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Abstract: This paper presents a Markov chain Monte Carlo (MCMC) algorithm to estimate parameters and latent stochastic processes in the asymmetric stochastic volatility (SV) model, in which the Box-Cox transformation of the squared volatility follows an autoregressive Gaussian distribution and the marginal density of asset returns has heavy-tails. To test for the significance of the Box-Cox transformation parameter, we present the likelihood ratio statistic, in which likelihood functions can be approximated using a particle filter and a Monte Carlo kernel likelihood. When applying the heavy-tailed asymmetric Box-Cox SV model and the proposed sampling algorithm to continuously compounded daily returns of the Australian stock index, we find significant empirical evidence supporting the Box-Cox transformation of the squared volatility against the alternative model involving a logarithmic transformation.

JEL Classification: C12; C15; C52

Keywords: Leverage effect; Likelihood ratio test; Markov chain Monte Carlo; Monte Carlo kernel likelihood; particle filter

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1 Introduction

The volatility of asset returns often exhibits a time-varying feature. One approach to modelling volatility is to employ the autoregressive conditional heteroskedasticity (ARCH) model developed by Engle (1982) or the generalized ARCH (GARCH) model by Bollerslev (1986). An alternative is the stochastic volatility (SV) model, in which the volatility is assumed to be a latent stochastic process. The SV model has received increased attention in the finance literature, because it provides an alternative approach to the Black-Scholes option pricing formula (Hull and White, 1987). Taylor (1982, 1986) shows that the SV model is often formulated in terms of stochastic differential equations,

$$\begin{aligned} d(\ln p_t) &= \alpha\ dt + \sigma_t\ dw_{1t}, \\ d(\ln \sigma_t^2) &= \lambda(\xi - \ln \sigma_t^2)\ dt + \sigma_w\ dw_{2t}, \end{aligned} \tag{1}$$

where p_t is the price of an asset at time t and $(w_{1t}, w_{2t})'$ is a bivariate standard Brownian motion. The correlation between dw_{1t} and dw_{2t}, denoted by $\rho = corr(dw_{1t}, dw_{2t})$, captures the leverage effect, which refers to the asymmetric behaviour that price movements are negatively correlated with volatility and is often observed in returns of equity prices (see, e.g., Nelson, 1991; Gallant, Rossi and Tauchen, 1992, 1993; Campbell and Kyle, 1993; Engle and Ng, 1993; among others). The empirical version of this model is typically formulated in discrete time as

$$\begin{aligned} y_t &= \sigma_t\ \varepsilon_t, \\ \ln \sigma_{t+1}^2 &= \mu + \phi \left(\ln \sigma_t^2 - \mu\right) + \sigma_u u_{t+1}, \end{aligned} \tag{2}$$

where y_t is the continuously compounded return, $\varepsilon_t \sim N(0,1)$, $u_t \sim N(0,1)$, $\ln \sigma_1^2 \sim N(0, \sigma_u^2/(1-\phi^2))$, and the correlation between ε_t and u_{t+1}, denoted by $\rho = corr(\varepsilon_t, u_{t+1})$, captures the leverage effect.[2] To reflect the asymmetric correlation between errors in the mean and volatility equations, this model is often termed the asymmetric log-normal SV (LSV) model, which was set up based on models of Clark (1973) and Tauchen and Pitts (1983) and was first documented by Taylor (1982).

The discrete-time log-normal SV model specifies the logarithmic squared volatility as an autoregressive Gaussian process, while there are some other specifications of the volatility process (see, e.g., Hull and White, 1987; Stein and Stein, 1991; Heston, 1993; Andersen, 1994; Jacquier, Polson and Rossi, 1994; Eraker, Johannes and Polson, 2003; among others). Yu, Yang and Zhang (2002) present an extension to the log-normal specification of squared volatilities, in which the Box-Cox transformation of the squared volatility is assumed to follow an autoregressive Gaussian distribution. Ignoring the leverage effect, they developed an MCMC algorithm to sample parameters and volatilities. They applied their model to daily returns of the dollar/pound exchange rate and found that the 90% Bayesian confidence interval of the Box-Cox transformation parameter does not cover zero. This is significant because a value of zero is equivalent to the logarithmic transformation of squared volatilities.

The purpose of the Box-Cox transformation of squared volatilities is to allow for skewness in the marginal distribution of squared volatilities, because there is no reason to assume that the underlying distribution is symmetric. The use of the logarithmic transformation of squared volatilities has a similar purpose. However, the Box-Cox transformation has the additional advantage that it allows the volatility process itself to choose a transformation parameter. It seems that the Box-Cox transformation of squared volatili-

[2] Taylor (1994) shows that the correlation between ε_t and u_{t+1} captures the leverage effect, and empirical evidence can be found in Ghysels, Harvey and Renault (1996), Harvey and Shephard (1996) and Bollerslev and Zhou (2003).

ties is a useful extension to the logarithmic transformation in SV models.

The sampling algorithm developed by Yu et al. (2002) is limited in two aspects. First, it does not incorporate the leverage effect, which is often observed in the distribution of equity returns (see, e.g., Eraker et al., 2003; Jacquier et al., 2004). Jacquier et al. (2004) point out that the leverage effect often induces skewness in the marginal distribution of returns on asset prices. This finding is consistent with the non-parametric evidence found by Gallant, Hsieh and Tauchen (1997). Second, the sampling algorithm of Yu et al. (2002) does not take account of the heavy-tailed feature of asset returns. Eraker et al. (2003) argue that asset returns often experience significant shocks, which result in a heavy-tailed marginal distribution of returns.

The aim of this paper is to provide a fully specified posterior density of the transformed volatilities and parameters, including the parameter capturing the leverage effect, and to show that all components can be estimated through a proposed sampling procedure. To incorporate the heavy-tailed feature into the marginal distribution of asset returns, we introduce into the mean equation a latent jump process, which is assumed to follow a χ^2 distribution. The mixture of χ^2 distributed jumps and Gaussian errors is a Student t distribution, which has been widely used to describe the heavy-tailed distribution of asset returns. A similar treatment of heavy-tailed distributions may be found in Geweke (1993) and Jacquier et al. (2004).

This paper aims to develop an MCMC algorithm, in which both the leverage effect and the heavy-tailed marginal distribution can be incorporated. Section 2 provides a description of the asymmetric Box-Cox SV model, the fully specified posterior density, conditional densities, and sampling algorithms designed to sample parameters and volatilities. In Section 3, we discuss how the likelihood ratio statistic, to test the significance of the Box-Cox transformation parameter, can be calculated using a particle filter and a Monte Carlo kernel likelihood. Section 4 presents an application of heavy-tailed asymmetric SV models and the proposed sampling algorithm to daily returns of the Australian stock index, and we find significant empirical evidence supporting the Box-Cox transformation of squared volatility against the alternative of a logarithmic transformation. Section 5 concludes the paper.

2 MCMC in the Box-Cox Transformed SV Model

2.1 The Basic Box-Cox Transformed SV Model

The SV model proposed by Yu et al. (2002) assumes that the Box-Cox transformation of squared volatility follows an autoregressive Gaussian process,

$$\begin{aligned} y_t &= \sigma_t \, \varepsilon_t, \\ h(\sigma_{t+1}^2, \delta) &= \mu + \phi \left[h(\sigma_t^2, \delta) - \mu \right] + \sigma_u u_{t+1}, \end{aligned} \tag{3}$$

where $\varepsilon_t \sim N(0,1)$, $u_t \sim N(0,1)$, and $h\left(\sigma_t^2, \delta\right)$ is defined by

$$h(x, \delta) = \begin{cases} (x^\delta - 1)/\delta & \text{if } \delta \neq 0 \\ \ln x & \text{if } \delta = 0 \end{cases}. \tag{4}$$

This model is called the basic BCSV model hereafter. Let $\alpha_t = h(\sigma_t^2, \delta)$ denote the Box-Cox transformed squared volatility. The model can be equivalently expressed as

$$\begin{aligned} y_t &= \sqrt{g(\alpha_t, \delta)} \, \varepsilon_t, \\ \alpha_t &= \mu + \phi(\alpha_{t-1} - \mu) + \sigma_u u_t, \end{aligned} \tag{5}$$

where

$$g(\alpha_t, \delta) = \begin{cases} (1+\delta\alpha_t)^{1/\delta} & \text{if } \delta \neq 0 \\ \exp(\alpha_t) & \text{if } \delta = 0 \end{cases}, \tag{6}$$

which is denoted as g_t hereafter.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)'$, $\mathbf{y} = (y_1, y_2, \cdots, y_n)'$, and $\theta = (\phi, \delta, \mu, \sigma_u)'$. The posterior density of $(\theta', \alpha')'$ is

$$\pi(\theta, \alpha|\mathbf{y}) \propto p(\mathbf{y}|\theta, \alpha)p(\alpha|\theta)p(\theta), \tag{7}$$

where $p(\mathbf{y}|\theta, \alpha)$ is the likelihood of $\mathbf{y}$ given $(\theta', \alpha')'$, $p(\alpha|\theta)$ is the density of α, and $p(\theta)$ is the prior density of θ. The sampling algorithm developed by Yu et al. (2002) involves breaking the joint posterior $\pi(\theta, \alpha|\mathbf{y})$ into two Gibbs' blocks denoted by $p(\theta|\alpha, \mathbf{y})$ and $p(\alpha|\theta, \mathbf{y})$, respectively. The sampling algorithm consists of sampling μ and σ_u^2 from the corresponding conditional posteriors, sampling ϕ and δ simultaneously through the random-walk Metropolis-Hastings algorithm, and sampling α via the single-move random-walk Metropolis-Hastings algorithm.

2.2 The Asymmetric BCSV Model

The basic BCSV model can be extended by allowing a nonzero correlation between ε_t and u_{t+1} that captures the leverage effect, and the model can be expressed as

$$\begin{aligned} y_t &= g_t^{1/2}\varepsilon_t, \\ \alpha_{t+1} &= \mu + \phi(\alpha_t - \mu) + \sigma_u u_{t+1}, \end{aligned} \tag{8}$$

where $\rho = corr(\varepsilon_t, u_{t+1})$, and g_t is defined in (6). Let $\theta = (\phi, \delta, \mu, \rho, \sigma_u)'$ represent the augmented parameter vector. To obtain the likelihood of $\mathbf{y}$ given $(\theta', \alpha')'$, we define

$$u_{t+1} = \rho\varepsilon_t + \sqrt{1-\rho^2}\eta_{t+1}, \tag{9}$$

for $t = 1, 2, \cdots, n-1$, where η_{t+1} is assumed to follow $N(0,1)$ and to be uncorrelated with ε_t. Equation (9) shows that $var(u_{t+1}) = 1$ and $cov(u_{t+1}, \varepsilon_t) = \rho$. Substituting (9) into (8), we obtain

$$\alpha_{t+1} = \mu + \phi(\alpha_t - \mu) + \rho\sigma_u g_t^{-1/2} y_t + \sqrt{1-\rho^2}\sigma_u \eta_{t+1}.$$

When incorporating the leverage effect into the log-normal SV model, Jacquier et al. (2004) re-parameterized ρ and σ_u to $\varphi = \rho\sigma_u$ and $\tau^2 = (1-\rho^2)\sigma_u^2$. We follow such a re-parameterization and obtain

$$\begin{aligned} y_t &= \sqrt{g_t}\ \varepsilon_t, \\ \alpha_{t+1} &= \mu + \phi(\alpha_t - \mu) + \varphi g_t^{-1/2} y_t + \tau\eta_{t+1}, \end{aligned} \qquad (10)$$

where $\alpha_1 \sim N(\mu, \tau^2/(1-\phi^2))$ and $y_n \sim N(0, g_n)$. Because ε_t and η_{t+1} are uncorrelated, the posterior of $(\theta', \alpha')'$ can be obtained in the same way as that in the basic BCSV model. Model (10) is referred to as the asymmetric BCSV model hereafter.

2.3 The Joint Posterior of Parameters and Volatilities

In order to obtain the posterior density of $(\theta', \alpha')'$, we need a prior for each parameter. Assume that $(\phi+1)/2 \sim Beta(\omega, \gamma)$ and $\tau^2 \sim IG(\zeta/2, S_\tau/2)$, which are, respectively, expressed explicitly as

$$\begin{aligned} p(\phi) &\propto \left(\frac{\phi+1}{2}\right)^{\omega-1}\left(1-\frac{\phi+1}{2}\right)^{\gamma-1}, \\ p(\tau^2) &\sim \left(\frac{1}{\tau^2}\right)^{\zeta/2+1}\exp\left\{-\frac{S_\tau/2}{\tau^2}\right\}, \end{aligned}$$

where ω, γ, ζ and S_τ are hyperparameters to be defined by users. The priors of the other parameters are, respectively, $\varphi|\tau^2 \sim N(\varphi_0, \tau^2/p_0)$, $\mu|\tau^2 \sim N(\mu_0, \tau^2/q_0)$, and $\delta \sim U(-2, 2)$ with φ_0, μ_0, p_0 and q_0 being hyperparameters.

The joint prior of θ is the product of these marginal priors. According to (7), the posterior density of $(\theta', \alpha')'$ is

$$\begin{aligned} \pi(\theta, \alpha|\mathbf{y}) &\propto p(\phi, \delta, \mu, \rho, \sigma) \times \prod_{t=1}^{n-1} p(y_t|\alpha_t, \theta) p(y_n|\theta) \times p(\alpha_1|\theta) \prod_{t=1}^{n-1} p(\alpha_{t+1}|\alpha_t, \theta) \\ &= p(\delta)(1+\phi)^{\omega-1/2}(1-\phi)^{\gamma-1/2} \left(\prod_{t=1}^{n} g_t^{-1/2}\right) \exp\left\{-\frac{1}{2}\sum_{t=1}^{n}\frac{y_t^2}{g_t}\right\} \\ &\quad \left(\frac{1}{\tau^2}\right)^{(n+\nu+2)/2+1} \exp\left\{-\frac{\kappa}{2\tau^2}\right\}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \kappa &= (1-\phi^2)(\alpha_1-\mu)^2 + \sum_{t=1}^{n-1}\left(\alpha_{t+1} - \mu - \phi(\alpha_t - \mu) - \varphi g_t^{-1/2} y_t\right)^2 \\ &\quad + p_0(\varphi-\varphi_0)^2 + q_0(\mu-\mu_0)^2 + S_\tau. \end{aligned}$$

After integrating out τ^2 from the joint posterior (11), we obtain the logarithmic posterior density of $(\phi, \delta, \mu, \varphi, \alpha')'$ given $\mathbf{y}$,

$$\begin{aligned} \log\ \pi(\phi, \delta, \mu, \varphi, \alpha|\mathbf{y}) &= \log\ p(\delta) + (\omega - 1/2)\log(1+\phi) + (\gamma-1/2)\log(1-\phi) \\ &\quad -\frac{1}{2}\sum_{t=1}^{n}\log(g_t) - \frac{1}{2}\sum_{t=1}^{n}\frac{y_t^2}{g_t} - \frac{n+\zeta+2}{2}\log(\kappa/2), \end{aligned} \tag{12}$$

while the conditional posterior of τ^2 is the inverted gamma density, $IG((n+\zeta+2)/2, \kappa/2)$. In the appendix, we present a different routine to obtain the joint posterior of $(\theta', \alpha')'$ given $\mathbf{y}$. These two approaches result in the same posterior density.

2.4 Conditional Posteriors

Once the posterior of $(\theta', \alpha')'$ is obtained, we can use the Gibbs sampler to sample each component of $(\theta', \alpha')'$ conditional on the other components. However, the mixing speed will generally be slow. If conditional posteriors of some parameters can be obtained, these parameters can be sampled, respectively, from their conditional posteriors directly. As a consequence, the overall mixing performance will be improved (see, e.g., Johannes and Polson, 2003, for a discussion on sampling techniques based on conditional posteriors).

2.4.1 Conditional Posterior of τ^2

As τ^2 can be integrated out from the joint posterior (11), the conditional posterior density of τ^2 is

$$\tau^2 \sim IG\left(\frac{n+\zeta+2}{2}, \frac{\kappa}{2}\right). \tag{13}$$

Given the other parameters and α, we can sample τ^2 directly from its conditional posterior.

2.4.2 Conditional Posterior of φ

For sampling φ based on (11), we found that the conditional posterior of φ is (up to a normalizing constant)

$$\begin{aligned}
& p(\varphi|\tau^2, \phi, \delta, \mu, \alpha, \mathbf{y}) \\
\propto \;& \exp\left\{-\frac{1}{2\tau^2}\left[\sum_{t=1}^{n-1}\left((\alpha_{t+1}-\mu)-\phi(\alpha_t-\mu)-\varphi g_t^{-1/2} y_t\right)^2 + p_0(\varphi-\varphi_0)^2\right]\right\} \\
= \;& \exp\left\{-\frac{1}{2\tau^2}\left[a_{11}\varphi^2 - 2a_{12}\varphi + a_{22} + p_0(\varphi^2 - 2\varphi_0\varphi + \varphi_0^2)\right]\right\} \\
\propto \;& \exp\left\{-\frac{1}{2\tau^2/(a_{11}+p_0)}\left(\varphi^2 - 2\frac{a_{12}+\varphi_0 p_0}{a_{11}+p_0}\varphi\right)\right\},
\end{aligned}$$

where

$$a_{11} = \sum_{t=1}^{n-1} y_t^2/g_t, \qquad a_{12} = \sum_{t=1}^{n-1}[\alpha_{t+1}-\mu-\phi(\alpha_t-\mu)]y_t/\sqrt{g_t}.$$

Hence the conditional posterior of φ is the Gaussian density,

$$\varphi \sim N\left(\frac{a_{12}+\varphi_0 p_0}{a_{11}+p_0}, \frac{\tau^2}{a_{11}+p_0}\right), \tag{14}$$

based on which φ can be sampled directly, given the other parameters and latent volatilities. Once τ^2 and φ are sampled, respectively, from their conditional posteriors, we can calculate ρ and σ through $\sigma^2 = \varphi^2 + \tau^2$ and $\rho = \varphi/\sigma$.

2.4.3 Conditional Posterior of μ

For sampling μ based on (11), we found that the conditional posterior of φ is (up to a normalizing constant)

$$
\begin{aligned}
& p(\mu|\tau^2, \phi, \delta, \varphi, \alpha, \mathbf{y}) \\
\propto \ & \exp\left\{ -\frac{1}{2\tau^2} \left[(1-\phi^2)(\alpha_1 - \mu)^2 + \sum_{t=1}^{n-1} (b_{t+1} - (1-\phi)\mu)^2 + q_0(\mu - \mu_0)^2 \right] \right\} \\
\propto \ & \exp\left\{ -\frac{(n-1)(1-\phi)^2 + (1-\phi^2) + q_0}{2\tau^2} \left(\mu^2 - 2\frac{(1-\phi^2)\alpha_1 + (1-\phi)\sum b_{t+1} + q_0\mu_0}{(n-1)(1-\phi)^2 + (1-\phi^2) + q_0}\ \mu \right) \right\},
\end{aligned}
$$

where $b_{t+1} = \alpha_{t+1} - \phi\alpha_t - \varphi g_t^{-1/2} y_t$ for $t = 1, 2, \cdots, n-1$. Then the conditional posterior of μ is the Gaussian density with mean and variance defined, respectively, by

$$
\begin{aligned}
\mu_* &= \frac{(1-\phi^2)\alpha_1 + (1-\phi)\sum_{t=1}^{n-1} b_{t+1} + q_0\mu_0}{(n-1)(1-\phi)^2 + (1-\phi^2) + q_0}, \\
\sigma_*^2 &= \frac{\tau^2}{(n-1)(1-\phi)^2 + (1-\phi^2) + q_0}.
\end{aligned}
\tag{15}
$$

Hence μ can be sampled directly from $N(\mu_*, \sigma_*^2)$, given the other parameters and α.

2.4.4 Sampling ϕ and δ

In order to sample ϕ and δ, we can use the random-walk Metropolis-Hastings algorithm, in which the proposal density is the standard normal and the acceptance probability is calculated based on the joint posterior (12).

2.4.5 Sampling α

We use the single-move random-walk Metropolis-Hastings algorithm to sample components of α sequentially, where the acceptance probability is calculated based on the joint posterior (11). Let $\alpha_{\backslash k}$ denote α with its kth component deleted, and $\pi(\alpha_k)$ denote the posterior of α_k conditional on $(\theta, \alpha_{\backslash k}, \mathbf{y})$, for $k = 1, 2, \cdots, n$. We found that $\pi(\alpha_k)$ has the following simple expressions.[3]

$$
\begin{aligned}
\pi(\alpha_1) &\propto -\frac{\log g_1}{2} - \frac{y_1^2}{2g_1} - \frac{1}{2\tau^2}\left[(1-\phi^2)(\alpha_1-\mu)^2 + \left(\alpha_2 - \mu - \phi(\alpha_1-\mu) - \varphi g_1^{-1/2} y_1\right)^2\right], \\
\pi(\alpha_n) &\propto -\frac{\log g_n}{2} - \frac{y_n^2}{2g_n} - \frac{1}{2\tau^2}\left[\alpha_n - \mu - \phi(\alpha_{n-1}-\mu) - \varphi g_{n-1}^{-1/2} y_{n-1}\right]^2, \\
\pi(\alpha_k) &\propto -\frac{\log g_k}{2} - \frac{y_k^2}{2g_k} - \frac{1}{2\tau^2}\sum_{t=k-1}^{k}\left[\alpha_{t+1} - \mu - \phi(\alpha_t-\mu) - \varphi g_t^{-1/2} y_t\right]^2, \qquad (16)
\end{aligned}
$$

for $k = 2, 3 \cdots, n-1$.

In summary, the sampling procedure of $(\theta', \rho, \nu, \alpha')'$ is as follows:

- sample (ϕ, δ) from their joint posterior (12) using the random-walk Metropolis-Hastings algorithm;
- sample τ^2 directly from the inverted Gamma density given by (13);
- sample φ directly from the Gaussian density given by (14);
- compute ρ and σ^2 based on τ^2 and φ;
- sample μ directly from the Gaussian density with mean and variance defined in (15);

[3]In the literature on SV models involving MCMC simulation, there are alternative algorithms for sampling α, such as the block-wise Metropolis-Hastings algorithm presented in Kim, Shephard and Chib (1998) and the accept/reject Metropolis-Hastings algorithm in Jacquier et al. (2004). We have been unable to modify these algorithms to incorporate the Box-Cox transformation involved in our model.

- sample α using the single-move random-walk Metropolis-Hastings algorithm with the acceptance probability computed through (16).

2.5 Heavy-Tailed Departure from Normality

In the context of SV models, heavy tails are often observed in the distribution of errors in the mean equation (see, e.g., Gallant, Hsieh and Tauchen, 1996; Jacquier et al., 2004). Motivated by the heavy-tailed asymmetric LSV model by Jacquier et al. (2004), we set up the following model,

$$\begin{aligned} y_t &= g_t^{1/2}\lambda_t^{1/2}\varepsilon_t, \\ \alpha_{t+1} &= \mu + \phi(\alpha_t - \mu) + \sigma_u u_{t+1}, \end{aligned} \tag{17}$$

where $\varepsilon_t \sim N(0,1)$, $u_{t+1} \sim N(0,1)$, $cov(\varepsilon_t, u_{t+1}) = \rho$, and $\lambda_t \sim IG(\nu/2, \nu/2)$, which is equivalent to the assumption that ν/λ_t follows a χ^2 distribution with ν degrees of freedom. This assumption implies that the marginal distribution of $v_t = \sqrt{\lambda_t}\varepsilon_t$ is Student t with ν degrees of freedom. The density of λ_t is

$$p(\lambda_t|\nu) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\left(\frac{1}{\lambda_t}\right)^{\nu/2+1}\exp\left\{-\frac{\nu/2}{\lambda_t}\right\}.$$

The heavy-tailed asymmetric BCSV model has a parameter vector $(\nu, \theta')'$ and two latent stochastic processes denoted by α and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)'$, respectively. Consider simulating from the joint posterior density $\pi(\theta, \alpha, \nu, \lambda|\mathbf{y})$ by successive sampling through conditional posteriors $p(\theta, \alpha|\nu, \lambda, \mathbf{y})$ and $p(\lambda, \nu|\theta, \alpha, \mathbf{y})$. The algorithm for drawing each set of conditionals is as follows.

2.5.1 Sampling θ and α

Given λ, the mean equation of the heavy-tailed asymmetric BCSV model is

$$y_t^* = y_t \lambda_t^{-1/2} = g_t^{1/2} \varepsilon_t.$$

The sampling algorithm for the asymmetric BCSV model discussed in Section 2.4 applies directly.

2.5.2 Sampling λ and ν

Given $(\nu, \theta', \alpha')'$, the posterior of λ is (up to a normalizing constant)

$$\begin{aligned} p(\lambda|\nu, \theta, \alpha, \mathbf{y}) &= \prod_{t=1}^{n} p(\lambda_t|y_t, \alpha_t, \nu) \propto \prod_{t=1}^{n} p(y_t|\lambda_t, \nu) p(\lambda_t|\nu) \\ &= \frac{1}{\sqrt{2\pi\lambda_t}} \exp\left\{-\frac{y_t^2/g_t}{2\lambda_t}\right\} \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \left(\frac{1}{\lambda_t}\right)^{\nu/2+1} \exp\left\{-\frac{\nu/2}{\lambda_t}\right\} \\ &\propto \left(\frac{1}{\lambda_t}\right)^{(\nu+1)/2+1} \exp\left\{-\frac{y_t^2/g_t + \nu}{2\lambda_t}\right\}. \end{aligned} \tag{18}$$

Hence we can sample λ_t directly from the inverted Gamma distribution,

$$\lambda_t \sim IG\left((\nu+1)/2, (y_t^2/g_t + \nu)/2\right), \qquad \text{for } t = 1, 2, \cdots, n.$$

When sampling ν, we set its prior density as $p(\nu) \propto \psi \exp(-\psi\nu)$, in which ω is the hyperparameter controlling the shape of the distribution.[4] Given λ, the posterior of ν is (up to a normalizing constant)

$$p(\nu|\lambda) \propto \psi \exp(-\psi\nu) \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \left(\frac{1}{\lambda_t}\right)^{\nu/2+1} \exp\left\{-\frac{\nu/2}{\lambda_t}\right\}, \tag{19}$$

[4]Geweke (1993) employs the same prior for the degree-of-freedom parameter in the linear regression model with t distributed errors. A reasonable prior for ν should prevent ν from getting too large during MCMC iterations, because the update of ν has a negligible effect when ν is already large. The purpose of such a prior is to put a low prior probability on the "problematic" region, where the likelihood is flat. One can also use $p(\nu) \propto 1/(1+\nu^2)$ employed by Bauwens and Lubrano (1998), or the Gaussian prior.

and ν can be sampled using the random-walk Metropolis-Hastings algorithm, in which the proposal density is the standard Gaussian density and the acceptance probability is computed through (19).

3 Testing the Significance of the Box-Cox Parameter

As the heavy-tailed asymmetric BCSV model nests the heavy-tailed asymmetric LSV model by setting $\delta = 0$, we wish to test the null hypothesis that $\delta = 0$ against the alternative that $\delta \neq 0$. Let $\Theta = (\nu, \theta')'$ denote the parameter vector of the heavy-tailed asymmetric BCSV model. The likelihood function of $\mathbf{y}$ given Θ plays an important role in our choice of test, the likelihood ratio (LR) test, but the likelihood is often intractable. In terms of LSV models, Kim et al. (1998) show that the likelihood can be approximated using the particle filter when the parameter vector is known. Our purpose is to derive the likelihood computed at the maximum likelihood estimate (MLE) of Θ, which cannot be directly obtained from the MCMC simulation.

Given the posterior sample of Θ that consists of sampled values of Θ during MCMC iterations, de Valpine (2004) presents a Monte Carlo kernel likelihood (MCKL), which is the importance-sampled kernel estimator of the likelihood up to a normalizing constant. De Valpine (2004) shows that the MLE of Θ is the argument that maximizes MCKL and can be obtained via a numerical maximization procedure. As the normalizing constant is often unknown, we cannot use the maximized MCKL to construct the LR statistic. This difficulty can be remedied by the particle filter algorithm, which can produce an approximate likelihood at the MLE of Θ.

3.1 Monte Carlo Kernel Likelihood

Let $\{\Theta^{(j)} : j = 1, 2, \cdots, m\}$ denote the posterior sample of Θ. De Valpine (2004) shows that the MCKL is (up to a normalizing constant)

$$\hat{L}_H(\Theta) = \frac{1}{m}\sum_{j=1}^{m} K_H(\Theta - \Theta^{(j)})\frac{1}{p(\Theta^{(j)})}, \tag{20}$$

where $p(\cdot)$ is the prior density of Θ, and $K_H(x) = |H|^{-1/2}K(H^{-1/2}x)$ with $K(\cdot)$ being a standard multivariate Gaussian kernel and H a symmetric positive definite $d \times d$ matrix known as the bandwidth matrix. The MLE of Θ is the argument that maximizes $\hat{L}_H(\Theta)$.

As the MCKL involves selecting an optimal bandwidth based on the posterior sample, de Valpine (2004) indicates that further work on automated bandwidth selection would facilitate the application of MCKL. To estimate the optimal bandwidth, we use the data-driven bandwidth selectors proposed by Zhang, King and Hyndman (2004) who present MCMC algorithms to estimate the optimal bandwidth for multivariate kernel density estimation under the likelihood cross-validation criterion. Their bandwidth selectors are superior to the normal reference rule, which was discussed in Scott (1992) and Bowman and Azzalini (1997), for bandwidth selection in multivariate kernel density estimation.

3.2 Particle Filter

By successive conditioning, the log-likelihood of $\mathbf{y}$ given Θ can be expressed as[5]

$$\ln L(\mathbf{y}|\Theta) = \ln L(y_1|\Theta) + \sum_{t=2}^{n} \ln L(y_t|\mathbf{y}_{t-1}, \Theta), \tag{21}$$

[5]The particle filter algorithm presented in Kim et al. (1998) is applicable to LSV models. We need to modify the algorithm and make it applicable to the heavy-tailed BCSV model. See Kim et al. (1998) for a summary on leading papers on the particle filter algorithm.

where $\mathbf{y}_t = (y_1, y_2, \cdots, y_t)'$ for $t = 1, 2, \cdots, n$. We draw latent variables according to the following equations,

$$\alpha_1^{(j)} \sim N(0, (1-\rho^2)\sigma_u^2/(1-\phi^2)),$$
$$\alpha_{t+1}^{(j)} \sim N\left(\mu + \phi(\alpha_t^{(j)} - \mu), \sigma^2\right),$$
$$\lambda_t^{(j)} \sim IG(\nu/2, \nu/2),$$

for $j = 1, 2, \cdots, M$, and $t = 1, 2, \cdots, n-1$. Given $\alpha_t^{(j)}$, $\alpha_{t+1}^{(j)}$ and $\lambda_t^{(j)}$, we have

$$y_t \sim N\left(\frac{\rho}{\sigma}\sqrt{g_t^{(j)}\lambda_t^{(j)}}\left[\alpha_{t+1}^{(j)} - \mu - \phi(\alpha_t^{(j)} - \mu)\right], g_t^{(j)}\lambda_t^{(j)}(1-\rho^2)\right),$$

for $t = 1, 2, \cdots, n-1$, and $y_n \sim N(0, g_n^{(j)}(1-\rho^2))$, where $g_t^{(j)}$ is computed according to (6) with α_t replaced by $\alpha_t^{(j)}$. According to the particle filter, the estimate of $L(y_t|\mathbf{y}_{t-1}, \Theta)$ is approximated by

$$\hat{L}\left(y_t|\mathbf{y}_{t-1}, \Theta\right) = \frac{1}{M}\sum_{j=1}^{M} p(y_t|\lambda_t^{(j)}, \alpha_t^{(j)}, \alpha_{t+1}^{(j)}), \tag{22}$$

where $p(y_t|\lambda_t^{(j)}, \alpha_t^{(j)}, \alpha_{t+1}^{(j)})$ is the density of y_t conditional on $\lambda_t^{(j)}$, $\alpha_t^{(j)}$ and $\alpha_{t+1}^{(j)}$. Then the likelihood of $\mathbf{y}$ given Θ can be approximated by substituting (22) into (21).

To construct the LR statistic, we compute the approximate likelihood at the MLE of Θ, which is obtained by maximizing the MCKL given in (20).

4 An Application to Daily Returns of the Australian Stock Index

4.1 Data

In this section, we present an application of the heavy-tailed asymmetric BCSV model to daily returns of the Australian All Ordinaries stock index, whose historical data were downloaded from Data Stream. As required by the modelling of stochastic volatilities, the continuously compounded daily returns are mean-corrected and variance-scaled. The data set contains 1140 observations from 1st January 2000 to 30th June 2004, excluding weekends and holidays.

4.2 Results Obtained through Both SV Models

We applied the heavy-tailed asymmetric BCSV model and the sampling algorithm to the data set. The hyperparameters required in the joint prior density were set, respectively, to $\omega = 20$, $\gamma = 1.5$, $\zeta = 2$, $S_\tau = 0.01$, $\varphi_0 = 0$, $\mu_0 = 0$, $p_0 = 0.5$ and $q_0 = 0.2$. The prior of δ is the uniform density over $(-2, 2)$, and the prior of ν is $\psi \exp(-\psi\nu)$ with $\psi = 0.2$.

The burn-in period contained 50,000 iterations, and the recorded period contained 500,000 iterations. To measure the mixing performance, we calculated the batch-mean standard error (BMSE) for each component of $(\nu, \theta', \alpha')'$, where the number of batches was 50 and there were 1,000 draws in each batch (see, e.g., Roberts 1996; Tse, Zhang and Yu, 2004). The ergodic average (or posterior mean) acts as an estimator of the corresponding component.

The retained draws of $(\nu, \theta')'$ are plotted in the first column of Figure 1, and the second and third columns provide autocorrelation functions and histograms of the corresponding parameters, respectively. Table 1 summarizes the MCMC output, which are the posterior mean, the 95% confidence interval, BMSE, and the standard deviation. The posterior estimates of the jump and latent volatility processes are presented in Figure 2. Both the BMSE and Figure 1 indicate that the proposed sampling algorithm has mixed very well.

Using the same prior densities of parameters without δ and fixing $\delta = 0$, we applied the heavy-tailed asymmetric LSV model and the sampling algorithm to the same data set. A summary of the posterior average of $(\nu, \theta')'$ is presented in the second panel of Table 1.

4.3 Significance of the Box-Cox Transformation Parameter

To compute the approximate likelihood function of $\mathbf{y}$ given $(\nu, \theta')'$, we followed de Valpine's (2004) suggestion of MCKL using a diagonal bandwidth matrix. We applied the sampling algorithm presented in Zhang et al. (2004) to the posterior sample of Θ to obtain the optimal bandwidth. The MLEs of $(\nu, \theta')'$ were obtained through a numerical maximization of the MCKL and are presented in Table 2.

Let $\hat{\Theta}_0$ and $\hat{\Theta}_1$ denote MLEs of Θ obtained under the null and alternative hypotheses, respectively. Using the particle filter with $M = 1,000,000$, we calculated the approximate likelihood values, denoted by $\hat{L}(\hat{\Theta}_0)$ and $\hat{L}(\hat{\Theta}_1)$, under the null and alternative hypotheses. The LR statistic is

$$LR = -2\left(\ln \hat{L}(\hat{\Theta}_0) - \ln \hat{L}(\hat{\Theta}_1)\right),$$

which approximately follows a χ^2 distribution with one degree of freedom. We found that the LR statistic is 35.7, and the corresponding p-value is 2.3×10^{-9}, which indicates strong significance of the Box-Cox transformation parameter.

4.4 Sensitivity of Priors

In order to examine the sensitivity of MCMC output to prior choices, we set the prior density of ν as $N(15, 25)$, which represents very flat prior information.[6] Then we applied the sampler to the same data set and found that the posterior averages of parameters and latent processes are quite similar to those obtained previously.

The prior density of δ is noninformative, and we found no obvious changes in the MCMC output when using a flat Gaussian prior namely $N(0, 10)$. As our choice of priors for $(\phi, \mu, \psi, \tau^2)'$ is consistent with the literature, we have not tried any other priors.

5 Conclusion

This paper presents an MCMC algorithm for sampling parameters and latent stochastic processes of volatilities and jumps of the heavy-tailed asymmetric SV model, in which the Box-Cox transformation of squared volatility is assumed to follow an autoregressive Gaussian distribution. The widely used logarithmic transformation of squared volatility is nested into the Box-Cox transformation by setting $\delta = 0$. We have presented the likelihood ratio statistic to test the significance of the Box-Cox transformation parameter. When applying the heavy-tailed asymmetric BCSV model to continuously compounded daily returns of the Australian stock index, we have found significant evidence supporting the Box-Cox transformation of squared volatility against the alternative of a logarithmic transformation.

[6] The empirical evidence found by Jacquier et al. (2004) showed that the posterior mean of ν for most return series is between 10 and 30, so we let the prior be centered at 15.

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Appendix: An Alternative Method to Obtain the Joint Posterior

The Box-Cox transformed SV model is described by,

$$y_t = g_t^{1/2}\varepsilon_t,$$

$$\alpha_{t+1} = \mu + \phi(\alpha_t - \mu) + u_{t+1},$$

where g_t is defined by (6). The variance-covariance matrix of $(\varepsilon_t, u_{t+1})'$ is

$$\Sigma = \begin{pmatrix} 1 & \rho\sigma_u \\ \rho\sigma_u & \sigma_u^2 \end{pmatrix},$$

for $t = 1, 2, \cdots, n-1$, $u_1 \sim N(0, \tau^2/(1-\phi^2))$, and $\varepsilon_n \sim N(0,1)$. Then we re-parameterize ρ and σ_u through $\varphi = \rho\sigma_u$ and $\tau^2 = (1-\rho^2)\sigma_u^2$, which is the same as the re-parameterization presented in Jacquier, Polson and Rossi (2004).

Let $\theta = (\phi, \delta, \mu, \rho, \sigma_u)'$, and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)'$. Given the joint prior, which is the same as that presented in Section 2, we obtain the posterior,

$$\begin{aligned} p(\theta, \alpha|\mathbf{y}) &\propto p(\phi, \delta, \mu, \rho, \sigma)p(y_1, y_2, \cdots, y_{n-1}; \alpha_2, \alpha_3, \cdots, \alpha_n|\theta)p(y_n|\theta)p(\alpha_1|\theta) \\ &= p(\phi, \delta, \mu, \rho, \sigma)\prod_{t=1}^{n-1} g_t^{-1/2} p(y_t\, g_t^{-1/2}, \alpha_{t+1}|\alpha_t, \theta)p(y_n|\theta)p(\alpha_1|\theta) \end{aligned}$$

$$= \quad p(\phi,\delta,\mu,\rho,\sigma)|\Sigma|^{-(n-1)/2}\left(\prod_{t=1}^{n-1} g_t^{-1/2}\right)\exp\left\{-\frac{1}{2}tr(\Sigma^{-1}A)\right\}p(y_n|\theta)p(\alpha_1|\theta),$$

where $|\Sigma| = \tau^2$, and

$$A = \sum_{t=1}^{n-1}(\varepsilon_t \;\; u_{t+1})'(\varepsilon_t \;\; u_{t+1}).$$

Jacquier, Polson and Rossi (2004) showed that

$$\Sigma^{-1} = \frac{1}{\tau^2}\begin{pmatrix} \varphi^2 & -\varphi \\ -\varphi & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{C}{\tau^2} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and it follows that

$$tr(\Sigma^{-1}A) = \frac{1}{\tau^2}tr(CA) + a_{11}.$$

Then the posterior of $(\theta', \alpha')'$ is

$$\begin{aligned} p(\theta,\alpha|\mathbf{y}) \quad &= \quad p(\phi,\delta,\mu,\rho,\sigma)\prod_{t=1}^{n-1} g_t^{-1/2}\left(\frac{1}{\tau^2}\right)^{(n-1)/2}\exp\left\{-\frac{1}{2\tau^2}tr(CA) - \frac{1}{2}a_{11}\right\} \\ &\quad\; p(y_n|\theta)p(\alpha_1|\theta), \end{aligned}$$

where

$$\begin{aligned} p(y_n|\theta) \quad &= \quad \frac{1}{\sqrt{2\pi g_n}}\exp\left\{-\frac{y_n^2}{2g_t}\right\}, \\ p(\alpha_1|\theta) \quad &= \quad \frac{1}{\sqrt{2\pi\tau^2/(1-\phi^2)}}\exp\left\{-\frac{(\alpha_1-\mu)^2}{2\tau^2/(1-\phi^2)}\right\}, \\ tr(CA) \quad &= \quad \sum_{t=1}^{n-1}\left(\alpha_{t+1} - \mu - \phi(\alpha_t - \mu) - \varphi g_t^{-1/2} y_t\right)^2. \end{aligned}$$

Substituting the priors into the above equation, we obtain the joint posterior

$$\begin{aligned} p(\theta,\alpha|\mathbf{y}) \quad &= \quad p(\delta)(1+\phi)^{\omega-1/2}(1-\phi)^{\gamma-1/2} \\ &\quad \left(\prod_{t=1}^{n} g_t^{-1/2}\right)\exp\left\{-\frac{1}{2}\sum_{t=1}^{n}\frac{y_t^2}{g_t}\right\}\left(\frac{1}{\tau^2}\right)^{(n+\zeta+2)/2+1}\exp\left\{-\frac{\kappa}{2\tau^2}\right\}, \end{aligned}$$

where

$$\kappa = (1-\phi^2)(\alpha_1-\mu)^2 + tr(CA) + p_0(\varphi-\varphi_0)^2 + q_0(\mu-\mu_0)^2 + S_\tau.$$

Hence the joint posterior obtained here is identical to (11) which is obtained through the transformation of ε_t and u_{t+1} defined in (9).

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Table 1: *Summary of the recorded draws of the parameter vector obtained from daily returns of the Australian Stock index. SD refers to the standard deviation computed through recorded draws.*

Model	Parameter	Mean	95% confidence interval	BMSE	SD
Heavy-tailed asymmetric BCSV model	ϕ	0.96423	(0.9324, 0.9868)	0.00056	0.01405
	δ	-0.51147	(-0.8405, -0.1456)	0.00814	0.17893
	μ	-0.47390	(-0.8416, -0.0844)	0.00437	0.18875
	ρ	-0.37693	(-0.5483, -0.1907)	0.00198	0.09180
	σ_u	0.20724	(0.1500, 0.2840)	0.00129	0.03371
	ν	16.95198	(9.1859, 30.2508)	0.18044	5.52484
Heavy-tailed Asymmetric log-normal SV model	ϕ	0.95504	(0.9217, 0.9798)	0.00057	0.01511
	δ	–	–	–	–
	μ	-0.34661	(-0.6016, -0.0440)	0.00323	0.14125
	ρ	-0.36621	(-0.5302, -0.1826)	0.00281	0.08849
	σ_u	0.20403	(0.1493, 0.2737)	0.00153	0.03310
	ν	16.26354	(8.8139, 29.0900)	0.23783	5.37950

Table 2: *MLEs obtained through the heavy-tailed asymmetric BCSV (HA-BCSV) model and the heavy-tailed asymmetric log-normal SV (HA-LSV) model.*

Parameter	HA-BCSV	HA-LSV
ϕ	0.98930	0.98995
δ	-0.49476	–
μ	-0.35167	-0.35660
ρ	-0.64464	-0.59756
σ_u	0.39782	0.35894
ν	20.41765	20.41758

Figure 1: *Plots of recorded draws of the parameter vector. Columns (from left to right) show the sampled paths, their autocorrelation functions and histograms, while rows (from top to bottom) represent ϕ, δ, μ, ρ, σ_u, and ν. In the first column, the x-axis represents iterations, and the y-axis represents the recorded value of a parameter. In the second column, the x-axis represents lags, and the y-axis represents the autocorrelation coefficient. In the last column, the x-axis represents parameter values, and the y-axis represents frequencies.*

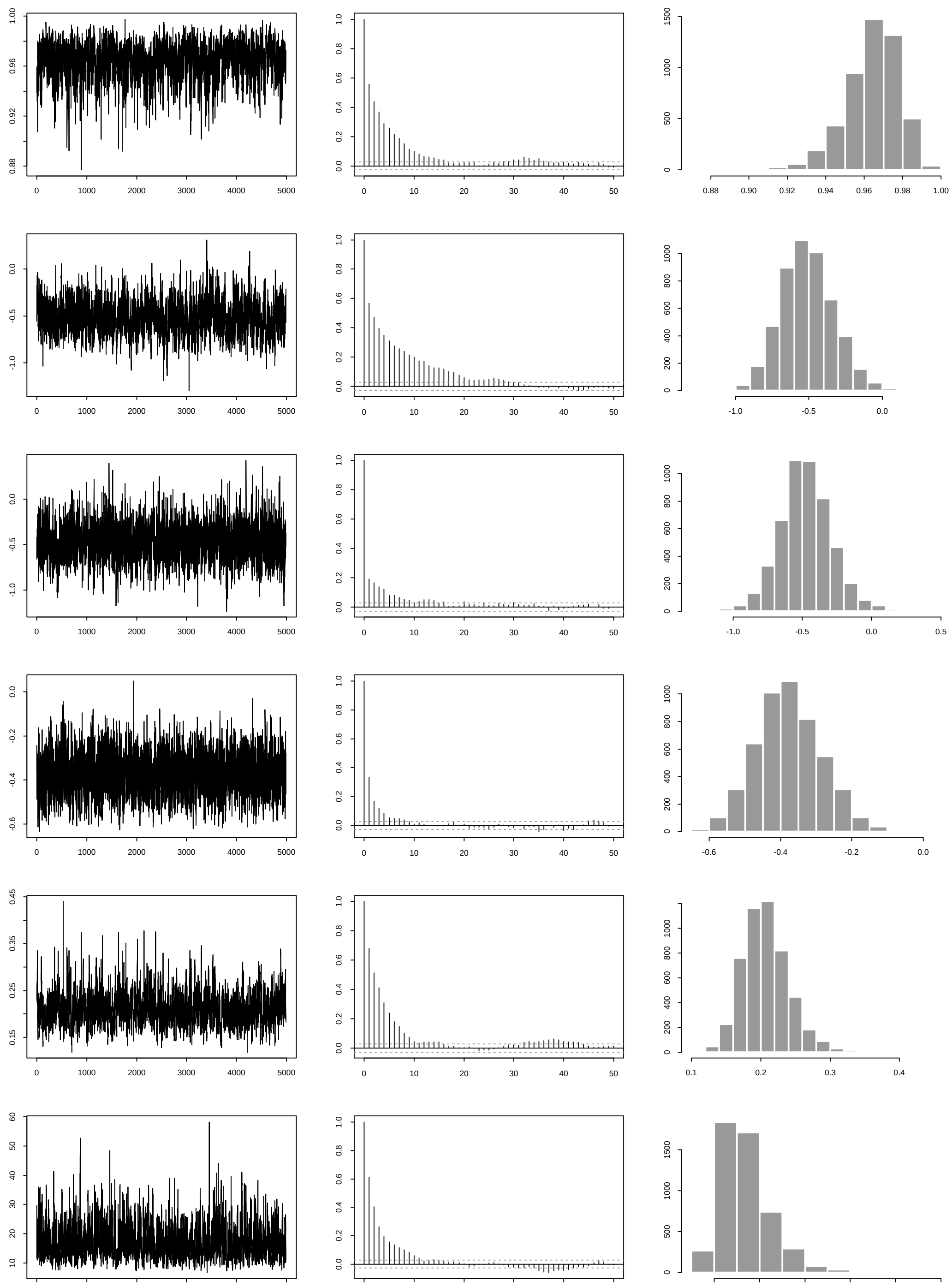

Figure 2: *Posterior estimates of the latent volatility and jump processes through the heavy-tailed asymmetric BCSV model. The first panel plots the estimated jump process estimated through, while the second panel plots the process of estimated volatilities. The x-axis represents time, while the y-axis represents the jump size and the estimated volatility, respectively.*