ISSN 1440-771X

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March 2012

Working Paper 07/12

Solving Replication Problems in Complete Market by Orthogonal Series Expansion

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Abstract

We reconsider the replication problem for contingent claims in a complete market under a general framework. Since there are various limitations in the Black–Scholes pricing formula, we propose a new method to obtain an explicit self–financing trading strategy expression for replications of claims in a general model. The main advantage of our method is that we propose using an orthogonal expansion method to derive a closed–form expression for the self–financing strategy that is associated with some general underlying asset processes. As a consequence, a replication strategy is obtained for a European option. Converse to the traditional Black-Scholes theory, we derive a pricing formula for a European option from the proposed replication strategy that is quite different from the Black-Scholes pricing formula. We then provide an implementation procedure and then a numerical example to show how the proposed trading strategy works in practice and then compare with a replication strategy based on the Black-Scholes theory.

Keywords: Approximation theory, Black-Scholes theory, complete market, stochastic process, time series

JEL Classifications: C13, C22, C45.

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1 Introduction

It is well known that in the Black–Scholes market described by the Black-Scholes model a European contingent claim H with maturity T is priced at time $t \in [0,T]$ as $c(X(t), \tau, K, r, \sigma)$ for a call option and $p(X(t), \tau, K, r, \sigma)$ for a put option, where $X(t)$ is the price process of the underlying asset that follows a geometric Brownian motion, $\tau = T - t$, r is interest rate, K is the strike price, σ is the volatility and functions c and p are given by

$$c(s, \tau, K, r, \sigma) = sN(d_1) - Ke^{-r\tau}N(d_2), \tag{1}$$

$$p(s, \tau, K, r, \sigma) = Ke^{-r\tau}N(-d_2) - sN(-d_1), \tag{2}$$

with $d_{1,2} = d_{1,2}(s, \tau, K, r, \sigma) = \frac{\ln(\frac{s}{K})+(r\pm\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$.

Meanwhile, the self-financing trading strategy of replication, commonly referred to as hedge ratio or, briefly, the delta of the option, is $\theta_t = c_s(X(t), \tau) = N(d_1(X(t), \tau))$ (see, for example, Musiela and Rutkowski (1997, p.131)). Accordingly, every contingent claim theoretically can be replicated by a self–financing trading strategy based on existing assets in the Black–Scholes market such that on one hand the risk of selling or buying claims in a complete market has been totally eliminated and on the other the claim is redundant in the sense that it can be written as a sum of an initial cost and a stochastic integral of the price process. It should be pointed out that much effort has been devoted in the literature to other models, including more general semimartingale models (see, for example, Barndorff-Nielson (1998)) and models associated with stochastic processes that are not semimartingales, such as fractional Brownian motion (see, for example, Cutland et al. (1995), Lin (1995)).

In this paper, we shall deal with a replication problem in a more general complete market than the Black-Scholes one by allowing that the drift and the volatility functions are unknown forms of arbitrary adapted processes. We propose a general way to obtain an explicit trading strategy expression for each replication issue in a complete market. Observe that the advantage of using an orthogonal series (Fourier) expansion in a Hilbert space with existing complete orthonormal basis is that the determination of a function in the space depends entirely on finding the coefficients in its expansion. Our methodology is that, in order to replicate a claim, we expand a stochastic process related to the dynamic strategy into an orthogonal series in a Hilbert space by means of the basis in the space. Therefore, under the martingale measure of the price process, since every European style contingent claim can be replicated, we simply employ a complete orthonormal basis of the Hilbert space in which the contingent claims are defined. Hence, the replication strategy of a contingent claim is expressed by means of the coefficients of a Fourier expansion. It is noted that our method is based on an expansion of stochastic processes developed in Dong and Gao (2011). Moreover, it is also noted that with our replication strategy we derive an alternative pricing formula for European options, which is quite different from the Black-Scholes option pricing formula.

The rest of the paper is organized as follows. Section 2 develops the replication strategy and pricing formula for contingent claims in a complete market. Section 3 derives an implementation procedure and then shows how to apply the proposed replication strategy in practice through using a numerical example. tSection 4 concludes and discusses several cases where our methodology may be applicable.

2 Replication in general complete markets

We firstly shall recall some basic and conventional notations, definitions and assumptions for our models in this section. Let us consider a general stochastic differential equation of the form

$$dX(t) = \mu_t dt + \sigma_t dW(t), \tag{3}$$

where $W(t)$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mu_t = \mu(t, X(t))$ and $\sigma_t = \sigma(t, X(t))$ are adaptable processes with respect to the information flow $\mathcal{F}_t^W$ generated by the Brownian motion $W(t)$, and both of them are unknown functions of t and $X(t)$. Obviously, we need to assume $\sigma_t \neq 0$ for all t. In addition, in order to ensure that the process is well-defined, the following conditions are necessary

$$\mathbb{P}\left(\int_0^t |\mu_s| ds < \infty, \quad \forall t\right) = 1, \quad \mathbb{P}\left(\int_0^t |\sigma_s|^2 ds < \infty, \quad \forall t\right) = 1.$$

A market, denoted by (X, B), consists of a risky asset with price process $X(t)$ and a riskless bank account with bounded stochastic interest rate $r(t, \omega)$ adapted with $\mathcal{F}_t^W$. The bank account $B(t)$ follows a stochastic model of the form

$$dB(t) = r(t, \omega) B(t) dt \quad \text{with} \quad B(0) = 1. \tag{4}$$

It follows that $B(t) = \exp\left(\int_0^t r(s, \omega) ds\right)$ and we therefore denote $D(t) = \frac{1}{B(t)}$ as the discount factor. In the sequel, let $r_t = r(t, \omega)$ for brevity. Let H be a contingent claim with maturity at T and satisfy that H is a $\mathcal{F}_T^W$ measurable random variable with finite second moment $E[H^2] < \infty$. One central problem in the literature in such a framework is the replication of contingent claims by means of dynamic trading strategy of underlying asset processes X and B.

Such a framework covers many examples. For example, if $\mu_t = \mu X(t)$, $\sigma_t = \sigma X(t)$ and $r_t = r > 0$, where μ, σ and r are constants, then the market (X, B) reduces to a classical Black-Scholes market. If $\mu_t = \mu X(t)$ and $r_t = r > 0$ where μ and r are constants, and $\sigma_t = \lambda_t X(t)$ where λ_t is a measurable adapted process, then the market (X, B) reduces to a generalised Black-Scholes market.

A dynamic trading portfolio strategy in the market is a (t, ω)-measurable process ϱ_t of the form $\varrho_t = (\theta_t, \eta_t)$ such that θ_t is $\mathcal{F}_t^W$-predictable and η_t is $\mathcal{F}_t^W$-adapted. The financial interpretation of a portfolio strategy (θ, η) is that θ_t is the number of units of asset X (it may be called a stock hereafter) held at time t, while η_t stands for the shares of bank accounts invested in the bank account at time t by an investor. The value of a portfolio $\varrho = (\theta, \eta)$ thereby is $V_t(\varrho) = \theta_t X(t) + \eta_t B(t)$. On the other hand, the cumulative gains from the trade on the stock up to time t are given by $G_t(\theta) = \int_0^t \theta_t dX(t)$ and the cumulative gains from the bank account up to time t are given by $Y_t(\theta) = \int_0^t \eta_t dB(t)$, provided that θ_t is integrable with respect to $X(t)$ and η_t is integrable with respect to $B(t)$ under the following assumption.

Assumption A: The shares of the stock θ_t and the number of bank account η_t satisfy

$$\int_0^T |\eta_t r_t B(t)| dt < \infty, \qquad \int_0^T [\theta_t \sigma_t]^2 dt < \infty, \qquad \mathbb{P} - a.s. \tag{5}$$

Remark 1. The first part of (5) is evident. The requirement of the second part is a consequence of $E \int_0^T \theta_t^2 d[X]_t < \infty$ that guarantees the integrability of θ_t with respect to the process $X(t)$, where $[X]_t$ is the variation process of $X(t)$. Also, this assumption is a standard requirement in the relevant literature (see, for example, Pham (2000), Øksendal (2000), Naulart and Schoutens (2001), Schweizer (2001) and Lim (2005)).

The cumulative costs incurred by $\varrho = (\theta, \eta)$ are thereby $C_t = V_t(\varrho) - G_t(\theta)$. A strategy is called *self–financing* if and only if C_t is a constant over time, i.e. $C_t = C_0 = V_0$. Hence,

$$V_t(\varrho) = V_0 + G_t(\theta). \tag{6}$$

The financial interpretation is that all changes in the wealth of the self-financing strategy, by the implication from (6) that $dV_t = \theta_t dX(t)$, are due to capital gains, as opposed to withdrawals of cash or infusions of new funds. Note that equation (6) indicates that for self-financing strategy the value process depends merely on θ_t. Hereafter we shall replace

$V_t(\varrho)$ by $V_t(\theta)$ in this case. It also implicitly assumes that the market in the question is frictionless, meaning that there are no transaction costs and no restrictions on short-selling.

Let $\Phi(\mathbb{P})$ denote the set of all self-financing strategies under $\mathbb{P}$. A contingent claim H is called attainable if there is a self-financing strategy θ_t^H such that $V_T(\theta^H) = H$, $\mathbb{P}$-a.s. or more precisely, if

$$H = H_0 + \int_0^T \theta_t^H dX(t), \qquad \mathbb{P} - a.s. \tag{7}$$

where H_0 is a constant. We say that a market is complete if every contingent claim is attainable. It is known that the Black-Scholes model is complete.

Let $\widehat{X}(t) = D(t)X(t)$ be the discounted process of $X(t)$. It follows from Itô Lemma that $d\widehat{X}(t) = D(t)[(\mu_t - r_t X(t))dt + \sigma_t dW(t)]$. Also, it follows from Lemma 12.2.3 of Øksendal (2000, p. 261) that if

$$E\left[\exp\left(\frac{1}{2}\int_0^T \Lambda_s^2 ds\right)\right] < \infty, \tag{8}$$

where $\Lambda_s = \frac{\mu_s - r_s X(s)}{\sigma_s}$, under the measure $\widetilde{\mathbb{P}}$ defined by Randon-Nikodým derivative

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left\{-\int_0^T \Lambda_s dW(s) - \frac{1}{2}\int_0^T \Lambda_s^2 ds\right\}, \tag{9}$$

$\widetilde{W}(t) := \int_0^t \Lambda_s ds + W(t)$ is a standard Brownian motion and hence $d\widehat{X}(t) = \widehat{\sigma}_t d\widetilde{W}(t)$, where $\widehat{\sigma}_t = D(t)\sigma_t$. Particularly, $\widetilde{\mathbb{P}}$ is an equivalent martingale measure since $\int_0^T \widetilde{E}[\widehat{\sigma}_t]^2 dt < \infty$, and the market is complete by Theorem 12.2.5 of Øksendal (2000, p. 263). Meanwhile, due to adaptivity of all processes with respect to $\mathcal{F}_t^W$, we have $\mathcal{F}_t^W = \mathcal{F}_t^X = \mathcal{F}_t^{\widetilde{W}}$, where the last two filtrations are generated by $X(t)$ and $\widetilde{W}(t)$, respectively.

Denote by $\widehat{V}_t(\theta) = D(t)V_t(\theta)$ the discounted value process of self-financing strategy θ_t. It follows from Lemma 12.2.2 of Øksendal (2000, p. 260) that $V_t = V_0 + \int_0^t \theta_t dX(t),\ t \in [0, T]$ if and only if $\widehat{V}_t = V_0 + \int_0^t \theta_t d\widehat{X}(t),\ t \in [0, T]$. This means that under condition (8), $\widehat{V}_t$ is a martingale because $\widehat{X}$ is a martingale too and therefore the strategy is $\widetilde{\mathbb{P}}$–admissible. Whence, the traditional pricing principle will give such H an arbitrage price. In addition, noting that for a hedger, her aim is to find θ_t such that $V_T = H$, the hedging problem (7) is equivalent to finding θ_t in the following equation

$$D(T)H = \widetilde{E}[D(T)H] + \int_0^T \theta_t d\widehat{X}(t). \tag{10}$$

It follows that the trading strategy θ_t, which replicates a claim H, satisfies that $D(T)H = \widetilde{E}[D(T)H] + \int_0^T \theta_t \widehat{\sigma}_t d\widetilde{W}(t)$. This makes sense since r_t is bounded and consequently Assumption A implies the integrability of $\theta_t\widehat{\sigma}_t$: $\widetilde{E}\int_0^T [\theta_t\widehat{\sigma}_t]^2 dt < \infty$.

Notice that the space of all stochastic processes in the form of $f(t, \widetilde{W}(t))$ satisfying $\widetilde{E}\int_0^T f^2(t, \widetilde{W}(t))dt < \infty$ is the Hilbert space $L^2([0,T]\times\mathbb{R}, \nu)$, where ν is the product measure of Lebesgue measure on $[0,T]$ and the probability measure of normal random variable $N(0,t)$. It is because of adaptivity that $\theta_t\widehat{\sigma}_t$ belongs to the Hilbert space. According to Dong and Gao (2011), we propose approximating $F(t, \widetilde{W}(t)) = \theta_t\widehat{\sigma}_t$ by an orthogonal expansion of the form:

$$F(t, \widetilde{W}(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\varphi_j(t)h_i(t, \widetilde{W}(t)), \tag{11}$$

where $h_i(t,x) = \frac{1}{\sqrt{i!}}H_i(\frac{x}{\sqrt{t}})$ with $H_i(\cdot)$ being the Hermite orthogonal polynomial system, $\varphi_j(t)$ is any complete orthonormal system on $[0,T]$, such as, the trigonometric system, the spline system or the wavelet system. Discussion about such expansions can be found in Dong and Gao (2011).

Therefore, the gains process of investment in $\widehat{X}$ can be written as

$$\begin{aligned}\widehat{G}_T(\theta) &= \int_0^T \theta_t d\widehat{X}(t) = \int_0^T F(t, \widetilde{W}(t))d\widetilde{W}(t) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\int_0^T \varphi_j(t)h_i(t, \widetilde{W}(t))d\widetilde{W}(t).\end{aligned} \tag{12}$$

Denote $\xi_{ij} = \int_0^T \varphi_j(t)h_i(t, \widetilde{W}(t))d\widetilde{W}(t)$. We then have the following theorem.

Theorem 1. *The system $\{1, \xi_{ij}\}$ is a complete orthonormal basis in the Hilbert space $\mathcal{H} = \{H : H$ is $\mathcal{F}_T^{\widetilde{W}}$ measurable and attainable and $\widetilde{E}[H^2] < \infty\}$, where $\widetilde{E}$ means the expectation operation under probability measure $\widetilde{\mathbb{P}}$.*

Before we give the proof of this theorem, we make the following remark.

Remark 2. Because $\mathcal{F}_T^{\widetilde{W}} = \mathcal{F}_T^W$ and the condition $\widetilde{E}[H^2] < \infty$ is equivalent to $E[H]^2 < \infty$, $\mathcal{H}$ can be written as $\mathcal{H} = \{H : H$ is $\mathcal{F}_T^W$ measurable and attainable and $E[H^2] < \infty\}$.

Proof. By Itô integral property, $\widetilde{E}[\xi_{ij}] = 0$ and

$$\widetilde{E}[\xi_{ij}]^2 = \widetilde{E}\int_0^T [\varphi_j(t)h_i(t, \widetilde{W}(t))]^2 dt = \int_0^T [\varphi_j(t)]^2 dt = 1.$$

Moreover, let $0 = s_0 < s_1 < \cdots < s_M = T$ be any partition on $[0,T]$ and $\triangle = \max_i(s_i - s_{i-1})$. We have for any i, j, m, n

$$\widetilde{E}[\xi_{ij}\xi_{nm}] = \widetilde{E}\left[\int_0^T \varphi_j(s)h_i(s, \widetilde{W}(s))d\widetilde{W}(s)\int_0^T \varphi_m(s)h_n(s, \widetilde{W}(s))d\widetilde{W}(s)\right]$$

$$
\begin{aligned}
&= \lim_{\triangle\to 0} \widetilde{E}\left[\sum_{k=1}^{M}\varphi_j(s_{k-1})h_i(s_{k-1},\widetilde{W}(s_{k-1}))(\widetilde{W}(s_k)-\widetilde{W}(s_{k-1}))\right.\\
&\qquad\left.\times\sum_{l=1}^{M}\varphi_m(s_{l-1})h_n(s_{l-1},\widetilde{W}(s_{l-1}))(\widetilde{W}(s_l)-\widetilde{W}(s_{l-1}))\right]\\
&= \lim_{\triangle\to 0} \widetilde{E}\sum_{k=1}^{M}\varphi_j(s_{k-1})\varphi_m(s_{k-1})h_i(s_{k-1},\widetilde{W}(s_{k-1}))h_n(s_{k-1},\widetilde{W}(s_{k-1}))\\
&\qquad\qquad\qquad\times(\widetilde{W}(s_k)-\widetilde{W}(s_{k-1}))^2\\
&= \lim_{\triangle\to 0}\sum_{k=1}^{M}\varphi_j(s_{k-1})\varphi_m(s_{k-1})\widetilde{E}[h_i(s_{k-1},\widetilde{W}(s_{k-1}))h_n(s_{k-1},\widetilde{W}(s_{k-1}))]\\
&\qquad\qquad\qquad\times(s_k-s_{k-1})\\
&=\delta_{in}\lim_{\triangle\to 0}\sum_{k=1}^{M}\varphi_j(s_{k-1})\varphi_m(s_{k-1})(s_k-s_{k-1}) = \delta_{in}\int_0^T\varphi_j(t)\varphi_m(t)dt=\delta_{in}\delta_{jm},
\end{aligned}
$$

where δ designates the Kronecker delta.

The completeness of $\{1,\xi_{ij}\}$ is due to that of $\varphi_j(s)h_i(s,\widetilde{W}(s))$ in the space $L^2([0,T]\times\mathbb{R},\nu)$ and attainability for any contingent claim $H\in\mathcal{H}$. □

The following corollary provides an expression of the proposed replication strategy and a pricing formula for European options.

Corollary 1. *Under Assumption A and condition* (8)*, in the market* (X,B) *a contingent claim* $H\in\mathcal{H}$ *is replicated by the self-financing strategy* θ_t^H*:*

$$
\theta_t^H=\frac{1}{\widehat{\sigma}_t}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\widetilde{E}[D(T)H\xi_{ij}]\varphi_j(t)h_i(t,\widetilde{W}(t)). \tag{13}
$$

More specifically, $H=H_0+\int_0^T\theta_t^H dX(t)$ *or equivalently,* $D(T)H=\widetilde{E}[D(T)H]+\int_0^T\theta_t^H d\widehat{X}(t)$*. Hence, the arbitrage price for* H *at time* t *is*

$$
Q^H(t)=B(t)\left(\widetilde{E}[D(T)H]+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\widetilde{E}[D(T)H\xi_{ij}]\xi_{ij}^{(t)}\right), \tag{14}
$$

where $\xi_{ij}^{(t)}=\int_0^t\varphi_j(t)h_i(t,\widetilde{W}(t))d\widetilde{W}(t)$*.*

Proof. Since $D(T)H=\widetilde{E}[D(T)H]+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}\xi_{ij}$, $c_{ij}=\widetilde{E}[D(T)H\xi_{ij}]$ by virtue of Theorem 1. In view of the expansion in (11), equation (13) holds.

Moreover, $\widehat{V}_t$ is a martingale and $V_T(\theta)=H$. According to (5.2.30) of Shreve (2004, p.218),

$$
Q^H(t)=B(t)\widehat{V}_t=B(t)\widetilde{E}[\widehat{V}_T|\mathcal{F}_t]=B(t)\widetilde{E}[D(T)H|\mathcal{F}_t]
$$

$$
\begin{aligned}
=&B(t)\left(\widetilde{E}[D(T)H]+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}\widetilde{E}[\xi_{ij}|\mathcal{F}_t]\right)\\
=&B(t)\left(\widetilde{E}[D(T)H]+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}B(t)c_{ij}\xi_{ij}^{(t)}\right),
\end{aligned}
$$

where $\xi_{ij}^{(t)}=\int_0^t\varphi_j(t)h_i(t,\widetilde{W}(t))d\widetilde{W}(t)$ since $\widetilde{E}[\xi_{ij}|\mathcal{F}_t]=\xi_{ij}^{(t)}$ by the property of the martingale $\{\xi_{ij}\}$. □

3 Implementation and numerical example

One thing we are concerned with is about how to use the expression of θ_t^H in practice. The main step of implementation of the proposed method is to convert the stochastic integrals ξ_{ij} into a series of usual integrals. To this end, we make the best use of the Itô lemma and specify the orthogonal sequence $\varphi_j(t)$ as follows:

$$
\varphi_0(t)=\frac{1}{\sqrt{T}},\qquad \varphi_j(t)=\sqrt{\frac{2}{T}}\cos\frac{j\pi t}{T},\quad j\geq 1.
$$

In addition, we also intensively use the relationship among Hermite polynomials to entail the following equations:

$$
h_i(t,x)=\sqrt{\frac{t}{i+1}}\frac{\partial}{\partial x}h_{i+1}(t,x),\ i\geq 0, \tag{15a}
$$

$$
\frac{x}{\sqrt{t}}h_i(t,x)=\sqrt{i+1}h_{i+1}(t,x)+\sqrt{i}h_{i-1}(t,x),\ i\geq 1. \tag{15b}
$$

To proceed, it is easy to figure out that $\xi_{00}=\frac{1}{\sqrt{T}}\widetilde{W}(T)$. Moreover, for $i=0$ and $j\geq 1$, Itô lemma gives

$$
\begin{aligned}
\xi_{0j}=&\int_0^T\varphi_j(t)d\widetilde{W}(t)=\varphi_j(T)\widetilde{W}(T)-\int_0^T\widetilde{W}(t)\varphi_j'(t)dt\\
=&(-1)^j\sqrt{\frac{2}{T}}\widetilde{W}(T)+\sqrt{\frac{2}{T}}\frac{j\pi}{T}\int_0^T\widetilde{W}(t)\sin\frac{j\pi t}{T}dt;
\end{aligned}
$$

and for $j=0$, $i\geq 1$, by virtue of (15) we have similarly

$$
\begin{aligned}
\xi_{i0}=&\int_0^T\varphi_0(t)h_i(t,\widetilde{W}(t))d\widetilde{W}(t)=\frac{1}{\sqrt{T}}\int_0^T h_i(t,\widetilde{W}(t))d\widetilde{W}(t)\\
=&\frac{1}{\sqrt{i+1}}h_{i+1}(T,\widetilde{W}(T))+\frac{1}{2\sqrt{T}}\frac{i}{\sqrt{i+1}}\int_0^T\frac{1}{\sqrt{t}}h_{i+1}(t,\widetilde{W}(t))dt.
\end{aligned}
$$

Furthermore, for all $i \geq 1$ and $j \geq 1$, let $F_{i+1,j}(t,x) = \varphi_j(t)\sqrt{\frac{t}{i+1}}h_{i+1}(t,x)$, Using Itô lemma as well as (15) yields

$$\xi_{ij} = \int_0^T \varphi_j(t)h_i(t,\widetilde{W}(t))d\widetilde{W}(t) = F_{i+1,j}(T,\widetilde{W}(T)) - \int_0^T \frac{\partial}{\partial t}F_{i+1,j}(t,\widetilde{W}(t))dt$$
$$- \frac{1}{2}\int_0^T \frac{\partial^2}{\partial x^2}F_{i+1,j}(t,\widetilde{W}(t))dt = (-1)^j\sqrt{\frac{2}{i+1}}h_{i+1}(T,\widetilde{W}(T))$$
$$+ \frac{1}{\sqrt{i+1}}\sqrt{\frac{2}{T}}\frac{j\pi}{T}\int_0^T \sqrt{t}\sin\frac{j\pi t}{T}h_{i+1}(t,\widetilde{W}(t))dt$$
$$+ \frac{1}{\sqrt{2T}}\frac{i}{\sqrt{i+1}}\int_0^T \frac{1}{\sqrt{t}}\cos\frac{j\pi t}{T}h_{i+1}(t,\widetilde{W}(t))dt.$$

To simplify the calculation, let $r > 0$ be a constant and hence so be $D(T)$. With these expressions of ξ_{ij}, we can calculate $c_{ij} = \widetilde{E}(H\xi_{ij})$ for any European option H as follows. Observe that Brownian motion $\widetilde{W}(t)$ has normal distribution $N(0,t)$ and independent increments so that $\widetilde{W}(t)$ and $\widetilde{W}(T) - \widetilde{W}(t)$ are independent for any $t \in [0,T]$. Whence, for European option, say $H = f(T,\widetilde{W}(T))$,

$$\begin{aligned} c_{00} =& \widetilde{E}[H\xi_{00}] = \widetilde{E}\left[f(T,\widetilde{W}(T))\frac{1}{\sqrt{T}}\widetilde{W}(T)\right] \\ =& \frac{1}{\sqrt{T}}\int_{-\infty}^{\infty} f(T,x)x\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{x^2}{2T}\right)dx \\ =& \int_{-\infty}^{\infty} f(T,\sqrt{T}x)x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \end{aligned}$$

for $i = 0$ and any $j \geq 1$;

$$\begin{aligned} c_{0j} =& (-1)^j\sqrt{2}\widetilde{E}\left[f(T,\widetilde{W}(T))\frac{1}{\sqrt{T}}\widetilde{W}(T)\right] \\ &+ \sqrt{\frac{2}{T}}\frac{j\pi}{T}\int_0^T \widetilde{E}[f(T,\widetilde{W}(T))\widetilde{W}(t)]\sin\frac{j\pi t}{T}dt \\ =& (-1)^j\sqrt{2}c_{00} + \sqrt{\frac{2}{T}}\frac{j\pi}{T}\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(T,x+y)x\sin\frac{j\pi t}{T} \\ &\times \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\frac{1}{\sqrt{2\pi(T-t)}}\exp\left(-\frac{y^2}{2(T-t)}\right)dxdydt \\ =& (-1)^j\sqrt{2}c_{00} + \frac{j}{\sqrt{2T}T}\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(T,\sqrt{t}x+\sqrt{T-t}y)\sqrt{t}x\sin\frac{j\pi t}{T} \\ &\times\exp\left(-\frac{x^2+y^2}{2}\right)dxdydt \end{aligned}$$

for $j = 0$ and $i \geq 1$;

$$c_{i0} = \widetilde{E}[H\xi_{i0}] = \widetilde{E}[f(T,\widetilde{W}(T))\xi_{i0}]$$

$$
\begin{aligned}
=&\frac{1}{\sqrt{i+1}}\widetilde{E}[f(T,\widetilde{W}(T))h_{i+1}(T,\widetilde{W}(T))] \\
&+\frac{1}{2\sqrt{T}}\frac{i}{\sqrt{i+1}}\int_0^T \frac{1}{\sqrt{t}}\widetilde{E}[f(T,\widetilde{W}(T))h_{i+1}(t,\widetilde{W}(t))]dt \\
=&\frac{1}{\sqrt{i+1}}\int_{-\infty}^{\infty} f(T,x)h_{i+1}(T,x)\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{x^2}{2T}\right)dx \\
&+\frac{1}{2\sqrt{T}}\frac{i}{\sqrt{i+1}}\int_0^T \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(T,x+y)h_{i+1}(t,x) \\
&\quad\times\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\frac{1}{\sqrt{2\pi(T-t)}}\exp\left(-\frac{y^2}{2(T-t)}\right)dxdydt \\
=&\frac{1}{i+1}\frac{1}{\sqrt{i!}}\int_{-\infty}^{\infty} f(T,\sqrt{T}x)H_{i+1}(x)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \\
&+\frac{1}{4\pi\sqrt{T}}\frac{i}{i+1}\frac{1}{\sqrt{i!}}\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{t}}f(T,\sqrt{t}x+\sqrt{T-t}y) \\
&\qquad\qquad\times H_{i+1}(x)\exp\left(-\frac{x^2+y^2}{2}\right)dxdydt,
\end{aligned}
$$

where we have used $h_i(t,x)=\frac{1}{\sqrt{i!}}H_i\left(\frac{x}{\sqrt{t}}\right)$ with $H_i(\cdot)$ being Hermite polynomials; and finally, for $i\geq 1$ and $j\geq 1$,

$$
\begin{aligned}
c_{ij}=&\widetilde{E}[H\xi_{ij}]=\widetilde{E}[f(T,\widetilde{W}(T))\xi_{ij}] \\
=&(-1)^j\sqrt{\frac{2}{i+1}}\widetilde{E}[f(T,\widetilde{W}(T))h_{i+1}(T,\widetilde{W}(T))] \\
&+\frac{1}{\sqrt{i+1}}\sqrt{\frac{2}{T}}\frac{j\pi}{T}\int_0^T\sqrt{t}\sin\frac{j\pi t}{T}\widetilde{E}[f(T,\widetilde{W}(T))h_{i+1}(t,\widetilde{W}(t))]dt \\
&+\frac{1}{\sqrt{2T}}\frac{i}{\sqrt{i+1}}\int_0^T\frac{1}{\sqrt{t}}\cos\frac{j\pi t}{T}\widetilde{E}[f(T,\widetilde{W}(T))h_{i+1}(t,\widetilde{W}(t))]dt \\
=&(-1)^j\frac{\sqrt{2}}{i+1}\frac{1}{\sqrt{i!}}\int_{-\infty}^{\infty}f(T,\sqrt{T}x)H_{i+1}(x)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \\
&+\frac{1}{i+1}\frac{1}{\sqrt{i!}}\frac{j}{\sqrt{2T}T}\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{t}\sin\frac{j\pi t}{T}f(T,\sqrt{t}x+\sqrt{T-t}y) \\
&\qquad\qquad\times H_{i+1}(x)\exp\left(-\frac{x^2+y^2}{2}\right)dxdydt \\
&+\frac{1}{\sqrt{2T}}\frac{i}{i+1}\frac{1}{\sqrt{i!}}\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{t}}\cos\frac{j\pi t}{T}f(T,\sqrt{t}x+\sqrt{T-t}y) \\
&\qquad\qquad\times\frac{1}{2\pi}H_{i+1}(x)\exp\left(-\frac{x^2+y^2}{2}\right)dxdydt.
\end{aligned}
$$

Thus, as the form of $H=f(\cdot,\cdot)$ is specified, we may calculate every coefficient c_{ij}, at least numerically. The following is an example from the text book Musiela and Rutkowski

(1997, p. 134). We compare our results with the one computed from using the Black-Scholes formula.

Table 3.1 The values of the coefficients c_{ij}

	j				
i	0	1	2	3	4
0	1.2933	0.0000	0.0001	-0.0012	-0.0003
1	0.3142	-0.1064	-0.0177	-0.0176	-0.0068
2	-0.1181	0.0677	0.0000	0.0075	-0.0000
3	-0.0175	0.0131	-0.0016	0.0014	-0.0003
4	0.0538	-0.0462	0.0115	-0.0051	0.0029
5	-0.0101	0.0097	-0.0033	0.0014	-0.0008
6	-0.0263	0.0269	-0.0113	0.0048	-0.0028
7	0.0140	-0.0151	0.0073	-0.0033	0.0019
8	0.0129	-0.0145	0.0079	-0.0038	0.0022

	j				
i	5	6	7	8	9
0	-0.0001	-0.0054	-0.0006	0.0006	0.0010
1	-0.0075	-0.0037	-0.0042	-0.0022	-0.0027
2	0.0026	-0.0000	0.0014	0.0000	0.0008
3	0.0006	-0.0001	0.0004	0.0000	0.0003
4	-0.0019	0.0013	-0.0009	0.0007	-0.0006
5	0.0005	-0.0003	0.0003	-0.0002	0.0002
6	0.0018	-0.0013	0.0009	-0.0007	0.0006
7	-0.0012	0.0009	-0.0006	0.0005	-0.0004
8	-0.0014	0.0010	-0.0007	0.0006	-0.0005

Example: Consider a call option on a stock with strike price \$30 and with three months to expiry. Suppose, in addition, the current stock price is \$31, the stock price volatility is

$\sigma = 10\%$ per annum, and the risk-free interest rate is $r = 5\%$ per annum with continuous compounding. We may suppose that the time interval is [0, 0.25], i.e., $T = 0.25$ years. Note that $H = (X_T - K)^+$ and we know the expression of X_T in terms of $\widetilde{W}(T)$. Therefore we can calculate the expectations $c_{ij} = \widetilde{E}(H\xi_{ij})$ displayed in Table 3.1.

Given truncation parameters I, J for i and j, we have the replication strategy for H at t, denoted by $\theta_t^H(I, J)$,

$$\theta_t^H(I, J) = \frac{D(T)}{\sigma \widehat{X}(t)} \sum_{i=0}^{I} \sum_{j=0}^{J} c_{ij} \varphi_j(t) h_i(t, \widetilde{W}(t)). \tag{16}$$

Table 3.2 gives some values of $\theta_0^H(I, J)$. It can be seen that as both I and J increase, the values converge to the value $\theta_0 = 0.82$ calculated in the book by the Black-Scholes formula. The maximum difference in absolute value is 0.0063, while the minimum difference in absolute value is 0.

Table 3.2 The values of $\theta_0^H(I, J)$

	J				
I	5	6	7	8	9
2	0.8263	0.8214	0.8200	0.8205	0.8208
4	0.8259	0.8217	0.8198	0.8207	0.8207
6	0.8256	0.8220	0.8197	0.8208	0.8206
8	0.8253	0.8222	0.8194	0.8209	0.8205

4 Conclusion and discussion

Through using an expansion of stochastic processes in a Hilbert space, an explicit solution of the self–financing replication strategy is given in terms of the basis in the space for the contingent claims defined on a complete financial market. As a consequence, we derive the arbitrage price for attainable claims using the martingale property of Iô integrals. This procedure that we derive the pricing formula from the replication policy is exactly converse to the one in Black-Scholes theory: In the Black-Scholes theory, the replication strategy is derived from the pricing formula, $\theta_t = c_s(X(t), T-t)$, so–called the delta of the option. This is because we take the advantage of the orthogonal series expansion of unknown functionals. The solutions for both are explicit, although they are expressed by means of infinite series. An implementation procedure for the proposed method has been provided and used in practice through using a numerical example. The numerical results show that our method is implementable and feasible.

Furthermore, since our method is based on expanding a process into an orthogonal series, in the case where there is a need to determine functionals of a stochastic process call for a stochastic process, the problem may be solved by determining the corresponding coefficients involved in the corresponding expansion of the process. Hence, there are many problems, for which our method may play a key role in providing solutions. Such problems include a) the hedging problems in an incomplete market with partial information available for a hedger who is guided by quadratic criterion; b) the hedging problems in an incomplete market with partial information available for a hedger who is guided by maximising her utility rather than quadratic criterion; c) the hedging problems in an incomplete market where the asset price processes are driven by some Lévy process rather than Brownian motion or compensated Poisson process; and d) optimal control problems with partial information. Such problems are left for future research.

5 Acknowledgments

The authors acknowledges the financial support from the Australian Research Council Discovery Grants Scheme under Grant Number: DP1096374.

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