$$

with strict inequality holding for some  $x$ .

A stronger condition for welfare improvement than SSD (or equivalently, GLD) is that of first-order stochastic dominance (FSD). The distribution for  $X$  first-order stochastically dominates  $Y$ , written  $X \geq_{FSD} Y$ , if and only if

$$F_X^{-1}(u) \geq F_Y^{-1}(u) \quad \text{for all } 0 \leq u \leq 1 \quad (7)$$

with strict inequality holding for some  $0 < u < 1$ . In this case the level of income from distribution  $X$  is greater than the level of income from distribution  $Y$  for all population proportions  $u$ .

Alternatively,  $X \geq_{FSD} Y$ , if and only if

$$F_X(x) \leq F_Y(x) \quad \text{for all } 0 < x < \infty \quad (8)$$

with strict inequality holding for some  $x$ . First-order stochastic dominance implies greater utility for all social welfare functions that are strictly increasing. Concavity is no longer required.

While we focus on computing posterior probabilities for Lorenz, first, and second-order stochastic dominance, our computations can be extended to third and higher-order stochastic dominance criteria. Third-order stochastic dominance is relevant for ordering distributions for social welfare functions where the marginal utility function is positive, decreasing, and strictly convex. See, for example, Chakravarty (2009) and Le Breton and Peluso (2009). The higher the degree of dominance, the greater the number of restrictions that need to be imposed on the social welfare function and the weaker the dominance inequality. That is, first-order stochastic dominance implies second-order stochastic dominance, and so on, but the converse is not true. Thus, if  $X \geq_{FSD} Y$ , the generalized Lorenz curve of  $X$  will also lie everywhere above  $Y$ .

Foster and Shorrocks (1988) showed that unidimensional stochastic dominance conditions are equivalent to unidimensional poverty orderings. Let  $z$  denote the ‘‘poverty line’’. First-order poverty dominance, in terms of the quantile function of the distribution, is defined as

$F_X^{-1}(u) \geq F_Y^{-1}(u)$  for all  $0 \leq u \leq F(z)$ , with strict inequality for some values of  $u$ . This is equivalent to the statement that the level of income of individuals is always (weakly) greater in distribution  $X$  than in  $Y$  up to population proportion given by  $F(z)$ . Similarly, second-order poverty dominance is defined as  $\int_0^u F_X^{-1}(t) dt \geq \int_0^u F_Y^{-1}(t) dt$  for all  $0 \leq u \leq F(z)$ , with strict inequality for some values of  $u$ . Davidson and Duclos (2000) show that poverty dominance at order  $s$  and poverty line  $z_s$  implies poverty dominance at order  $s+1$  for a poverty line  $z_{s+1} > z_s$ . Thus, there exists a higher order of dominance such that dominance is achieved for any finite  $z$ . Barrett et al. (2016) consider testing for poverty dominance in terms of poverty gap profiles – see, for example, Jenkins and Lambert (1997). For a given poverty line, this dominance criterion is equivalent to restricted generalized Lorenz dominance.

### 3. BAYESIAN ASSESSMENT OF DOMINANCE RELATIONSHIPS.

We now examine how to assess the dominance conditions within a Bayesian framework. Given two parametric distributions, each with known parameter values, one way to assess each form of dominance (LD, GLD or FSD) is to compute  $L(u)$ ,  $\mu L(u)$  and  $F^{-1}(u)$  for both distributions for a grid of values for  $u$  in the interval  $(0,1)$ . If the grid contains a relatively large number of values, and the dominance inequality being considered is satisfied for all those values, then it is reasonable to conclude that the condition is satisfied for all  $u$ , and hence dominance holds. Alternatively, for GLD and FSD one could compute  $F(x)$  and  $\int_0^x F(t) dt$  for both distributions for a grid of values of  $x$  in the interval  $(0, x_{\max})$  where  $x_{\max}$  is a value deemed to be sufficiently large, and examine whether the relevant inequalities are satisfied for all these values.

Suppose now that the distributions for  $X$  and  $Y$  have unknown parameter vectors  $\theta_x$  and  $\theta_y$  that are estimated using income distribution data. Since these parameters are not known with certainty, any conclusion about whether one distribution dominates another cannot be made with

certainty. In Bayesian inference, uncertainty about whether one distribution dominates another, or whether one function is greater than another at a particular point, can be expressed in terms of a probability statement. To obtain or estimate such probability statements, assume we have draws on  $\theta_x$  and  $\theta_y$  from the two posterior densities  $p(\theta_x | \mathbf{x})$  and  $p(\theta_y | \mathbf{y})$ . Taking FSD as an example, it will be useful to distinguish between  $\Pr[F_X^{-1}(u) \geq F_Y^{-1}(u)]$  for a given value of  $u$  and  $\Pr[F_X^{-1}(u) \geq F_Y^{-1}(u)]$  for all values of  $0 < u < 1$ . An estimate of the first probability is given by the proportion of draws of  $\theta_x$  and  $\theta_y$  for which  $F_X^{-1}(u) \geq F_Y^{-1}(u)$  for a given  $u$ . We will denote this probability by  $\Pr[X \geq_{FSD} Y | u]$ . The probability  $X \geq_{FSD} Y$  for the range  $(0, 1)$  is given by the proportion of values  $\theta_x$  and  $\theta_y$  for which the inequality holds *for all values of  $u$* . We denote this probability by  $\Pr[X \geq_{FSD} Y]$ . In practice we can consider a grid of  $u$  values within the interval  $(0, 1)$  and count the number of parameter draws where the inequality holds for all  $u$  in the grid. Since

$$\Pr[X \geq_{FSD} Y] \leq \min_u \Pr[X \geq_{FSD} Y | u] \quad (9)$$

a finer grid can be taken in the region that counts: those values of  $u$  where  $\Pr[X \geq_{FSD} Y | u]$  reaches a minimum. Similar probability statements can be made for LD and GLD.

In our empirical application we used 999 values of  $u$  between 0.001 and 0.999 with increments of 0.001. Graphing  $\Pr[X \geq_{FSD} Y | u]$  against  $u$  – curves that we call “probability curves” – provides useful information about the range(s) of the income distribution that have the largest impact on the probability of dominance. The inequality in (9) means that a probability curve is a powerful tool for visually determining the probability of dominance over the whole range of population proportions as well as within sub-intervals of  $u$ . Examining a probability

curve is particularly useful for assessing restricted intervals such as poverty dominance as well as the sensitivity of poverty dominance to specification of a poverty line.

Having found  $\Pr[X \geq_{FSD} Y]$  we can reverse the process to find  $\Pr[Y \geq_{FSD} X]$ . The probability that neither  $X$  nor  $Y$  dominates is given by  $1 - \Pr[X \geq_{FSD} Y] - \Pr[Y \geq_{FSD} X]$ .

Suppose that we have a sequence of  $M$  draws on  $\theta_X$  and a sequence of  $M$  draws on  $\theta_Y$ , and we assess probability using  $M$  pairwise comparisons. Estimates of the dominance probabilities may change with different orderings of the draws on  $\theta_X$  and  $\theta_Y$ . To ensure that the probability of dominance is robust to the order of the  $M$  parameter draws, we randomly rearrange the order of the draws and repeat the procedure  $C$  times. This gives a set of  $C$  dominance probabilities,  $\Pr(X \geq_{FSD} Y)_t$  for  $t=1,2,\dots,C$ , and a set of  $C$  probability curves. We use the mean probability of dominance,  $\overline{\Pr(X \geq_{FSD} Y)} = \sum_{t=1}^C \Pr(X \geq_{FSD} Y)_t / C$ , as our estimate of the probability of dominance. We will also define the maximum and minimum values of the set of  $C$  dominance probabilities as the upper and lower bounds, respectively, for the dominance probability estimate. These bounds give an indication of how sensitive the dominance probability estimate is to the choice of pairwise ordering procedure. We can perform similar operations on the set of  $C$  probability curves to obtain bounds on the probability curve.

#### 4. ESTIMATION AND DOMINANCE ASSESSMENT WITH A GAMMA MIXTURE

To minimize the dependence of the posterior probabilities for dominance on the assumed parametric functional form for the income distributions, we use a mixture of gamma densities which has the ability to approximate well any functional form. Our specification and estimation procedure follows that in Wiper et al. (2001). Other mixtures, such as a log-normal mixture would

also provide a good approximation and could be used within the same framework.<sup>1</sup> An income distribution modelled as a gamma mixture with  $K$  components can be written as

$$p(x | \mathbf{w}, \boldsymbol{\mu}, \mathbf{v}) = \sum_{k=1}^K w_k G(x | v_k, v_k / \mu_k) \quad (10)$$

where  $x$  is a random income draw from the probability density function (pdf)  $p(x | \mathbf{w}, \boldsymbol{\mu}, \mathbf{v})$ , with parameter vectors,  $\mathbf{w} = (w_1, w_2, \dots, w_K)'$ ,  $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_K)'$ , and  $\mathbf{v} = (v_1, v_2, \dots, v_K)'$ . The pdf  $G(x | v_k, v_k / \mu_k)$  is a gamma density with mean  $\mu_k > 0$  and shape parameter  $v_k > 0$ ,

$$G(x | v_k, v_k / \mu_k) = \frac{(v_k / \mu_k)^{v_k}}{\Gamma(v_k)} x^{v_k-1} \exp\left(-\frac{v_k}{\mu_k} x\right) \quad (11)$$

Including the mean  $\mu_k$  as one of the parameters in the pdf makes the parameterization in (11) different from the standard textbook one, but it is convenient for later analysis. Details of the prior pdfs and the MCMC algorithm proposed by Wiper et al. (2001) for drawing observations from the posterior pdf  $p(\boldsymbol{\mu}, \mathbf{v}, \mathbf{w} | \mathbf{x})$ , where  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  is a random sample from (10), are given in Appendix A.

Having obtained MCMC draws  $(\boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$ ,  $m = 1, 2, \dots, M$  from the posterior distribution  $p(\boldsymbol{\mu}, \mathbf{v}, \mathbf{w} | \mathbf{x})$ , we can proceed to set up the machinery to assess various types of dominance. Associated with each draw, we have a gamma mixture density  $p(x | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$  that can be obtained from (10). If interest centers on the predictive density for  $x$ , it can be obtained by averaging  $p(x | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$  over all the parameter draws. However, for assessing dominance our interest centres on (i) the Lorenz curves  $L_x(u | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$ , (ii) the generalized Lorenz curves

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<sup>1</sup> One potential disadvantage of a mixture of lognormal densities is that, when the traditional noninformative prior is used for the parameters, the predictive density for income, obtained by integrating out the parameters, is a mixture of “log- $t$  distributions” which do not have finite moments.

$$GL_X(u | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)}) = \boldsymbol{\mu}^{(m)} L_X(u | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$$

with  $\boldsymbol{\mu}^{(m)} = \sum_{k=1}^K w_k^{(m)} \boldsymbol{\mu}_k^{(m)}$ , and (iii) the quantile functions  $F^{-1}(u | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$ , for each of the parameter draws  $m = 1, 2, \dots, M$ .

The first challenge is to obtain a value of the quantile function  $F^{-1}(u | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$  for a given proportion  $u$ , and a given set of parameters  $(\boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$ . Inverting the distribution function for a mixture of gamma densities is not straightforward. A closed form solution cannot be obtained, and solving the required nonlinear equation numerically for a large number of MCMC draws is not only time consuming, but also fraught with dangers of non-convergence. To overcome these problems we developed the algorithm described in Appendix B.

The quantile function values can be used to assess first-order stochastic dominance. For Lorenz and generalized Lorenz dominance we also need to evaluate the first moment distribution function evaluated at quantiles  $\hat{y}$  computed from the algorithm in Appendix B. Dropping the  $(m)$  superscript for convenience, the first moment distribution function for the mixture of gamma densities is

$$\begin{aligned} F_X^{(1)}(\hat{y} | \boldsymbol{\mu}, \mathbf{v}, \mathbf{w}) &= \frac{1}{\boldsymbol{\mu}} \int_0^{\hat{y}} x p(x | \boldsymbol{\mu}, \mathbf{v}, \mathbf{w}) dx \\ &= \frac{1}{\boldsymbol{\mu}} \sum_{k=1}^K w_k \int_0^{\hat{y}} x G(x | v_k, v_k / \boldsymbol{\mu}_k) dx \\ &= \frac{1}{\boldsymbol{\mu}} \sum_{k=1}^K w_k \int_0^{\hat{y}} \frac{(v_k / \boldsymbol{\mu}_k)^{v_k}}{\Gamma(v_k)} x^{(v_k+1)-1} \exp\left(-\frac{v_k x}{\boldsymbol{\mu}_k}\right) dx \\ &= \frac{1}{\boldsymbol{\mu}} \sum_{k=1}^K w_k \int_0^{\hat{y}} \frac{\boldsymbol{\mu}_k (v_k / \boldsymbol{\mu}_k)^{v_k+1}}{\Gamma(v_k+1)} x^{(v_k+1)-1} \exp\left(-\frac{v_k x}{\boldsymbol{\mu}_k}\right) dx \\ &= \frac{1}{\boldsymbol{\mu}} \sum_{k=1}^K w_k \boldsymbol{\mu}_k F_k(\hat{y} | v_k + 1, v_k / \boldsymbol{\mu}_k) \end{aligned}$$

where  $F_k(\cdot | v_k + 1, v_k / \mu_k)$  is the distribution function of a gamma density with parameters  $(v_k + 1)$  and  $v_k / \mu_k$ .

## 5. DATA AND ESTIMATION RESULTS

The data used to illustrate the methodology are household income data obtained for Indonesian urban regions for the years 1999, 2002, 2005, and 2008. These data were created using the household expenditure data obtained from the National Socio-Economic Survey.<sup>2</sup> Appropriate price and equivalence-scale adjustments were made to ensure comparability over the years. Summary statistics for the data are presented in Table 1. The units are thousands of rupiah per month.

If one makes judgments on the ordering of the distributions on the basis of the means or medians, the population becomes better off as it moves from 1999 to 2005, but drops back in 2008, a likely consequence of the global financial crisis. Similarly, using the standard deviation and Gini coefficient to measure inequality suggests inequality increases from 1999 to 2005, but then decreases in 2008. Assuming a welfare function where mean income contributes positively and the standard deviation or Gini contributes negatively, it is not clear whether the distribution in any one year is preferable. Comparing the mean and median incomes, and checking the mean against the maximum values, reveals an extremely long right tail, with a small number of households with very high incomes. For example, in 2002 the mean income was 432, the maximum income was 24903, and the proportion of households with incomes greater than 2000 was only 0.0074.

Goodness-of-fit criteria were used to choose a suitable number of mixture components up to a maximum of five. The parameters for the prior densities were as follows. We set  $\varphi_k = 1$  for all  $k$ , implying a flat prior for the mixture weights on each component. For the exponential prior

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<sup>2</sup> The authors are grateful to Ari Handayani for providing the data used in Handayani (2013).

on scale parameters  $\nu_k$  we set  $\theta = 0.01$  for all  $k$ . A 95% probability interval for this prior is (2.54, 202.86) implying a large range of values are possible. For the  $\mu_k$  we set  $\alpha_k = 2.2$  for all  $k$ , and  $\beta_1 = 60, \beta_2 = 100, \beta_3 = 150, \beta_4 = 200$  and  $\beta_5 = 250$ . The 95% probability intervals for  $\mu_1, \mu_2, \dots, \mu_5$  are (10.31, 202.86), (17.06, 320.84), (24.93, 488.68), (33.85, 626.13), and (42.47, 813.53), respectively. All these intervals suggest that the priors are relatively uninformative.

To assess goodness-of-fit for components  $K = 1, 2, \dots, 5$ , we drew initial MCMC samples of 15000, of which 5000 were discarded as a burn in. Maximum likelihood estimates for the Singh-Maddala and Dagum distributions were also obtained as a benchmark for comparison with the gamma mixtures. For each of the four years, three criteria were used to compare the estimated distribution functions  $\widehat{F}(x_j)$ , with the empirical distributions  $F_0(x_j) = j/n$ , where  $j$  refers to the  $j^{\text{th}}$  observation after ordering them from lowest to highest. The  $\widehat{F}(x_j)$  were computed at the posterior means for the gamma mixture components and at the maximum likelihood estimates for the Singh-Maddala and Dagum distributions. The three goodness-of-fit criteria are

$$d = \max_j \left| \widehat{F}(x_j) - F_0(x_j) \right|$$

$$RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^n \left( \widehat{F}(x_i) - F_0(x_i) \right)^2}$$

$$MAE = \frac{1}{n} \sum_{i=1}^n \left| \widehat{F}(x_i) - F_0(x_i) \right|$$

The results in Table 2 provide unambiguous support for a gamma mixture with 4 components. It is superior to all other models for every measure of goodness of fit, and for all years. There is also strong support for the choice of a mixture model over the commonly used Singh-Maddala and Dagum distributions. We therefore chose the 4-component gamma mixture to obtain the draws used for dominance assessment in Section 6. We re-estimated the 4-

component mixture with 50 different MCMC chains. For each chain, 11000 draws were generated, the first 1000 draws were discarded as a burn-in, and only every 50th draw of the subsequent 10000 draws was retained. With 50 chains and 200 retained observations from each chain, a total of  $M = 10000$  independent draws were retained for subsequent analysis. All parameters showed evidence of converging. A relatively small number of independent MCMC draws was chosen in preference to a large-number of correlated draws to reduce computing time for inverting the distribution functions and carrying out the dominance comparisons. The estimated densities for the 4-component mixtures are presented together with corresponding histograms in Figures 1, 2, 3, and 4. The shape of each distribution has been captured well by the 4-component mixtures, confirming the ability of the mixture distribution to capture the essential features of each distribution. Posterior means and standard deviations for each of the parameters for all components and all years are given in Table 3. In general, the weight for the last component is very small.

## 6. DOMINANCE RESULTS

We begin a discussion of the dominance results by visually comparing the estimated distribution functions (Figure 5), the generalized Lorenz curves (Figure 6), and the Lorenz curves (Figure 7) for each of the 4 years. In Figure 5, it appears that the distributions in 2002, 2005 and 2008 all FSD the 1999 distribution. However, it seems that there is no clear ranking between 2002, 2005, and 2008. In Figure 6 the generalized Lorenz curves for 2002, 2005, and 2008 are also everywhere above the 1999 distribution, consistent with the fact that FSD implies GLD. A similar but less clear remark can be made about the Lorenz curves in Figure 7. Here the 1999 distribution Lorenz dominates 2002, 2005, and 2008, except for grid points where  $u$  is close to zero or one.

When comparing the four years of income distributions, there are six possible pairwise comparisons that can be made. For each comparison, we calculate the probability that  $X$  dominates  $Y$ , the probability that  $Y$  dominates  $X$ , and the corresponding upper and lower bounds

for the estimated probabilities. These probabilities are reported in Table 4, along with bounds computed using  $C = 1000$ . The values in the table can be used to determine the probability that neither  $X$  nor  $Y$  dominates, given by  $1 - \Pr(X \geq_D Y) - \Pr(Y \geq_D X)$ . In each of the three segments in the table, the last two columns provide evidence of how the income distribution has changed from one period to the next. We consider each pair of years in turn.

For 1999 and 2002, the dominance probabilities in the table make clear predictions. In terms of FSD and GLD, the probability that 2002 dominates 1999 is greater than 0.99. While for LD, the probability that 1999 dominates 2002 is 0.9996. The level of welfare measured by FSD and the level of relative inequality described by LD have increased significantly from 1999 to 2002. With respect to GLD, we can conclude that growth over the period has been sufficiently large enough to compensate for the increase in inequality, making social welfare unambiguously greater in 2002. In contrast, when comparing 2002 and 2005, we find the probability that neither year is dominant is very close to 1 for FSD, GLD and LD. There is a only very low probability (0.0037) that 2002 dominates 2005 in terms of LD. We obtain a similar result for FSD and GLD when comparing 2005 to 2008; the probability that neither year is dominant is very close to 1. However, in terms of LD, the probability that 2008 dominates 2005 is close to 1. There are also positive probabilities (0.9996, 0.1915, and 0.3331) that 1999 Lorenz dominates 2002, 2005, and 2008, respectively. These results lead us to conclude that the level of inequality has increased from 1999 to 2005, and decreased from 2005 to 2008. The only clear improvement in terms of FSD and GLD is from 1999 to 2002 (0.9990), and 1999 to 2008 (0.9997). It is somewhat surprising that the FSD and GLD probabilities of 2005 over 1999 (0.2529 and 0.4676, respectively) are far less than those for 2002 over 1999 (both over 0.99). Since both the mean and median for 2005 are greater than those for 2002, we might expect the FSD and GLD probabilities for 2005 over 1999 to be greater than those for 2002 over 1999. Another surprising result comes from comparing the Lorenz dominance probabilities and the estimated Gini coefficients for 2002,

2005 and 2008. The Gini coefficient for 2005 (0.3797) is higher than those for both 2002 (0.3509) and 2008 (0.3583). Hence, we may expect similar LD probabilities for 2008 over 2005, and 2002 over 2005. However, the probability 2008 Lorenz dominates 2005 is 1, while the probability 2002 Lorenz dominates 2005 is only 0.0037. These examples illustrate the usefulness of using posterior probabilities in addition to summary statistics from the distributions. They provide us with some evidence that there may be subtle changes occurring in different parts of the distribution over time that are not captured by the summary statistics. These aspects can be investigated further by using probability curves to assess the probability of dominance at different sub-intervals of the population proportions.

Recall that the probability curves provide us with the probability of dominance at each population proportion  $u$ . Therefore we can use these curves to see which population proportions have the greatest effect on the probability of dominance, or lack of it. Furthermore we are able to show how the probability of dominance will change if we restrict our focus to a particular segment of the population, such as the poorest 10% or 20%. The probability of dominance within the restricted range will necessarily be less than or equal to the minimum value of the probability curve for that restricted range. Additionally, the probability curve for dominance in one direction is the mirror image of the probability curve of dominance in the other direction. For example,  $\Pr(2002 \geq_{FSD} 1999 | u)$  is the mirror image of  $\Pr(1999 \geq_{FSD} 2002 | u)$ .

The probability curves for 2005 over 1999 are given in Figure 8. The lower and upper bounds of the probability curves are also included but are visually indistinguishable from the mean curve. By observing Figure 8 we can see that the FSD and GLD probabilities for 2005 over 1999 are very sensitive to the starting point of the population proportion  $u$ . If we restrict our attention to  $u \geq 0.005$ , then, instead of being only 0.25 and 0.47, both dominance probabilities are greater than 0.95, bringing them much in line with the FSD and GLD probabilities for 2002 over 1999. Thus, the probability curves enable us to isolate the cause of

significant differences between the dominance probabilities. Between 2002 and 2005 the incomes of the poorest 0.5% of the sample must have declined sufficiently such that we are unable to establish either a FSD or GLD relationship for 2005 over 1999. Such a precise statement cannot be made using only the summary statistics.

To identify the cause of the other surprising result, we can view the probability curves for 2002 over 2005 and 2008 over 2005, given in Figures 9 and 10, respectively. If we restrict the analysis to approximately  $u < 0.98$ , instead of being only 0.0037, the probability that 2002 Lorenz dominates 2005 is greater than 0.9, a similar LD probability to 2008 over 2005 when unrestricted. We can use this information to infer several interesting things. The richest 2% of the sample in 2002 must be sufficiently richer than the richest 2% in 2008 to prevent us from making a definitive statement about how income inequality changed from 2002 to 2005. In other words, we cannot establish a Lorenz dominance relationship for 2002 over 2005 *only* because of the level of incomes of the richest 2%. This captures an important aspect of the distributions that we could not establish simply through the use of Gini coefficients as summary statistics. It is, however, consistent with another summary statistic; the maximum income for 2002 (24,903) is almost double that of 2008 (13,182).

This more thorough investigation of what initially were surprising results demonstrates the usefulness of examining more than simple summary statistics when making statements about welfare changes. The probability curves help reveal the complete picture of changes in the income distribution. Also, the example underscores the importance of precisely estimating the quantile functions close to the bounds. This was a significant issue encountered in Chotikapanich and Griffiths (2006) when using the Singh-Maddala and Dagum distributions.

The probability curves are not only useful for isolating population proportions that are critical for dominance assessment. They can also be used to investigate dominance over restricted regions. Poverty orderings, for example, are concerned with dominance below a

poverty line. A dominance probability within a restricted range is only likely to differ from a corresponding unrestricted dominance probability if the minimum of the probability curve occurs outside the restricted region. Moreover, the probability curve shows how the restricted dominance probability is likely to change as the restricted interval (for example, the poverty line) changes.

Table 5 contains dominance results for the lowest 10% of the population. Comparing the unrestricted and restricted probabilities of dominance for 2002 over 2005 illustrates how striking the difference can be. The unrestricted probabilities did not allow us to make any statements about how welfare differs across years, since the probability of FSD, GLD and LD for 2002 over 2005 were 0, 0, and 0.0037, respectively. After restricting the range, however, these probabilities increased to 0.5330, 0.9754, and 1, respectively. Examining the probability curves for FSD and GLD in Figure 9 reveals that restricting the range to less than 10% would further increase the probability of dominance. Similar remarks can be made for GLD dominance of 2008 over 2005 where the probability increased from 0.000 to 0.591; see Figure 10.

### *6.1. Comparison of Bayesian and sampling theory results*

It is instructive to compare the Bayesian posterior probabilities with some sampling theory results and highlight any differences that may emerge. In Table 6 we report the probabilities of dominance alongside the  $p$ -values obtained using the Barrett and Donald (2003) tests for FSD and GLD and the Barrett et al. (2014) test for LD. The  $p$ -values are those obtained assuming a null hypothesis of dominance is true. In terms of the general conclusions about dominance that are likely to be made from each of the approaches, there is agreement in most cases. However, there are some glaring differences that deserve closer investigation.

Comparing 2005 and 2008, we find that, for FSD and GLD, the sampling theory  $p$ -values (0.8907 and 0.7610 for 2005 dominating 2008 and 0.0011 and 0.0000 for 2008

dominating 2005) would lead us to conclude that 2005 dominates 2008. However, the Bayesian approach gives a probability of 0.9999 that neither year dominates.<sup>3</sup> Examining the probability curves in Figure 10, we find that the Bayesian result is largely driven by the poorest segment of the population. For FSD, ignoring the poorest 10% would give a dominance probability for 2005 over 2008 of approximately 0.5. Ignoring the poorest 10% and the middle 25-60% would increase this probability to 1. For GLD, ignoring the poorest 20% would give a dominance probability of approximately 0.85. A similar outcome occurs when examining FSD and GLD for 2008 and 2002, and 2005 and 2002. The Bayesian probabilities that neither dominates are almost 1, whereas the sampling theory  $p$ -values of 0.4264 and 0.4625 for FSD and 0.6100 and 0.6880 for SSD, along with zero values in the other direction, suggest that 2002 is dominated by both 2005 and 2008. Taking 2002 and 2005 as an example, the probability curves in Figure 9 suggest that the probability of 2005 dominating 2002 is 1 if we ignore the poorest 16% of the population for FSD and the poorest 36% for GLD. It is perhaps dangerous to make a general conclusion from just these few examples, but it appears that the sampling theory tests are likely to suggest dominance if there is a large range of population segments where the probability of dominance is 1, even if there are some limited ranges where the probability is close to 0.

The same issue can arise with LD. The  $p$ -values of 1 for 1999 dominating 2005 and 0 for 2005 dominating 1999 suggest 1999 is dominant, whereas the Bayesian probability that neither dominates is 0.8085. The probability curve in Figure 8 shows that it is the richest 10% of the population that caused the discrepancy. Ignoring them would increase the Bayesian probability of dominance to 1.

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<sup>3</sup> One should resist the temptation to treat the  $p$ -values as sampling theory probabilities of dominance. The  $p$ -value for GLD of 2005 over 2008 of 0.7610 is less than that for FSD which is 0.8907. The probability of GLD must be at least as great as the probability of FSD.

## 7. CONCLUSIONS

The development of statistical inference for assessing how income distributions have changed over time in what might be considered a desirable way has attracted a great deal of attention within the sampling theory framework. Hypothesis testing procedures have been developed for, among other things, Lorenz dominance, generalized Lorenz dominance and first-order stochastic dominance. This paper provides an alternative to the existing hypothesis tests, by defining a novel approach to assessing dominance relationships within a framework of Bayesian inference. A desirable feature of this framework is the reporting of results in terms of probabilities – a natural way to express our uncertainty. Furthermore, these probabilities can be provided for dominance in either direction, as well as the probability that dominance does not occur. They overcome the problem of giving favorable treatment to what is chosen as the null hypothesis in sampling theory inference. By employing a flexible gamma mixture model we minimize the sensitivity of dominance results to the chosen distribution. Additionally, we suggest an approach for calculating upper and lower bounds for the posterior probability of dominance, allowing inference to be made with greater confidence about the accuracy of the estimates. The methodology is applied to data for Indonesia for the years 1999, 2002, 2005, and 2008. In general, the dominance test results led us to conclude that the level of inequality increased from 1999 to 2005, and decreased from 2005 to 2008. There was also strong evidence of an increase in welfare from 1999 to 2002, but little evidence of improvements from 2002 to 2005 and from 2005 to 2008. The introduction of probability curves that trace how the probability of dominance at a particular population proportion changes as the population proportion changes enabled us to isolate segments of the population having the greatest impact on overall dominance and to explain seemingly contradictory outcomes from the Bayesian and sampling theory approaches. Future research will extend the framework to multivariate frameworks and to orderings of more than two distributions.

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#### APPENDIX A: GIBBS SAMPLER FOR PARAMETERS OF GAMMA MIXTURE

The gamma mixture density function is given by  $p(x | \mathbf{w}, \boldsymbol{\mu}, \mathbf{v}) = \sum_{k=1}^K w_k G(x | v_k, v_k / \mu_k)$ . Let  $Z_1, Z_2, \dots, Z_n$  be indicator variables such that  $Z_i = k$  when the  $i$ -th observation comes from the  $k$ -th component in the mixture. Then,  $P(Z_i = k | \mathbf{w}) = w_k$  for  $k = 1, 2, \dots, K$ , with  $w_k > 0$  and  $\sum_{k=1}^K w_k = 1$ . Conditional on  $Z_i = k$ , the distribution of  $x_i$  is  $G(v_k, v_k / \mu_k)$ . Following Wiper et al. (2001), we use the following prior distributions for each of the parameters:

$$p(\mathbf{w}) = D(\boldsymbol{\phi}) \propto w_1^{\phi_1 - 1} w_2^{\phi_2 - 1} \dots w_K^{\phi_K - 1} \quad (\text{Dirichlet})$$

$$p(v_k) \propto \exp\{-\theta v_k\}, \quad k = 1, 2, \dots, K \quad (\text{exponential})$$

$$p(\mu_k) = GI(\alpha_k, \beta_k) \propto \mu_k^{-(\alpha_k + 1)} \exp\left\{-\frac{\beta_k}{\mu_k}\right\}, \quad k = 1, 2, \dots, K \quad (\text{inverted gamma})$$

These prior densities combine nicely with the likelihood function and are sufficiently flexible to represent vague prior information which can be dominated by the sample data. Because we are interested in the gamma mixture distribution, not its individual components, identification of the components and ordering of the means or scale parameters are not required (Geweke, 2007). However, if identification of the components is desired, it can be imposed a priori using the restriction  $\mu_1 < \mu_2 < \dots < \mu_K$ .

The conditional posterior densities for use in a Gibbs sampler are:

$$1.) \quad P(Z_i = k | \mathbf{x}, \mathbf{w}, \mathbf{v}, \boldsymbol{\mu}) = \frac{p_{ik}}{p_{i1} + p_{i2} + \dots + p_{iK}} \quad \text{where} \quad p_{ik} = w_k \frac{(v_k / \mu_k)^{v_k}}{\Gamma(v_k)} x_i^{v_k - 1} \exp\left\{-\frac{v_k x_i}{\mu_k}\right\};$$

2.)  $p(\mathbf{w} | \mathbf{x}, \mathbf{z}, \mathbf{v}, \boldsymbol{\mu}) = D(\boldsymbol{\varphi} + \mathbf{n})$  where  $\mathbf{z} = (k_1, k_2, \dots, k_n)'$  are the realized components for each of the observations, and  $\mathbf{n} = (n_1, n_2, \dots, n_K)'$ , with  $n_k$  being the number of observations for which  $Z_i = k$ . Also,  $\sum_{k=1}^K n_k = n$ ;

3.)  $p(\boldsymbol{\mu}_k | \mathbf{x}, \mathbf{z}, \mathbf{w}, \mathbf{v}) = GI(\alpha_k + n_k v_k, \beta_k + S_k v_k)$  where  $S_k = \sum_{i:Z_i=k} x_i$ ;

4.)  $p(v_k | \mathbf{x}, \mathbf{z}, \mathbf{w}, \boldsymbol{\mu}) \propto \frac{v_k^{n_k v_k}}{[\Gamma(v_k)]^{n_k}} \exp\left\{-v_k \left(\theta + \frac{S_k}{\boldsymbol{\mu}_k} + n_k \log \boldsymbol{\mu}_k - \log P_k\right)\right\}$  with  $P_k = \prod_{i:Z_i=k} x_i$ .

This density is not a recognizable form and requires a Metropolis step. A candidate  $\tilde{v}_k^{(t+1)}$  is drawn from a gamma density  $G(r, r/v_k^{(t)})$  with mean equal to the previous draw  $v_k^{(t)}$  and is accepted with probability

$$\min \left\{ 1, \frac{p(\tilde{v}_k^{(t+1)} | \mathbf{x}, \mathbf{z}^{(t+1)}, \mathbf{w}^{(t+1)}, \boldsymbol{\mu}^{(t+1)}) p(\tilde{v}_k^{(t+1)}, v_k^{(t)})}{p(v_k^{(t)} | \mathbf{x}, \mathbf{z}^{(t+1)}, \mathbf{w}^{(t+1)}, \boldsymbol{\mu}^{(t+1)}) p(v_k^{(t)}, \tilde{v}_k^{(t+1)})} \right\}$$

where  $p(v_k^{(t)}, \tilde{v}_k^{(t+1)})$  is the gamma density used to generate  $\tilde{v}_k^{(t+1)}$ . The value of  $r$  is chosen by experimentation to give an acceptance rate of approximately 0.25 to 0.4.

#### APPENDIX B: ALGORITHM FOR OBTAINING GAMMA MIXTURE QUANTILES

1. Implement for each parameter draw  $m = 1, 2, \dots, M$ .
2. Generate a large number of draws from  $p(x | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$ , say 100,000, and sort them from lowest to highest. Let the vector of these draws be denoted by  $\mathbf{x}$  and the  $j$ th ordered value by  $x_j$ .
3. Find the 100,000 cumulative proportions  $F(\mathbf{x} | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$ .
4. Implement for proportions  $u_i, i = 1, 2, \dots, n$ .
5. For a given  $u_i$ , find the smallest  $j$ , call it  $j_i$ , for which  $F(x_{j_i} | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)}) \geq u_i$ .
6. Set a tolerable error  $\varepsilon$ . We used  $\varepsilon = 10^{-4}$  in our empirical work.

7. Find an initial value  $\hat{y}_1 = F^{-1}(u_i | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$  as a draw from the uniform distribution  $U(x_{j_{i-1}}, x_{j_i})$ .
8. Compute  $F_X(\hat{y}_1 | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)})$ . If  $|F_X(\hat{y}_1 | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)}) - u_i| > \varepsilon$ , go to step 9; otherwise, go back to 4. (Step 9 was seldom needed in our empirical work.)
9. We improve on the initial value  $\hat{y}_1$  as follows. If  $F_X(\hat{y}_1 | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)}) - u_i > 0$ , generate a random draw  $\hat{y}_2 = U(x_{j_{i-1}}, \hat{y}_1)$ . If  $F_X(\hat{y}_1 | \boldsymbol{\mu}^{(m)}, \mathbf{v}^{(m)}, \mathbf{w}^{(m)}) - u_i < 0$ , generate  $\hat{y}_2 = U(\hat{y}_1, x_{j_i})$ .
10. Repeat step 8 with the new  $\hat{y}$ .
11. Go back to step 1.

Table 1: Sample Statistics (in Rp '000 per month)

	1999	2002	2005	2008
Sample mean	332.63	432.36	477.04	454.08
Median	270.40	337.80	357.95	355.27
Minimum	44.12	57.65	38.32	59.84
Maximum	5973.92	24,902.67	30,216.51	13,181.99
Standard deviation	249.32	477.29	511.19	401.23
Gini Coefficient	0.3184	0.3509	0.3797	0.3583
Sample size	25,175	29280	24,687	26,648
Proportion more than 2000	0.00242	0.0074	0.0134	0.0099

Table 2: Goodness-of-fit comparisons

	1999						
	$K=1$	$K=2$	$K=3$	$K=4$	$K=5$	Dagum	S-M
<i>RMSE</i>	0.0478	0.0140	0.0059	0.0037	0.0080	0.0084	0.0125
<i>MAE</i>	0.0414	0.0120	0.0050	0.0030	0.0078	0.0076	0.0111
<i>d</i>	0.0766	0.0258	0.0199	0.0083	0.0119	0.0130	0.0197
2002							
<i>RMSE</i>	0.0559	0.0189	0.0065	0.0046	0.0086	0.0079	0.0117
<i>MAE</i>	0.0491	0.0157	0.0055	0.0037	0.0081	0.0069	0.0103
<i>d</i>	0.0853	0.0333	0.0116	0.0107	0.0131	0.0128	0.0198
2005							
<i>RMSE</i>	0.0556	0.0172	0.0046	0.0023	0.0086	0.0079	0.0122
<i>MAE</i>	0.0488	0.0145	0.0040	0.0018	0.0081	0.0067	0.0106
<i>d</i>	0.0864	0.0325	0.0093	0.0062	0.0132	0.0151	0.0209
2008							
<i>RMSE</i>	0.0472	0.0156	0.0047	0.0026	0.0034	0.0112	0.0133
<i>MAE</i>	0.0414	0.0134	0.0041	0.0021	0.0030	0.0097	0.0114
<i>d</i>	0.0726	0.0269	0.0086	0.0064	0.0068	0.0191	0.0224

Table 3: Posterior means and standard deviations (in parenthesis) of mixture parameter estimates and mixture weights

		Component			
		1	2	3	4
1999	$\mu$	176.5033 (4.8098)	294.4166 (10.2637)	515.4174 (25.5746)	1296.5664 (138.7104)
	$w$	0.2497 (0.0366)	0.4976 (0.0336)	0.2361 (0.0332)	0.0166 (0.0036)
	$\nu$	13.9886 (1.1937)	8.1546 (0.5342)	5.0413 (0.4295)	2.1930 (0.3500)
2002	$\mu$	234.8722 (3.8026)	430.4595 (8.7793)	918.7395 (39.9731)	4141.7860 (637.1302)
	$w$	0.3466 (0.0256)	0.5383 (0.0245)	0.1105 (0.0111)	0.0045 (0.0009)
	$\nu$	9.9992 (0.4361)	6.1073 (0.2747)	3.4621 (0.2771)	1.2346 (0.2343)
2005	$\mu$	232.9290 (4.5749)	441.1039 (11.6864)	901.9546 (57.0056)	2557.9964 (313.5324)
	$w$	0.3073 (0.0281)	0.5277 (0.0308)	0.1492 (0.0200)	0.0157 (0.0038)
	$\nu$	9.2294 (0.4858)	5.5929 (0.3738)	3.3478 (0.3371)	1.5555 (0.2293)
2008	$\mu$	198.3503 (5.7395)	361.0210 (15.6557)	625.4886 (41.1883)	1411.7252 (95.0639)
	$w$	0.2014 (0.0264)	0.4674 (0.0424)	0.2800 (0.0468)	0.0512 (0.0079)
	$\nu$	12.6651 (0.9888)	7.4202 (0.5833)	5.2249 (0.5254)	2.2095 (0.1686)

Table 4: Dominance probabilities, with bounds in parentheses <sup>a</sup>

	08 D 05	05 D 08				
FSD	0.0000 (0.0000,0.0000)	0.0001 (0.0000,0.0004)				
GLD	0.0000 (0.0000,0.0000)	0.0001 (0.0000,0.0005)				
LD	1.0000 (0.9997,1.0000)	0.0000 (0.0000,0.0000)				
	08 D 02	02 D 08	05 D 02	02 D 05		
FSD	0.0000 (0.0000,0.0001)	0.0000 (0.0000,0.0000)	0.0001 (0.0000,0.0004)	0.0000 (0.0000,0.0000)		
GLD	0.0148 (0.0117,0.0183)	0.0000 (0.0000,0.0000)	0.0002 (0.0000,0.0006)	0.0000 (0.0000,0.0000)		
LD	0.0000 (0.0000,0.0000)	0.0000 (0.0000,0.0000)	0.0000 (0.0000,0.0000)	0.0037 (0.0021,0.0056)		
	08 D 99	99 D 08	05 D 99	99 D 05	02 D 99	99 D 02
FSD	0.9980 (0.9966,0.9991)	0.0000 (0.0000,0.0000)	0.2529 (0.2432,0.2624)	0.0000 (0.0000,0.0000)	0.9943 (0.9923,0.9962)	0.0000 (0.0000,0.0000)
GLD	0.9997 (0.9989,1.0000)	0.0000 (0.0000,0.0000)	0.4676 (0.4595,0.4772)	0.0000 (0.0000,0.0000)	0.9990 (0.9979,0.9998)	0.0000 (0.0000,0.0000)
LD	0.0000 (0.0000,0.0000)	0.3331 (0.3247,0.3421)	0.0000 (0.0000,0.0000)	0.1915 (0.1829,0.2020)	0.0000 (0.0000,0.0000)	0.9996 (0.9988,1.0000)

<sup>a</sup> For each pairwise comparison, dominance probabilities for both directions are reported. These probabilities can be used to find the probability that neither distribution is dominant. For example, when comparing 2002 with 2008 for FSD, there is a probability of one that neither dominates. When comparing 1999 with 2005 for FSD, there is a probability of  $(1-0.2529) = 0.7471$  that neither dominate.

Table 5: Restricted dominance probabilities (for the lowest 10% of incomes), with bounds in parenthesis

	08 D 05	05 D 08				
FSD	0.0108 (0.0081,0.0136)	0.0001 (0.0000,0.0005)				
GLD	0.5910 (0.5796,0.6000)	0.0001 (0.0000,0.0005)				
LD	1.0000 (0.9999,1.0000)	0.0000 (0.0000,0.0000)				
	08 D 02	02 D 08	05 D 02	02 D 05		
FSD	0.0027 (0.0014,0.0042)	0.1378 (0.1306,0.1447)	0.0002 (0.0000,0.0006)	0.5330 (0.5250,0.5419)		
GLD	0.0288 (0.0240,0.0330)	0.1297 (0.1229,0.1360)	0.0002 (0.0000,0.0006)	0.9754 (0.9710,0.9788)		
LD	0.0000 (0.0000,0.0000)	0.8426 (0.8357,0.8506)	0.0000 (0.0000,0.0000)	1.0000 (1.0000,1.0000)		
	08 D 99	99 D 08	05 D 99	99 D 05	02 D 99	99 D 02
FSD	0.9980 (0.9966,0.9991)	0.0000 (0.0000,0.0000)	0.2529 (0.2432,0.2624)	0.0000 (0.0000,0.0000)	0.9943 (0.9923,0.9962)	0.0000 (0.0000,0.0000)
GLD	0.9997 (0.9989,1.000)	0.0000 (0.0000,0.0000)	0.4676 (0.4595,0.4772)	0.0000 (0.0000,0.0000)	0.9999 (0.9979,0.9998)	0.0000 (0.0000,0.0000)
LD	0.0000 (0.0000,0.0000)	1.0000 (1.0000,1.0000)	0.0000 (0.0000,0.0000)	1.0000 (1.0000,1.0000)	0.0000 (0.0000,0.0000)	1.0000 (1.0000,1.0000)

Table 6: A comparison of sampling theory  $p$ -values with stochastic dominance probabilities

		08 D 05	05 D 08				
FSD	probability	0.0000	0.0001				
	$p$ -value	0.0011	0.8907				
GLD	probability	0.0000	0.0001				
	$p$ -value	0.0000	0.7610				
LD	probability	1.0000	0.0000				
	$p$ -value	1.0000	0.0000				
		08 D 02	02 D 08	05 D 02	02 D 05		
FSD	probability	0.0000	0.0000	0.0001	0.0000		
	$p$ -value	0.4264	0.0000	0.4625	0.0000		
GLD	probability	0.0148	0.0000	0.0002	0.0000		
	$p$ -value	0.6100	0.0000	0.6480	0.0000		
LD	probability	0.0000	0.0000	0.0000	0.0037		
	$p$ -value	0.0930	0.0090	0.0000	0.2590		
		08 D 99	99 D 08	05 D 99	99 D 05	02 D 99	99 D 02
FSD	probability	0.9980	0.0000	0.2529	0.0000	0.9943	0.0000
	$p$ -value	0.9713	0.0000	0.9988	0.0000	0.9580	0.0000
GLD	probability	0.9997	0.0000	0.4676	0.0000	0.9990	0.0000
	$p$ -value	1.0000	0.0000	0.8590	0.0000	0.9990	0.0000
LD	probability	0.0000	0.3331	0.0000	0.1915	0.0000	0.9996
	$p$ -value	0.0000	0.9970	0.0000	1.0000	0.0000	0.9940

Figure 1: Histogram and estimated density with parameters set at posterior means for 1999

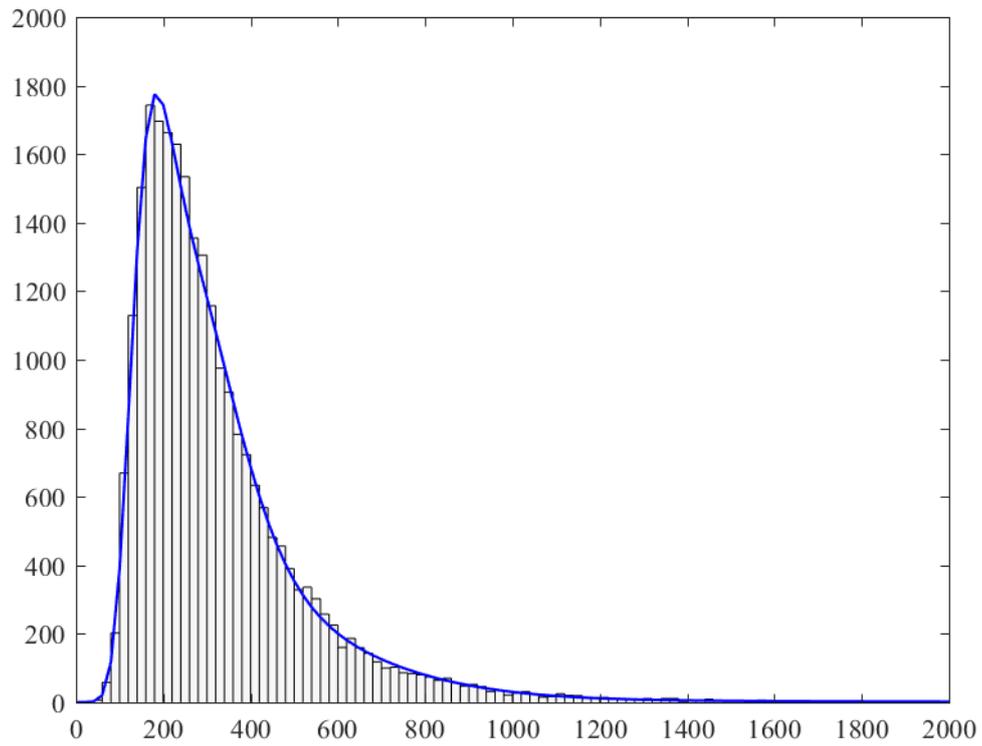


Figure 2: Histogram and estimated density with parameters set at posterior means for 2002

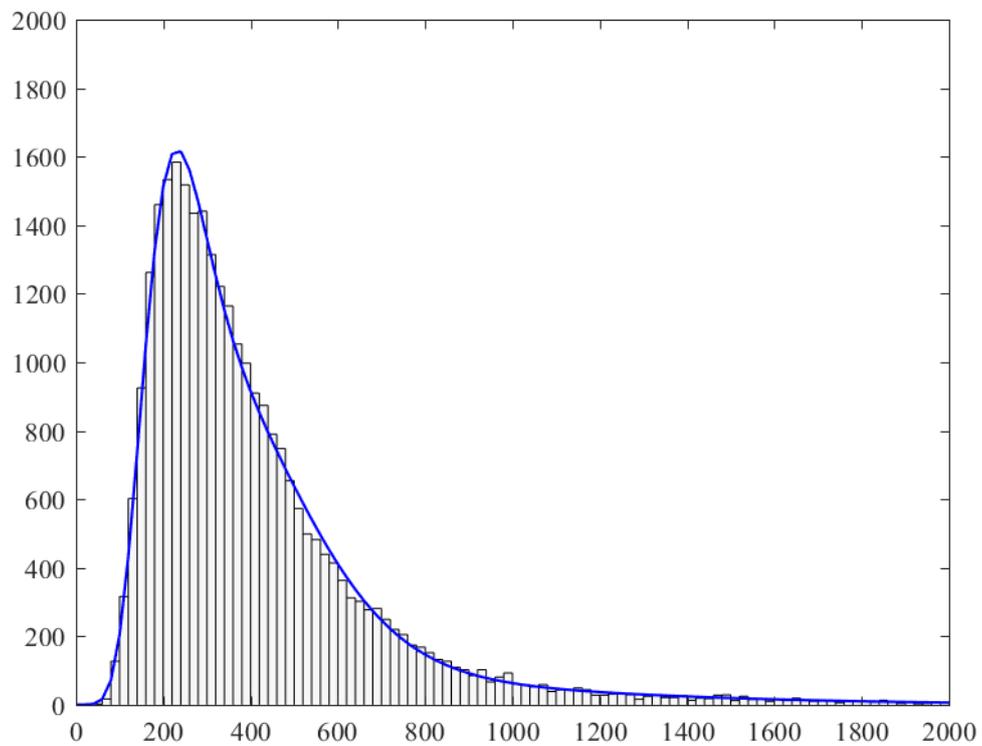


Figure 3: Histogram and estimated density with parameters set at posterior means for 2005

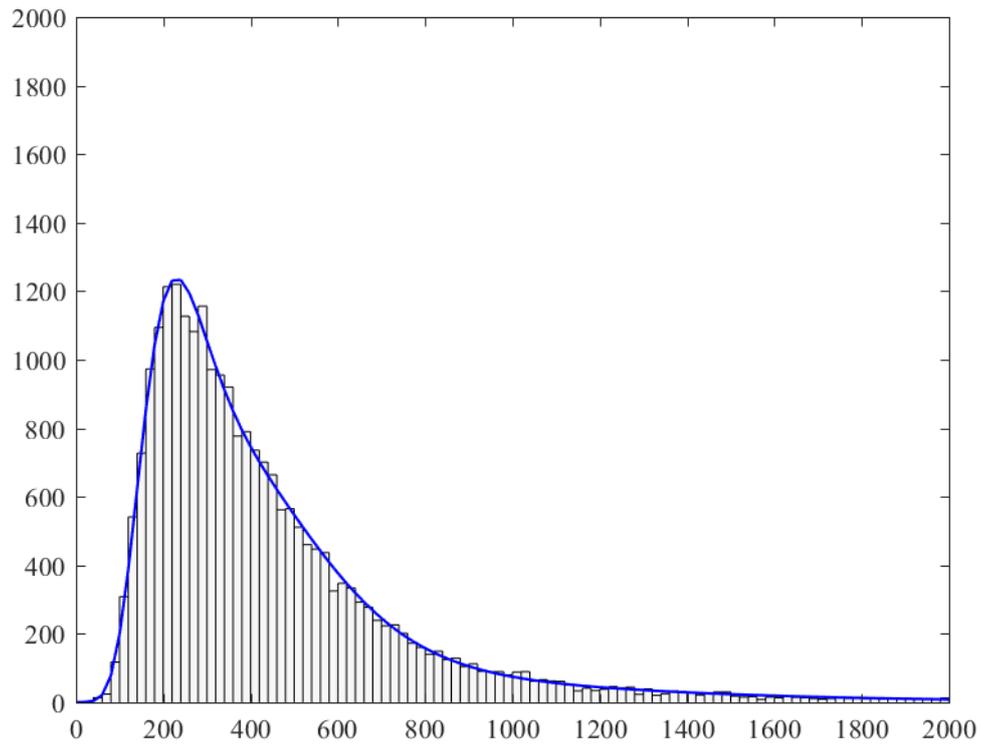


Figure 4: Histogram and estimated density with parameters set at posterior means for 2008

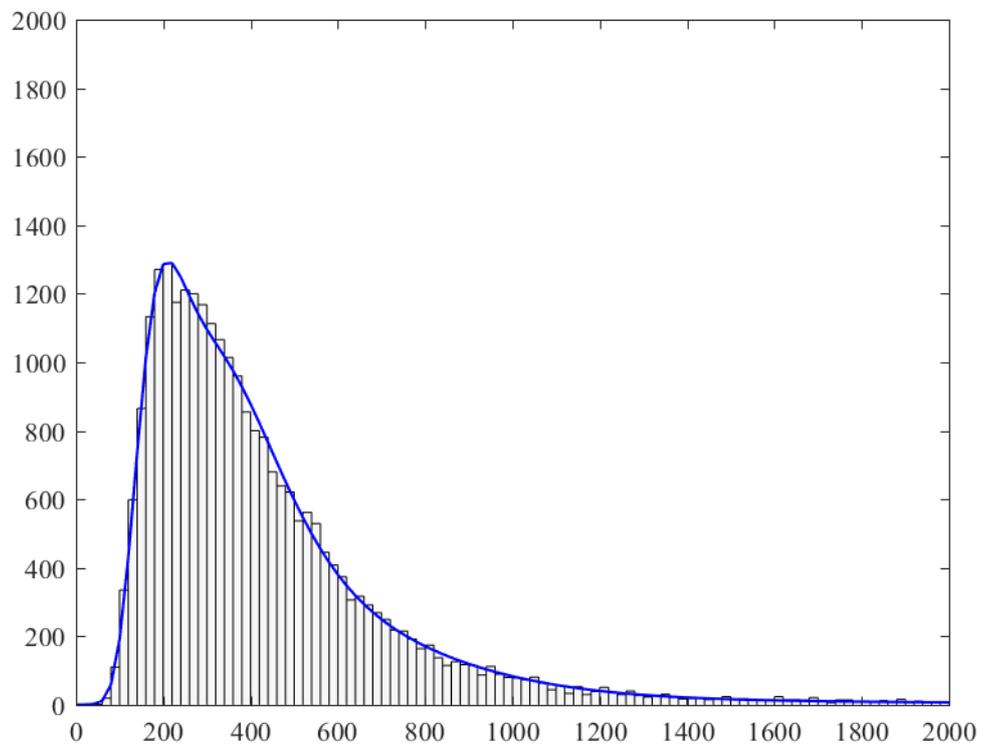


Figure 5: Estimated distribution functions with parameters set at posterior means

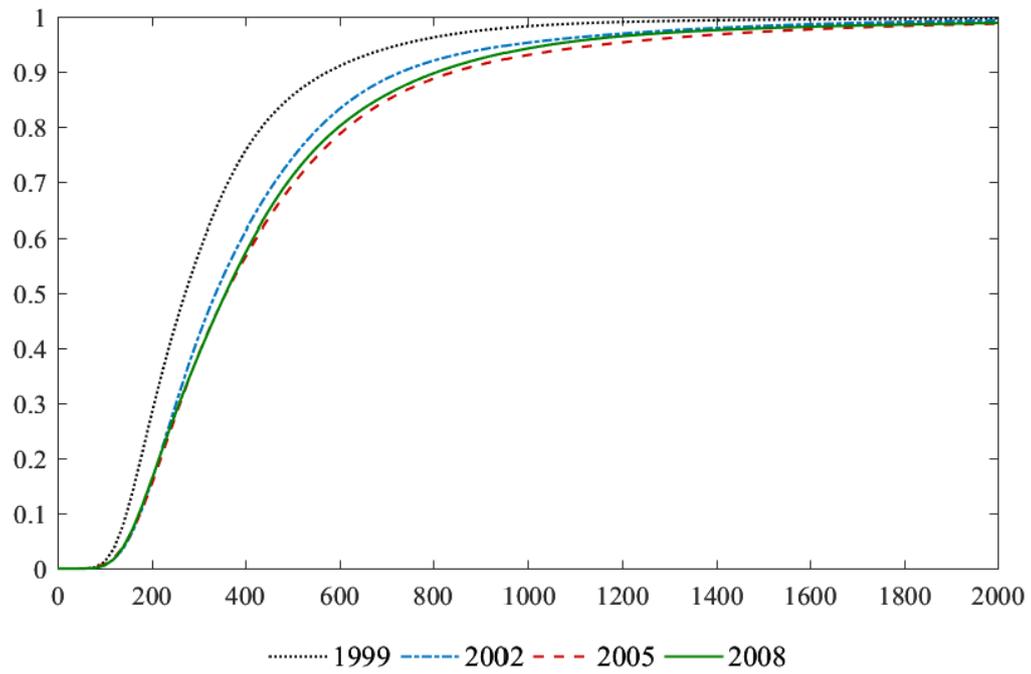


Figure 6: Estimated generalized Lorenz curves with parameters set at posterior means

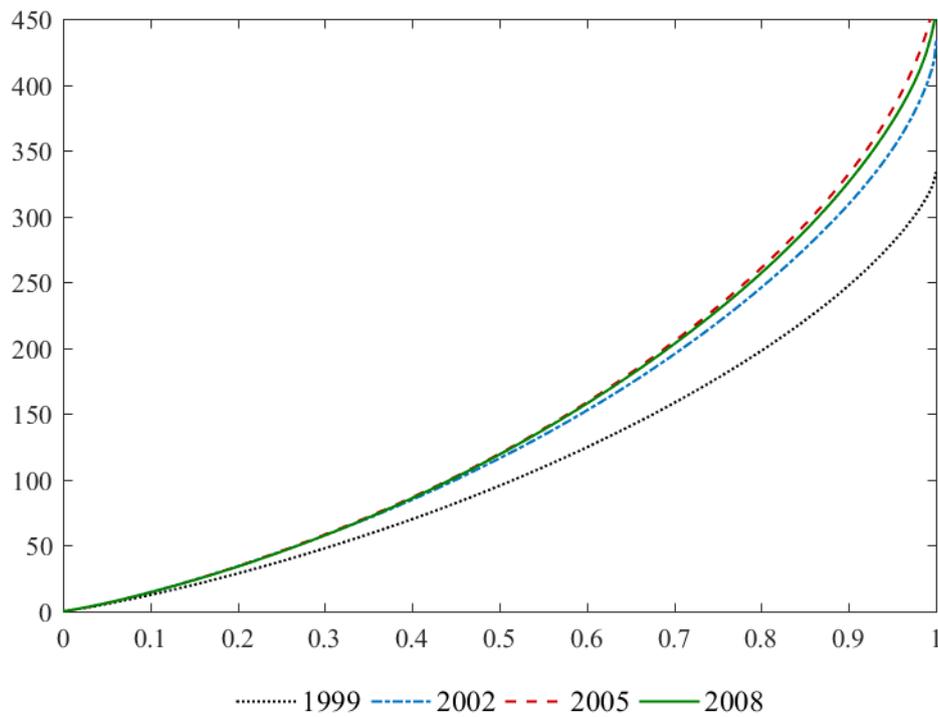


Figure 7: Estimated Lorenz curves with parameters set at posterior means

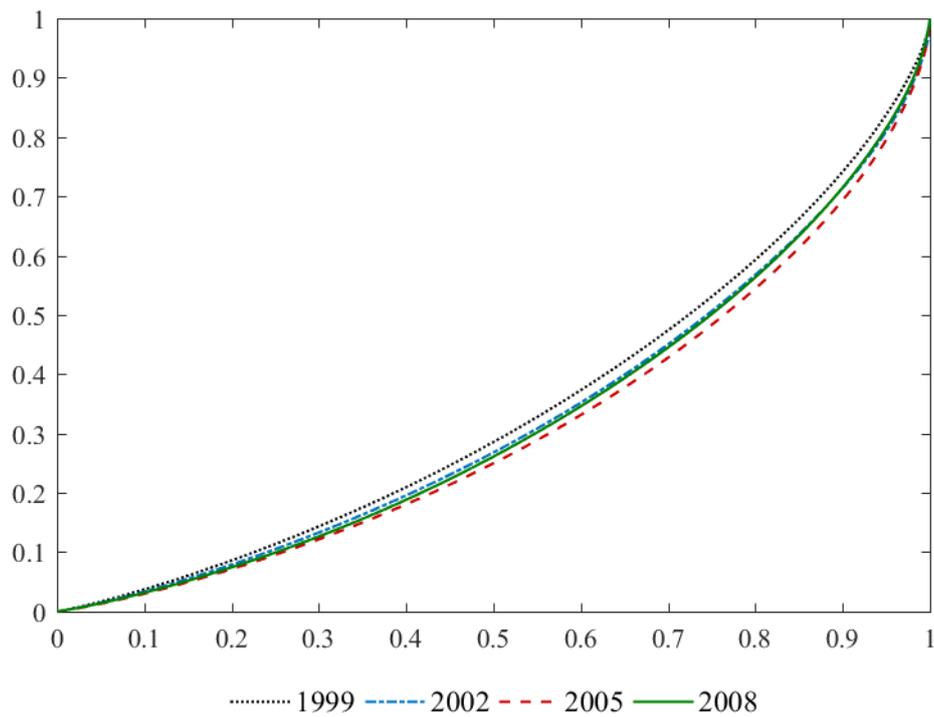


Figure 8: Estimated probability curve of FSD, GLD, and LD for the comparison of 1999 and 2005, with their lower and upper bound

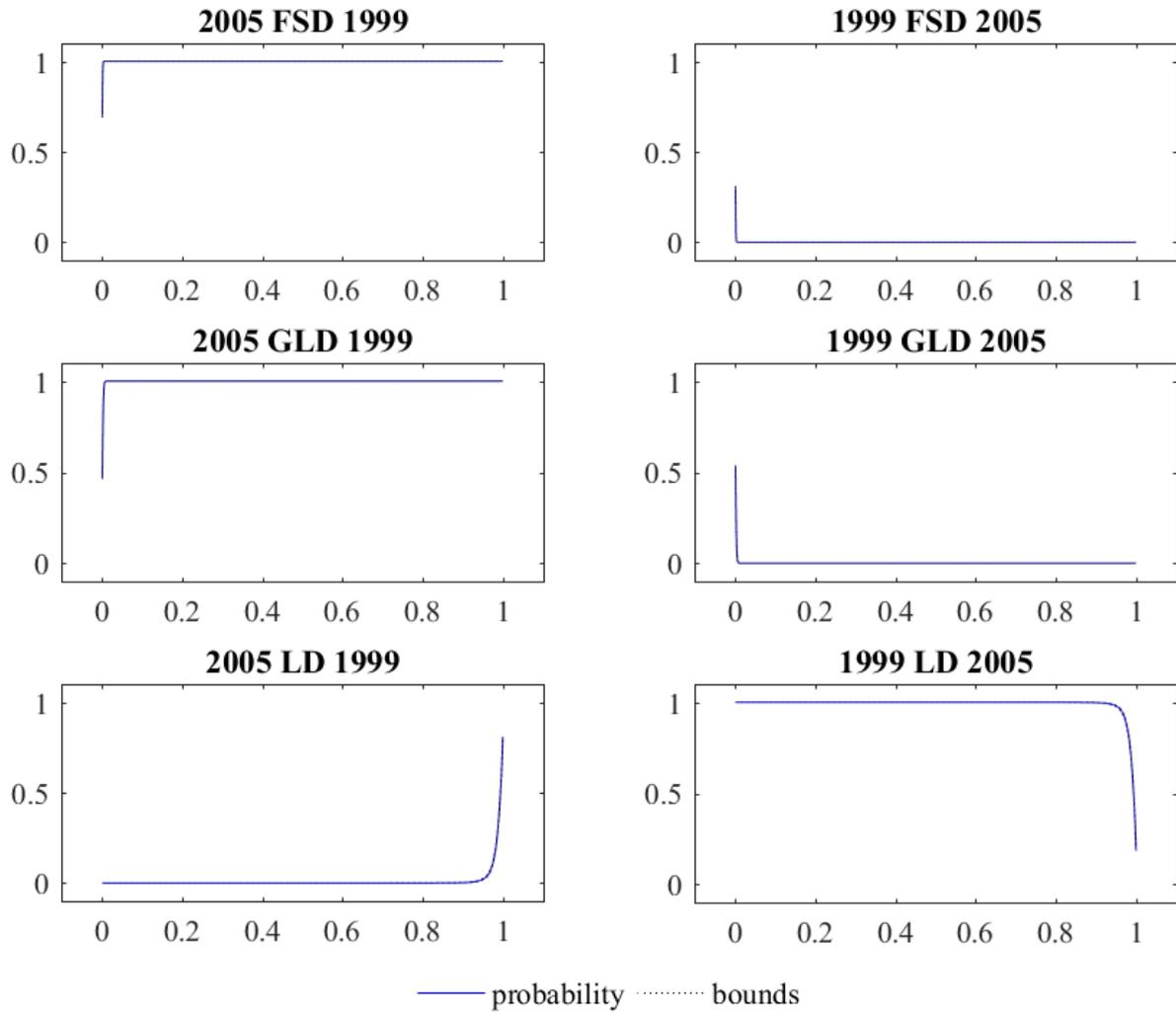


Figure 9: Estimated probability curve of FSD, GLD, and LD for the comparison of 2002 and 2005, with their lower and upper bound

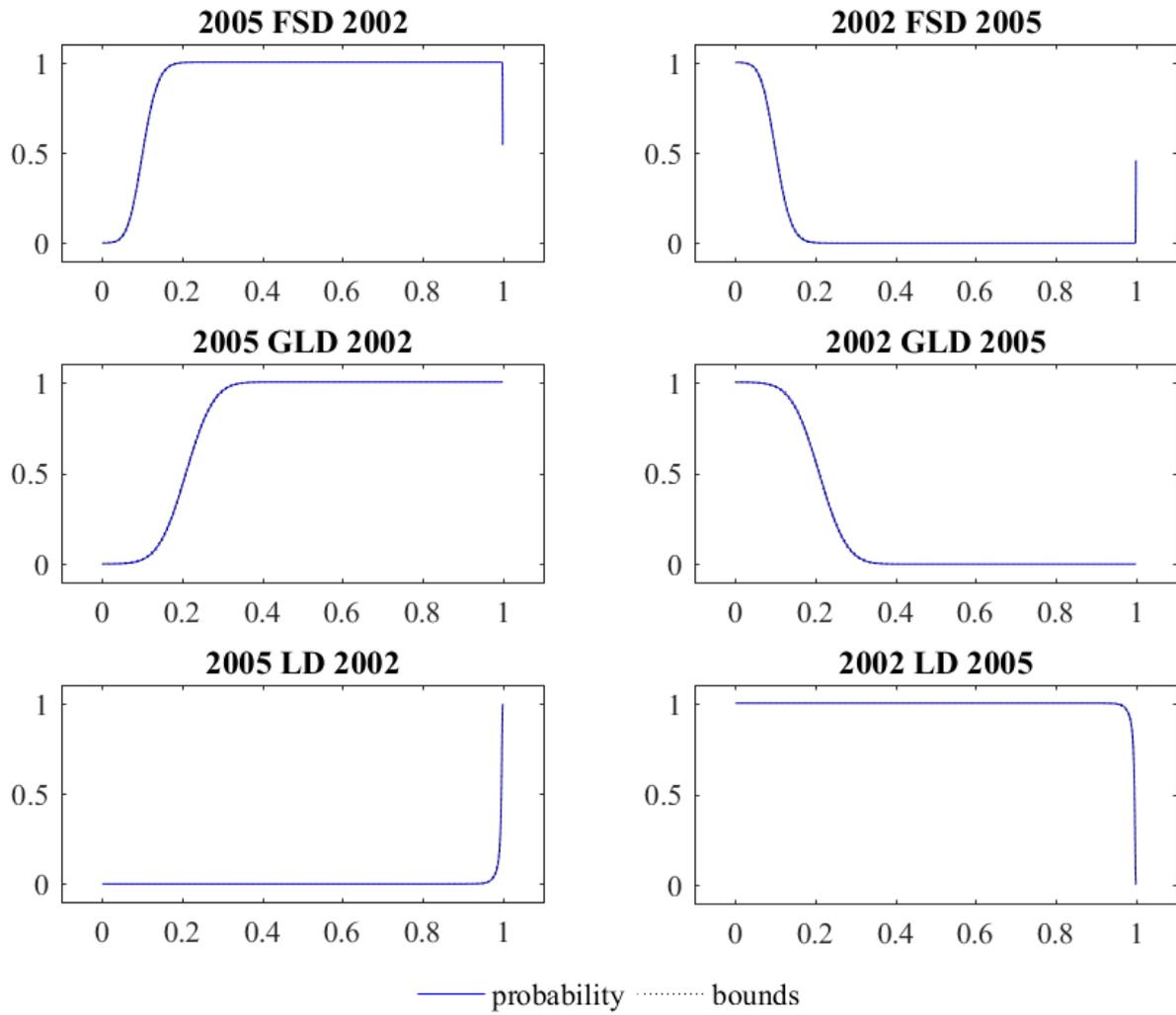


Figure 10: Estimated probability curve of FSD, GLD, and LD for the comparison of 2005 and 2008, with their lower and upper bound

