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Abstract

In the context of a general regression model in which some regression coefficients are of interest and others are purely nuisance parameters, we derive the density function of a maximal invariant statistic with the aim of testing for the inclusion of regressors (either linear or non-linear) in linear or semi-linear models. This allows the construction of the locally best invariant test, which in two important cases is equivalent to the one-sided *t* test for a regression coefficient in an artificial linear regression model.

Key words: Invariance; linear regression model; locally best invariant test; non-linear regression model; nuisance parameters; *t* test.

JEL CLASSIFICATION: C2, C12

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1. Introduction

Statistical models and particularly those used by econometricians, involve a large number of influences. These kinds of models contain two types of parameters, those of interest and those not of immediate interest that are known as nuisance parameters. Their presence can cause unexpected complications for statistical inference. Kalbfleisch and Sprott (1970) discussed methods of eliminating nuisance parameters from the likelihood function so that inference can be made about the parameters of interest. In this paper, we use invariance arguments in order to deal with nuisance parameters and derive maximal invariant likelihoods for semi-linear regression models with the aim of testing for the inclusion of regressors (either linear or non-linear) in linear or semi-linear models.

In practice, many statistical problems including testing of hypotheses, display symmetries, which impose additional restrictions on the choice of an appropriate statistical test. Among others, Lehmann (1959a, 1959b, 1986) suggested the use of invariance arguments to overcome the problem of nuisance parameters. The idea behind invariance is that if the hypothesis testing problem under consideration has a particular invariance property, then we should restrict attention to only those tests that share this invariance property. The class of all invariant functions can be obtained as the totality of functions of a maximal invariant. A maximal invariant is a statistic which takes the same value for the observed data vectors that are connected by transformations and different values for those data vectors that are not connected by transformations. Consequently any invariant test statistic can be written as a function of the maximal invariant. This means we can treat the maximal invariant as the observed data, find its density and then construct appropriate tests based on this density.

The aim of this paper is to derive the density function of the maximal invariant statistic in the context of the general regression model and then construct a locally best invariant (LBI) test for a non-linear regressor.

The plan of this paper is as follows. First we derive the density function of the maximal invariant statistic for the general regression model in Section 2. In Section 3, we construct the LBI test statistic for a non-linear regressor using the density function. Finally, some concluding remarks are made in Section 4.

2. Derivation of the density function

Consider the model,

$$y = X_1\beta_1 + g(X_2, \beta_2) + u \tag{2.1}$$

where X_1 is an $n \times q$ nonstochastic matrix, X_2 is an $n \times p$ nonstochastic matrix and $g(X_2, \beta_2)$ is an $n \times 1$ known function of β_2 and X_2. Note $g(X_2, \beta_2) = X_2\beta_2$ is the special case of the linear regression model.

Our interest is in testing $H_0{:}\beta_2 = 0$ against $H_a{:}\beta_2 > 0$ in the context of the above general regression model. It is assumed that $u \sim N(0, \sigma^2 I_n)$ where σ^2 is unknown. This testing problem is invariant under the class of transformations

$$y \to \gamma_0 y + X_1\gamma \tag{2.2}$$

where γ_0 is a positive scalar and γ is a $q \times 1$ vector.

Let $M_1 = I - X_1(X_1'X_1)^{-1}X_1'$ and P be any $m \times n$ matrix such that $PP' = I_m$, $P'P = M_1$ where $m = n - q$. Multiplying both sides of (2.1) by PM_1 and noting that $PM_1 = P$ and $M_1X_1 = 0$ we get

$$Py = Pg(X_2, \beta_2) + Pu. \tag{2.3}$$

Thus $Py \sim N(Pg(X_2, \beta_2), \sigma^2 I_m)$. Let $z = Py$. Note that $w = z/(z'z)^{1/2}$ is a maximal invariant statistic (see King, 1980). Our aim is to find its density function.

Let $r^2 = z'z$ be the usual squared distance of z from the origin. Now, we change z to the *m*-dimensional polar co-ordinates $(r, \theta_1, \theta_2, \ldots, \theta_{m-1})$ via

$$\begin{aligned} z_1 &= r\cos\theta_1, \\ z_j &= r\Big(\prod_{k=1}^{j-1}\sin\theta_k\Big)\cos\theta_j;\ \text{ for } 2 \le j \le m-1, \\ z_m &= r\prod_{k=1}^{m-1}\sin\theta_k, \end{aligned} \tag{2.4}$$

where $0 \le r \le \infty$, $0 \le \theta_k \le \pi$, for $k = 1, 2, \ldots, (m-2)$ and $0 \le \theta_{m-1} \le 2\pi$.

The Jacobian of this transformation is

$$J_m(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) = \left|\frac{\partial(z_1, z_2, \ldots, z_m)}{\partial(r, \theta_1, \theta_2, \ldots, \theta_{m-1})}\right| = r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k^{\,m-1-k}$$

(Miller, 1964, p.13). Observe that $z = rw$.

The joint density function of z after the above change of variables, becomes

$$f(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) = (2\pi\sigma^2)^{-m/2}\exp\{-\frac{1}{2\sigma^2}(r^2 w'w - 2w'rPg(X_2, \beta_2) + g'(X_2, \beta_2)P'Pg(X_2, \beta_2)\}r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k^{\,m-1-k}. \tag{2.5}$$

To find the density function of w, we first must find the marginal density function of $(\theta_1, \theta_2, \ldots, \theta_{m-1})$. This can be obtained by integrating out r in (2.5),

$$f(\theta_1,\theta_2,\ldots,\theta_{m-1}) = (2\pi\sigma^2)^{-m/2}\int_0^\infty \exp\{-\frac{1}{2\sigma^2}(r^2 w'w - 2w'rPg(X_2,\beta_2) +$$

$$g'(X_2,\beta_2)P'Pg(X_2,\beta_2)\}\; r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}dr\;.$$

$$= (2\pi)^{-m/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\;\exp(b(w,\beta_2))$$

$$\int_0^\infty \exp\{-\frac{1}{2}(\lambda - a(w,\beta_2))^2\}\lambda^{m-1}d\lambda$$

where

$$a(w,\beta_2) = w'Pg^*(X_2,\beta_2)\,, \tag{2.6}$$

$$b(w,\beta_2) = \frac{1}{2}\{g^{*\prime}(X_2,\beta_2)P'ww'Pg^*(X_2,\beta_2) - g^{*\prime}(X_2,\beta_2)P'Pg^*(X_2,\beta_2)\}\,, \tag{2.7}$$

$$g^*(X_2,\beta_2) = \frac{g(X_2,\beta_2)}{\sigma}\,, \tag{2.8}$$

$\lambda = \dfrac{r}{\sigma}$ and $w'w = 1$.

Using Mathematica (Wolfram, (1993)), we found that

$$\int_0^\infty \exp\{-\frac{1}{2}(\lambda - a(w,\beta_2))^2\}\lambda^{m-1}d\lambda$$

$$= \{\Gamma(\frac{m}{2})1F1[\frac{m}{2},\frac{1}{2},\frac{a^2(w,\beta_2)}{2}] + \sqrt{2}a(w,\beta_2)\Gamma(\frac{1+m}{2})$$

$$1F1[\frac{1+m}{2},\frac{3}{2},\frac{a^2(w,\beta_2)}{2}]\}\{2^{\frac{m}{2}-1}\exp(-\frac{a^2(w,\beta_2)}{2})\},$$

where 1F1[.,.,.] is the confluent hypergeometric function, which has the form

$$1F1[c,d,\delta] = 1 + \frac{c\delta}{d} + \frac{c(c+1)}{d(d+1)}\frac{\delta^2}{2!} + \ldots = \sum_{k=0}^{\infty}\frac{(c)_k}{(d)_k}\frac{\delta^k}{k!}\,. \tag{2.9}$$

Therefore the marginal density function of $\theta_1, \theta_2, \ldots, \theta_{m-1}$ is

$$f(\theta_1, \theta_2, \ldots, \theta_{m-1}) = \frac{1}{2}\Gamma(\frac{m}{2})\pi^{-m/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\exp\{c(w,\beta_2)\}$$

$$\{1F1[\frac{m}{2},\frac{1}{2},\frac{a^2(w,\beta_2)}{2}] + \sqrt{2}a(w,\beta_2)\eta 1F1[\frac{1+m}{2},\frac{3}{2},\frac{a^2(w,\beta_2)}{2}]\}, \quad (2.10)$$

where

$$c(w,\beta_2) = b(w,\beta_2) - \frac{a^2(w,\beta_2)}{2} = -\frac{1}{2}g^{*\prime}(X_2,\beta_2)M_1 g^*(X_2,\beta_2) \qquad (2.11)$$

and

$$\eta = \frac{\Gamma(\frac{1+m}{2})}{\Gamma(\frac{m}{2})}. \qquad (2.12)$$

Consequently the density function of w is,

$$f(w) = \frac{1}{2}\Gamma(\frac{m}{2})\pi^{-\frac{m}{2}}\exp\{c(w,\beta_2)\}\ \{1F1[\frac{m}{2},\frac{1}{2},\frac{a^2(w,\beta_2)}{2}] +$$

$$\sqrt{2}a(w,\beta_2)\eta 1F1[\frac{1+m}{2},\frac{3}{2},\frac{a^2(w,\beta_2)}{2}]\}, \qquad (2.13)$$

where $c(w,\beta_2)$ and η are defined by (2.11) and (2.12). Using this density function we can construct the LBI test statistic for testing $H_0: \beta_2 = 0$.

3. Construction of the test

We are interested in testing $H_0 : \beta_2 = 0$ against $H_a : \beta_2 > 0$ in the context of (2.1). Our interest is in the case of $p = 1$, i.e. where β_2 is a scalar. An LBI test of H_0 against H_a is that with critical region of the form

$$\left.\frac{\partial \log f(w)}{\partial \beta_2}\right|_{\beta_2 = 0} \geq c_\alpha \qquad (3.1)$$

provided the left hand side is not a constant (see Ferguson, 1967, King and Hiller, 1985 and Wu and King, 1994).

Taking logs on both sides of (2.13) we get

$$\log f(w) = -\log 2 - \frac{m}{2}\log \pi - \frac{1}{2} g^{*\prime}(X_2, \beta_2) M_1 g^*(X_2, \beta_2) +$$

$$\log\{1\mathrm{F}1[\frac{m}{2}, \frac{1}{2}, \frac{a^2(w,\beta_2)}{2}] + \sqrt{2}a(w,\beta_2)\eta 1\mathrm{F}1[\frac{1+m}{2}, \frac{3}{2}, \frac{a^2(w,\beta_2)}{2}]\}.$$

Now

$$\frac{\partial \log f(w)}{\partial \beta_2} = -\frac{\partial}{\partial \beta_2}(g^*(X_2,\beta_2)') M_1 g^*(X_2,\beta_2) + \{1\mathrm{F}1[\frac{m}{2}, \frac{1}{2}, \frac{a^2(w,\beta_2)}{2}] +$$

$$\sqrt{2}a(w,\beta_2)\eta 1\mathrm{F}1[\frac{1+m}{2}, \frac{3}{2}, \frac{a^2(w,\beta_2)}{2}]\}^{-1}\{m\ 1\mathrm{F}1[\frac{m+2}{2}, \frac{3}{2}, \frac{a^2(w,\beta_2)}{2}]$$

$$\frac{\partial}{\partial \beta_2}\left(\frac{a^2(w,\beta_2)}{2}\right) + \sqrt{2}\eta \frac{\partial}{\partial \beta_2}\left(a(w,\beta_2)\right) 1\mathrm{F}1[\frac{1+m}{2}, \frac{3}{2}, \frac{a^2(w,\beta_2)}{2}] +$$

$$\sqrt{2}a(w,\beta_2)\eta \frac{1+m}{3} 1\mathrm{F}1[\frac{m+3}{2}, \frac{5}{2}, \frac{a^2(w,\beta_2)}{2}] \frac{\partial}{\partial \beta_2}\left(\frac{a^2(w,\beta_2)}{2}\right)\}$$

$$= -\frac{2}{\sigma^2}\ h(X_2,\beta_2)M_1 g(X_2,\beta_2) + \{1\text{F}1[\frac{m}{2},\frac{1}{2},\frac{a^2(w,\beta_2)}{2}] +$$

$$\sqrt{2}a(w,\beta_2)\eta 1\text{F}1[\frac{1+m}{2},\frac{3}{2},\frac{a^2(w,\beta_2)}{2}]\}^{-1}\{m\ 1\text{F}1[\frac{m+2}{2},\frac{3}{2},\frac{a^2(w,\beta_2)}{2}]$$

$$\frac{1}{\sigma^2} w'Pg(X_2,\beta_2)(w'P)'\ h(X_2,\beta_2) + \frac{\sqrt{2}}{\sigma}\eta(w'P\ h(X_2,\beta_2))'$$

$$1\text{F}1[\frac{1+m}{2},\frac{3}{2},\frac{a^2(w,\beta_2)}{2}] + \sqrt{2}a(w,\beta_2)\eta\frac{1+m}{3}1\text{F}1[\frac{m+3}{2},\frac{5}{2},\frac{a^2(w,\beta_2)}{2}]$$

$$\frac{1}{\sigma^2} w'Pg(X_2,\beta_2)P'w\ h(X_2,\beta_2) \tag{3.2}$$

where $h(X_2,\beta_2) = \frac{\partial}{\partial\beta_2}\left(g(X_2,\beta_2)'\right)$.

We have following important special cases of $\frac{\partial \log f(w)}{\partial\beta_2}$ when $g(X_2,\beta_2)$ is evaluated under $\beta_2 = 0$.

Case 1: When $g(X_2,\beta_2)\big|_{\beta_2=0} = 0$ but $h(X_2,0)$ is non-zero.

If we evaluate (3.2) at $\beta_2 = 0$ noting that $g(X_2,\beta_2)\big|_{\beta_2=0} = 0$, we have

$$\left.\frac{\partial \log f(w)}{\partial\beta_2}\right|_{\beta_2=0} = \frac{\sqrt{2}}{\sigma}\eta\ w'Ph(X_2,0)\ .$$

Hence the LBI test rejects H_0 for

$$s = w'Ph(X_2,0) \ge d_\alpha \tag{3.3}$$

where d_α is an appropriate critical value. To understand how this test might be best applied in practice, consider testing $H_0{:}\beta_2 = 0$ against $H_a{:}\beta_2 > 0$ in the artificial regression model

$$y = X_1\beta_1 + \beta_2 h(X_2,0) + u\,. \tag{3.4}$$

The OLS estimator of β_2 is

$$\hat{\beta}_2 = (h(X_2,0)' M_1 h(X_2,0))^{-1} h(X_2,0)' M_1 y\,,$$

and the unbiased OLS estimator of the error variance is

$$\hat{\sigma}^2 = y'(M_1 - M_1 h(X_2,0)(h(X_2,0)' M_1 h(X_2,0))^{-1} h(X_2,0)' M_1) y / (m-1).$$

The standard t test statistic is

$$t = \frac{\hat{\beta}_2}{\hat{\sigma}(h(X_2,0)' M_1 h(X_2,0))^{-1/2}}$$

$$= \frac{(m-1)^{1/2}(h(X_2,0)' M_1 h(X_2,0))^{-1} h(X_2,0)' M_1 y}{\{y'(M_1 - M_1 h(X_2,0)(h(X_2,0)' M_1 h(X_2,0))^{-1} h(X_2,0)' M_1) y\}^{1/2} \{h(X_2,0)' M_1 h(X_2,0)\}^{-1/2}}.$$

Replacing s from (3.3) in t we have

$$t = \frac{(m-1)^{1/2}\{h(X_2,0)' M_1 h(X_2,0)\}^{-1/2} s}{\{1 - (h(X_2,0)' M_1 h(X_2,0))^{-1} s^2\}^{1/2}}; \tag{3.5}$$

where $h(X_2,0)' M_1\, h(X_2,0)$ is a positive scalar. Observe that (3.5) is a monotonic increasing function of the test statistic s. Thus we may conclude that our LBI test is equivalent to the t test of β_2 in the artificial regression (3.4). This solves the problem of finding an appropriate critical value for the LBI test based on (3.3). In particular note that when $g(X_2, \beta_2) = X_2\beta_2$ then $h(X_2,0) = X_2$ and the LBI test is equivalent to the t test of β_2 in the linear regression

$$y = \beta_1 X_1 + \beta_2 X_2 + u\,, \tag{3.6}$$

as might be expected.

Case 2: When $g(X_2,\beta_2)\big|_{\beta_2=0} = l = (1,1,\ldots,1)'$ **(say).**

If we evaluate (3.2) at $\beta_2 = 0$ noting that $g(X_2,\beta_2)\big|_{\beta_2=0} = l$ we have

$$\frac{\partial \log f(w)}{\partial \beta_2}\bigg|_{\beta_2=0} = -\frac{1}{\sigma^2} h(X_2,0) M_1 l + \{1\text{F}1[\frac{m}{2},\frac{1}{2},\frac{a^2(w,0)}{2}] +$$

$$\frac{\sqrt{2}}{\sigma}(w'Pl)\eta 1\text{F}1[\frac{1+m}{2},\frac{3}{2},\frac{a^2(w,0)}{2}]\}^{-1}\{1\text{F}1[\frac{m+2}{2},\frac{3}{2},\frac{a^2(w,0)}{2}]$$

$$\{\frac{m}{\sigma^2}(w'Pl)(w'P)'h(X_2,0)\} + \frac{\sqrt{2}}{\sigma}\ \eta\ (w'P)'\ h(X_2,0)1\text{F}1[\frac{1+m}{2},\frac{3}{2},\frac{a^2(w,0)}{2}]$$

$$+\frac{\sqrt{2}}{3\sigma^3}(1+m)(w'Pl)\eta 1\text{F}1[\frac{m+3}{2},\frac{5}{2},\frac{a^2(w,0)}{2}]\{(w'Pl)(wP)'h(X_2,0)\}\}$$

Hence the LBI test rejects H_0 for

$$\{1\text{F}1[\frac{m}{2},\frac{1}{2},\frac{a^2(w,0)}{2}] + \frac{\sqrt{2}}{\sigma}(w'Pl)\eta 1\text{F}1[\frac{1+m}{2},\frac{3}{2},\frac{a^2(w,0)}{2}]\}^{-1}$$

$$\{1\text{F}1[\frac{m+2}{2},\frac{3}{2},\frac{a^2(w,0)}{2}]\{\frac{m}{\sigma^2}(w'Pl)k(w'P)'h(X_2,0)\} +$$

$$\frac{\sqrt{2}}{\sigma}\ \eta\ (w'P)\ h(X_2,0)1\text{F}1[\frac{1+m}{2},\frac{3}{2},\frac{a^2(w,0)}{2}] + \frac{\sqrt{2}}{3\sigma^3}(1+m)(w'Pl)$$

$$\eta 1\text{F}1[\frac{m+3}{2},\frac{5}{2},\frac{a^2(w,0)}{2}]\{(w'Pl)(w'P)'h(X_2,0)\}\} \geq c_\alpha,$$

where c_α is an appropriate critical value. Note that if $g(X_2,\beta_2)\big|_{\beta_2=0} = l$ then $a = w'Pl$ and sum of the elements $w'P$ is zero provided there is an intercept in the regression. In this case the test reduces to (3.3) and again, our LBI test is equivalent to the t test of β_2 in the artificial regression (3.4).

4. Concluding remarks

In this paper, we derived the density function of the maximal invariant statistic of the non-linear part of a semi-linear regression model. This density was then used to construct the LBI test of the non-linear parameter. In two important general cases, the test is easily applied because it is equivalent to a *t* test of a regression coefficient in an artificial linear regression.

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