Challenging the Robustness of Optimal Portfolio Investment with Moving Average-Based Strategies

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CHALLENGING THE ROBUSTNESS OF OPTIMAL PORTFOLIO INVESTMENT WITH MOVING AVERAGE-BASED STRATEGIES

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Abstract. The aim of this paper is to compare the performance of a theoretically optimal portfolio with that of a moving average-based strategy in the presence of parameters misspecification. The setting we consider is that of a stochastic asset price model where the trend follows an unobservable Ornstein-Uhlenbeck process. For both strategies, we provide the asymptotic expectation of the logarithmic return as a function of the model parameters. Then, numerical examples are given, showing that an investment strategy using a moving average crossover rule is more robust than the optimal strategy under parameters misspecification.

Introduction

There exist three main approaches to investing in financial markets (see Blanchet-Scalliet et al. (2007)). The first one is based on fundamental economic principles (see Tideman (1972) for details). The second one is based on historical time series of prices and volumes (see Taylor and Allen (1992), Brown and Jennings (1989) and Edwards et al. (2007) for details). The third one relies on mathematical models and stochastic control and was introduced in Merton (1969). Assuming that the risky asset follows a geometric Brownian motion, Merton derived the optimal investment rules for an investor maximizing his expected utility function. Many generalisations of this problem have been studied in the literature (see Karatzas and Zhai (2001), Bren-dle (2006), Lakner (1998), Sass and Haussmann (2004), or Rieder and Bauerle (2005) for example), contributing to a rigorous theory for optimal investment in financial markets.

There is however a serious drawback to this strand of research, namely, the calibration problem. In Bel Hadj Ayed et al. (2017a), the authors assess the feasibility of forecasting trends modeled by an unobserved mean-reverting diffusion. They show that, due to a weak signal-to-noise ratio, a bad calibration is extremely likely. Therefore, the robustness of the optimal strategy under parameters misspecification is a very natural and practical point to address.

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In this paper, we consider a stochastic asset price model where the trend is an unobservable Ornstein Uhlenbeck process. The purpose of this work is to characterize and compare the respective performances of the optimal strategy and of a moving average crossover strategy under parameters misspecification. A few related works can be found in the literature: for instance, using the same risky asset model, Zhu and Zhou (2009) analyse the performance of a trading strategy based on a geometric moving average rule. Also, the authors in Blanchet-Scalliet et al. (2007) assume that the drift is an unobservable, piecewise constant process jumping at a random time. They analyse the performance of the optimal trading strategy under parameters misspecification and compare it, using Monte Carlo simulations, with that of a simple investment rule based on moving averages.

The paper is organized as follows: the first section presents the model, recalls some results from filtering theory and rewrites the Kalman filter estimator as a corrected exponential average.

In the second section, the optimal trading strategy under parameters misspecification is investigated. The stochastic differential equation driving the optimal portfolio’s logarithmic return is established. Using this result, we provide, in closed form, the asymptotic expectation of the logarithmic return as a function of the signal-to-noise-ratio and of the trend mean reversion speed. We conclude this section by giving conditions on the model and the strategy parameters that guarantee a positive asymptotic expected logarithmic return and the existence of an optimal duration.

In the third section, we consider a moving average crossover strategy. Again, the stochastic differential equation for the logarithmic return is provided, and the asymptotic expectation of the logarithmic return as a function of the model parameters is given in closed form.

In the fourth section, numerical simulations are performed. First, the optimal durations of the Kalman filter and of the optimal strategy under parameters misspecification are illustrated over different regimes for the trend. We then compare the respective performances of a moving average crossover strategy and of a classical optimal strategy used in the industry (with a duration $\tau = 1$ year) over several theoretical regimes. We also compare their performances under Heston stochastic volatility model using Monte Carlo simulations. These numerical simulations actually show that the approach based on moving average crossover is more robust than the optimal strategy under parameters misspecification.

Finally, an empirical analysis based on real data is performed, leading to the same conclusions.
1. Setup

This section begins by presenting the model, which corresponds to an unobserved mean-reverting diffusion. After that, we reformulate this model in a completely observable environment (see Liptser and Shiriaev [1977] for details). This setting introduces the conditional expectation of the trend, knowing the past observations. Then, we recall the asymptotic continuous time limit of the Kalman filter and we rewrite this estimator as a corrected exponential average.

1.1. The model. Consider a financial market living on a stochastic basis \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), where \(\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}\) is the natural filtration associated to a two-dimensional (uncorrelated) Wiener process \((W^S, W^\mu)\), and \(\mathbb{P}\) is the objective probability measure. The dynamics of the risky asset \(S\) is given by

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma_S dW^S_t, \quad (1)
\]

\[
d\mu_t = -\lambda \mu_t dt + \sigma_\mu dW^\mu_t, \quad (2)
\]

with \(\mu_0 = 0\). We also assume that \((\lambda, \sigma_\mu, \sigma_S) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*\). The parameter \(\lambda\) is called the trend mean reversion speed. Indeed, \(\lambda\) can be seen as the "force" that pulls the trend back to zero. Denote by \(\mathbb{F}^S = \{\mathcal{F}^S_t\}\) be the natural filtration associated to the price process \(S\). An important point is that only \(\mathbb{F}^S\)-adapted processes are observable, which implies that agents in this market do not observe the trend \(\mu\).

1.2. The observable framework. As stated above, the agents can only observe the stock price process \(S\). Since, the trend \(\mu\) is not \(\mathbb{F}^S\)-measurable, the agents do not observe it directly. Indeed, the model \([1)-(2)]\) corresponds to a system with partial information. The following proposition gives a representation of the model \([1)-(2)]\) in an observable framework (see Liptser and Shiriaev [1977] for details).

**Proposition 1.** The dynamics of the risky asset \(S\) is also given by

\[
\frac{dS_t}{S_t} = E \left[ \mu_t | \mathcal{F}^S_t \right] dt + \sigma_S dN_t, \quad (3)
\]

where \(N\) is a \((\mathbb{P}, \mathbb{F}^S)\) Wiener process.

**Remark 1.1.** In the filtering theory (see Liptser and Shiriaev [1977] for details), the process \(N\) is called the innovation process. To understand this name, note that:

\[
dN_t = \frac{1}{\sigma_S} \left( \frac{dS_t}{S_t} - E \left[ \mu_t | \mathcal{F}^S_t \right] dt \right).
\]

Then, \(dN_t\) represents the difference between the current observation and what we expect knowing the past observations.
1.3. Optimal trend estimator. The system $1$-$2$ corresponds to a Linear Gaussian Space State model (see Brockwell and Davis (2002) for details). In this case, the Kalman filter gives the optimal estimator, which corresponds to the conditional expectation $E[\mu_t|F^S_t]$. Since $(\lambda, \sigma_\mu, \sigma_S) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*$, the model $1$-$2$ is a controllable and observable time invariant system. In this case, it is well known that the estimation error variance converges to an unique constant value (see Kalman et al. (1962) for details). This corresponds to the steady-state Kalman filter. The following proposition (see Bel Hadj Ayed et al. (2017a) for a proof) provides a continuous-time representation of the steady-state Kalman filter:

**Proposition 2.** The steady-state Kalman filter has a continuous time limit depending on the asset returns:

$$d\tilde{\mu}_t = -\lambda \beta \tilde{\mu}_t dt + \lambda (\beta - 1) \frac{dS_t}{S_t},$$

where

$$\beta = \left(1 + \frac{\sigma_\mu^2}{\lambda^2 \sigma_S^2}\right)^{\frac{1}{2}}.$$ (5)

The steady-state Kalman filter can also be rewritten as a corrected exponential average:

**Proposition 3.**

$$\hat{\mu}_t = m^* \tilde{\mu}_t^*,$$

where $m^* = \frac{\beta - 1}{\beta}$ and $\tilde{\mu}_t^*$ is the exponential average given by:

$$d\tilde{\mu}_t^* = -\frac{1}{\tau^*} \tilde{\mu}_t^* dt + \frac{1}{\tau^*} \frac{dS_t}{S_t},$$

with an average duration $\tau^* = \frac{1}{\lambda \beta}$.

2. Optimal strategy under parameters misspecification

In this section, the optimal trading strategy under parameters misspecification is considered. We first give the stochastic differential equation driving the optimal portfolio’s logarithmic return, and then provide, in closed form, the asymptotic expectation of the same.

2.1. Context. Consider the financial market defined in Section 1 with constant risk free rate and no transaction costs. Let $P$ be a self financing portfolio given by:

$$dP_t = \frac{\omega_t}{P_t} dS_t,$$

$$P_0 = x.$$
where $\omega_t$ is the fraction of wealth invested in the risky asset (also named the control variable). The agent aims at maximizing his expected logarithmic utility on an admissible domain $\mathcal{A}$ for the allocation process. In this section, we assume that the agent is not able to observe the trend $\mu$. Formally, $\mathcal{A}$ represents all the $\mathcal{F}^S$-progressive and measurable processes and the problem is:

$$\omega^* = \arg \sup_{\omega \in \mathcal{A}} \mathbb{E} \left[ \ln (P_t) \mid P_0 = x \right].$$

The solution of this problem is well known and easy to compute (see Lakner (1998) for example). Indeed, it has the following form:

$$\omega^*_t = \frac{E \left[ \mu_t \mathcal{F}^S_t \right]}{\sigma^2 S}.$$

Using the steady-state Kalman filter, the optimal allocation becomes:

$$\omega^*_t = \frac{\hat{\mu}_t}{\sigma^2 S},$$

where $\hat{\mu}$ is given by Equation 4.

The case of perfect parameters specification is developed in Bel Hadj Ayed et al. (2017b). In practice however, the parameters are unknown and must be estimated. In Bel Hadj Ayed et al. (2017a), the authors assess the feasibility of forecasting trends modeled by an unobserved mean-reverting diffusion. They show that, due to a weak signal-to-noise ratio, a bad calibration is very likely. Since, from Proposition 3, the steady state Kalman filter is a corrected exponential moving average of past returns, it follows that a misspecification of the parameters $\left( \lambda, \sigma_{\mu} \right)$ is equivalent to a misspecification of the factor $\frac{\beta - 1}{\beta}$ and the duration $\tau^*$. Suppose that an agent make an error $\epsilon = (\epsilon_m, \epsilon_\tau)$ on the parameters and, thinking that the optimal duration is $\tau_\epsilon$, considers the following misspecified Kalman filter:

$$d\tilde{\mu}_t = -\frac{1}{\tau_\epsilon} \tilde{\mu}_t \, dt + \frac{1}{\tau_\epsilon} \frac{dS_t}{S_t},$$

$$\tilde{\mu}_0 = 0.$$

Using this estimator, the agent will invest following:

$$\frac{dP_t}{P_t} = m_\epsilon \tilde{\mu}_t \frac{dS_t}{S_t},$$

$$P_0 = x,$$

where $m_\epsilon > 0$. The following lemma gives the law of this filter $\tilde{\mu}^*$:

**Lemma 2.1.** The exponential moving average of Equation 9 is given by:

$$\tilde{\mu}^*_t = e^{-\frac{t}{\tau_\epsilon}} \left( \int_0^t e^{\frac{s}{\tau_\epsilon}} \mu_s \, ds + \sigma_S \int_0^t e^{\frac{s}{\tau_\epsilon}} \, dW^S_s \right).$$
Moreover, this filter is a centered Gaussian process, whose variance is:

\[
\mathbb{V} [\tilde{\mu}_t] = \frac{\sigma^2_S}{2\tau_e} \left( 1 - e^{-2t} \right) + \frac{\sigma^2_\mu}{\tau^2_e \lambda \left( \frac{1}{\tau_e} - \lambda \right)} \cdot \left( \frac{1}{\tau_e} e^{\frac{-2t}{\tau_e}} - \frac{1}{2} \right) \\
+ \frac{1 - e^{-t(\lambda + \frac{1}{\tau_e})}}{\frac{1}{\tau_e} + \lambda} + \frac{2e^{-t(\lambda + \frac{1}{\tau_e})} - e^{-2t(\lambda + \frac{1}{\tau_e})} - e^{-\frac{4t}{\tau_e}}}{\frac{1}{\tau_e} - \lambda}.
\]

Proof. Applying Itô’s lemma to the function \( f(\tilde{\mu}_t, t) = \tilde{\mu}_t e^\frac{t}{\tau_e} \) and using Equation (1), it follows that:

\[
df(\tilde{\mu}_t, t) = e^\frac{t}{\tau_e} \left( \mu_t dt + \sigma_S dW^S_t \right).
\]

The integral of this stochastic differential equation from 0 to \( t \) gives Equation (13). Therefore, \( \tilde{\mu}_t \) is a Gaussian process. Its mean is null (because \( \mu_0 = 0 \)). Since \( \mu \) and \( W^S \) are supposed to be independent, the variance of the process \( \tilde{\mu}_t \) is equal to the sum of

\[
\mathbb{V} \left[ e^{\frac{-t}{\tau_e}} \int_0^t e^\frac{s}{\tau_e} \mu_s ds \right]
\]

and

\[
\mathbb{V} \left[ e^{\frac{-t}{\tau_e}} \sigma_S \int_0^t e^\frac{s}{\tau_e} dW^S_s \right].
\]

The first term is computed using:

\[
\mathbb{V} \left[ \int_0^t e^{\frac{s}{\tau_e}} \mu_s ds \right] = \int_0^t \int_0^t e^{\frac{s_1+s_2}{\tau_e}} \mathbb{E} [\mu_{s_1} \mu_{s_2}] ds_1 ds_2.
\]

Since \( \mu \) is a centered Ornstein Uhlenbeck, for all \( s, t \geq 0 \), we have:

\[
\mathbb{E} [\mu_s \mu_t] = \mathbb{Cov} [\mu_s, \mu_t] = \frac{\sigma^2_\mu}{2\lambda} e^{-\lambda(s+t)} \left( e^{2\lambda s t} - 1 \right).
\]

Finally, the second term is computed using:

\[
\mathbb{V} \left[ \int_0^t e^{ks} dW^S_s \right] = \frac{1}{2k} \left( e^{2kt} - 1 \right),
\]

with \( k > 0 \). \( \square \)

2.2. Portfolio dynamic. The following proposition gives the stochastic differential equation of the misspecified optimal portfolio:

**Proposition 4.** Equation (11) leads to:

\[
d\ln(P_t) = \frac{m_\epsilon \tau_e}{2\sigma^2_S} \mu^2_t d\tilde{\mu}_t^2 + m_\epsilon \frac{\mu^2_t}{\sigma^2_S} \left( 1 - \frac{m_\epsilon}{2} \right) - \frac{1}{2\tau_e} dt. \tag{14}
\]

Proof. Equation (11) is equivalent to (by Itô’s lemma):

\[
d\ln(P_t) = \frac{m_\epsilon \mu^2_t}{\sigma^2_S} dS_t - \frac{m^2_\epsilon \tilde{\mu}^2_t}{2\sigma^2_S} dt.
\]

Using Equation (6),

\[
d\ln(P_t) = \frac{m_\epsilon \tau_e}{\sigma^2_S} \tilde{\mu}_t d\tilde{\mu}_t + \frac{m^2_\epsilon \tilde{\mu}^2_t}{\sigma^2_S} - \frac{1}{2} \frac{m^2_\epsilon \tilde{\mu}^2_t}{\sigma^2_S} dt.
\]
Itô’s lemma on Equation (6) gives:
\[ d\tilde{\mu}_t^2 = 2\tilde{\mu}_t d\tilde{\mu}_t + \frac{\sigma^2 S}{\tau^2} dt. \]

Using this equation, the dynamics of the logarithmic return follow. □

Remark 2.2. Proposition 4 shows that the returns of the optimal strategy can be broken down into two terms. The first one represents an option on the square of the realized returns and is called the Option profile. The second term is called the Trading Impact. These terms are introduced and discussed in Bruder and Gaussel (2011) in a general diffusion model for the risky asset.

2.3. Expected logarithmic return. The following theorem gives the asymptotic expected logarithmic return of the misspecified optimal strategy.

**Theorem 2.3.** Consider the portfolio given by Equation (11). There holds:
\[ \lim_{T \to \infty} \mathbb{E} \left[ \ln \left( \frac{P_T}{P_0} \right) \right] = m_e \tau_e \left( \beta^2 - 1 \right) \frac{(2 - m_e) - m_e \left( \tau_e + \frac{1}{\lambda} \right)}{4 \tau_e \left( \tau_e + \frac{1}{\lambda} \right)}, \]
\[ (15) \]
where \( \beta \) is given by Equation (5).

**Proof.** Using Proposition 4 it follows that:
\[ \mathbb{E} \left[ \ln \left( \frac{P_T}{P_0} \right) \right] = \frac{m_e \tau_e}{2 \sigma^2 S} \mathbb{E} \left[ \tilde{\mu}_T^2 \right] + m_e \int_0^T \left( \frac{\mathbb{E} \left[ \tilde{\mu}_t^2 \right] (2 - m_e)}{2 \sigma^2 S} - \frac{1}{2 \tau_e} \right) dt. \]

Moreover, \( \mathbb{E} \left[ \tilde{\mu}_t^2 \right] \) is given by Lemma 2.1. Then, integrating the expression from 0 to \( T \) and letting \( T \) tend to \( \infty \), the result follows. □

The following result is a corollary of the previous theorem. It represents the asymptotic expected logarithmic return as a function of the signal-to-noise-ratio and of the trend mean reversion speed \( \lambda \).

**Corollary 2.4.** Consider the portfolio given by Equation (11). In this case:
\[ \lim_{T \to \infty} \mathbb{E} \left[ \ln \left( \frac{P_T}{P_0} \right) \right] = m_e \frac{2 \tau_e (2 - m_e) SNR - m_e \left( \lambda \tau_e + 1 \right)}{4 \tau_e \left( \lambda \tau_e + 1 \right)}, \]
\[ (16) \]
where SNR is the signal-to-noise-ratio:
\[ SNR = \frac{\sigma^2 \mu}{2 \lambda \sigma^2 S}. \]
\[ (17) \]

Moreover:

1. If \( m_e < 2 \), for a fixed parameter value \( \lambda \), this asymptotic expected logarithmic return is an increasing function of SNR.
For a fixed parameter value SNR, it is a decreasing function of $\lambda$.

**Proof.** Since $\beta = \sqrt{1 + \frac{2SNR}{\lambda}}$, the use of this expression in Equation (15) gives the result. $\square$

**Remark 2.5.** Assume that the agent achieves a perfect calibration and uses $m^* = m_{e=0} = \frac{\beta - 1}{2}$ and $\tau^* = \tau_{e=0} = \frac{1}{\lambda \beta}$. In this case, we recover the result of Bel Hadj Ayed et al. (2017):

$$\lim_{T \to \infty} E \left[ \ln \left( \frac{P_T}{P_0} \right) \right] = \frac{1}{2} \left( SNR + \lambda - \sqrt{\lambda (\lambda + 2SNR)} \right),$$

where SNR is defined in Equation (17).

The following proposition provides conditions on the trend parameters and the duration $\tau_e$ that guarantee a positive asymptotic expected logarithmic return and the existence of an optimal duration.

**Proposition 5.** Consider the portfolio given by Equation (11) and suppose that $m_e < 2$. In this case, the asymptotic expected logarithmic return is positive if and only if:

1. $\frac{SNR}{\lambda} > \frac{2m_e}{2-m_e}$.
2. $\tau_e > \tau_{\min}$, where:

$$\tau_{\min} = \frac{m_e}{2 (2 - m_e) SNR - \lambda m_e}.$$  

Moreover, there exists an optimal duration $\tau_{\min} < \tau_{opt} < \infty$ if and only if $\frac{SNR}{\lambda} > \frac{2m_e}{2-m_e}$ and:

$$\tau_{opt} = \frac{m_e + \sqrt{(2 - m_e)2m_e \frac{SNR}{\lambda}}}{2 (2 - m_e) SNR - \lambda m_e}.$$

**Proof.** Using Equation (16), the first part of the proposition follows. Since the asymptotic expected logarithmic return of the mis-specified strategy is positive after $\tau_{min}$ and tends to zero if $\tau_e$ tends to the infinity, there exists an optimal duration $\tau_{opt}$. This optimal value is obtained as a critical point by cancelling the derivative of Equation (16) with respect to $\tau$. $\square$

**Remark 2.6.** Even if the investor’s estimate of the SNR is not higher than the real SNR, the constraint $\frac{SNR}{\lambda} > \frac{2m_e}{2-m_e}$ may not be satisfied. Indeed, an over-estimation of the trend mean reversion speed $\lambda$ can lead to a negative performance.

### 3. MOVING AVERAGE CROSSOVER STRATEGY

In this section, we consider a moving average crossover strategy based on geometric moving averages. For such a portfolio, we write down the equation of its logarithmic return and provide in closed form the asymptotic expectation of the logarithmic return.
3.1. **Context.** In the same financial market as in Section 2 let \( G(t, L) \) be the geometric moving average at time \( t \) of the stock prices on a window \( L \):

\[
G(t, L) = \exp \left( \frac{1}{L} \int_{t-L}^{t} \log(S_u) \, du \right).
\]  

(21)

Let \( Q \) be a self financing portfolio given by:

\[
\frac{dQ_t}{Q_t} = \frac{\theta_t}{S_t} \, dS_t,
\]

(22)

\[
Q_0 = x,
\]

(23)

where \( \theta_t \) is the fraction of wealth invested by the agent in the risky asset:

\[
\theta_t = \gamma + \alpha 1_{G(t, L_1) > G(t, L_2)}
\]

with \( \gamma, \alpha \in \mathbb{R} \) and \( 0 < L_1 < L_2 < t \). This trading strategy is a combination of a fixed strategy and a pure moving average crossover strategy.

3.2. **Portfolio dynamic.** The following proposition gives the stochastic differential equation of the cross moving average portfolio.

**Proposition 6.** Equation (22) leads to:

\[
\begin{align*}
\frac{d\ln(Q_t)}{Q_t} & = \left( \gamma + \alpha 1_{G(t, L_1) > G(t, L_2)} \right) \mu_t - \frac{\gamma^2 \sigma^2}{2} \sigma^2_t + \frac{\alpha^2 + 2\alpha \gamma}{2} 1_{G(t, L_1) > G(t, L_2)} \right) dt \\
& \quad + \left( \gamma + \alpha 1_{G(t, L_1) > G(t, L_2)} \right) \sigma_t \sigma dW_t^S.
\end{align*}
\]

**Proof.** Applying Itô’s lemma to the process \( \ln(Q) \) and using \( 1_{G(t, L_1) > G(t, L_2)} = 1_{G(t, L_1) > G(t, L_2)} \), Proposition [6] follows. \( \square \)

3.3. **Expected logarithmic return.** The following theorem characterizes the asymptotic expected logarithmic return of the portfolio associated to the moving average crossover trading strategy.

**Theorem 3.1.** Consider the portfolio given by Equation (22). In this case:

\[
\begin{align*}
\lim_{T \to \infty} \mathbb{E} \left[ \ln \left( \frac{Q_T}{Q_0} \right) \right] & = -\frac{\gamma^2 \sigma^2}{2} - \frac{\alpha^2 + 2\alpha \gamma}{2} \sigma^2 \Phi \left( \frac{m_{(L_1, L_2, \sigma_s)}}{\sqrt{s(L_1, L_2, \lambda, \sigma_\mu, \sigma_s)}} \right) \\
& \quad + \frac{\alpha \sigma^2_{\mu}}{2 \lambda^2 L_1 L_2 \sqrt{s(L_1, L_2, \lambda, \sigma_\mu, \sigma_s)}} \Phi' \left( \frac{m_{(L_1, L_2, \sigma_s)}}{\sqrt{s(L_1, L_2, \lambda, \sigma_\mu, \sigma_s)}} \right).
\end{align*}
\]
where $\Phi$ is the cumulative distribution function of the standard normal variable and:

$$m_{(L_1, L_2, \sigma_S)} = \frac{-\sigma_S^2}{4} (L_2 - L_1),$$

$$s_{(L_1, L_2, \lambda, \sigma, \mu, \sigma_S)} = -\sigma_S^2 (L_2 - L_1)^2 - \frac{\sigma^2}{\lambda^4} \left( \frac{1}{L_1} - \frac{1}{L_2} \right)$$

$$+ \frac{\sigma_S^2}{\lambda^5} \left[ \frac{1}{L_1^2} (1 - e^{-\lambda L_1}) + \frac{1}{L_2^2} (1 - e^{-\lambda L_2}) - \frac{1}{L_1 L_2} \left( 1 - e^{-\lambda (L_2 - L_1)} - e^{-\lambda (L_2 + L_1)} \right) \right].$$

**Proof.** Since the processes $\mu$ and $W^S$ are centered, Proposition 6 implies that:

$$\mathbb{E} \left[ \ln \left( \frac{Q_T}{Q_0} \right) \right] = -\frac{\gamma^2 \sigma_S^2}{2} (T - L_2)$$

$$+ \alpha \int_{L_2}^{T} \mathbb{E} \left[ \mu_t \mathbf{1}_{G(t, L_1) > G(t, L_2)} \right] dt$$

$$- \frac{(\alpha^2 + 2 \alpha \gamma) \sigma_S^2}{2} \int_{L_2}^{T} \mathbb{E} \left[ \mathbf{1}_{G(t, L_1) > G(t, L_2)} \right] dt,$$

where $T > L_2$. Let $t > L_2$ and consider the following process:

$$X_t = m_1(t) - m_2(t), \quad (24)$$

where $\forall i \in \{1, 2\}$:

$$m_i(t) = \frac{1}{L_i} \int_{t-L_i}^{t} \log \left( S_u \right) du.$$  

Then $X$ is a Gaussian process. Based on Lemma 2 in [Zhu and Zhou (2009)], $\forall t > L_2$:

$$\{ G(t, L_1) > G(t, L_2) \} \iff \{ X_t > 0 \}, \quad (25)$$

$$\mathbb{E} \left[ \mathbf{1}_{G(t, L_1) > G(t, L_2)} \right] = \Phi \left( \frac{\mathbb{E} [X_t]}{\sqrt{\text{Var} [X_t]}}, \right), \quad (26)$$

$$\mathbb{E} \left[ \mu_t \mathbf{1}_{G(t, L_1) > G(t, L_2)} \right] = \text{Cov} [X_t, \mu_t] \frac{\text{Var} [X_t]}{\sqrt{\text{Var} [X_t]}} \Phi' \left( -\frac{\mathbb{E} [X_t]}{\sqrt{\text{Var} [X_t]}} \right) \quad (27)$$

The following lemma gives the mean, the asymptotic variance of the process $X$ and the covariance function between the processes $X$ and $\mu$. 

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Lemma 3.2. Consider the process $X$ defined in Equation (24). In this case, $\forall t > L_2$:

$$
\mathbb{E} [X_t] = -\frac{\sigma^2 S}{4} (L_2 - L_1),
$$

(28)

$$
\lim_{t \to \infty} \text{Var} [X_t] = s(L_1, L_2, \lambda, \sigma, \sigma_S),
$$

(29)

$$
\text{Cov} [X_t, \mu_t] = g(t, L_1) - g(t, L_2),
$$

(30)

where $s(L_1, L_2, \lambda, \sigma, \sigma_S)$ is defined in Theorem 3.1 and

$$
g (t, L) = -\frac{\sigma^2 e^{-\lambda t}}{\lambda^2 L} \left( \lambda L + \sinh (\lambda (t - L)) - \sinh (\lambda t) \right).
$$

(31)

Proof of Lemma 3.2. Since:

$$
\mathbb{E} [m_i (t)] = -\frac{\sigma^2 S}{4} (2t - L_i),
$$

Equation (28) follows. Moreover:

$$
\text{Cov} [m_1 (t), m_2 (t)] = \frac{1}{L_1 L_2} \int_{t-L_1}^{t} \int_{t-L_2}^{t} \text{Cov} [\ln S_u, \ln S_v] \, du \, dv,
$$

Since

$$
\text{Cov} [\ln S_u, \ln S_v] = \int_{0}^{u} \int_{0}^{v} \text{Cov} [\mu_s, \mu_t] \, ds \, dt + \sigma^2 S \min (u, v),
$$

and the drift $\mu$ is an Ornstein Uhlenbeck process:

$$
\text{Cov} [\mu_s, \mu_t] = \frac{\sigma^2 e^{-\lambda (s+t)}}{2\lambda} \left( e^{2\lambda \min(s,t)} - 1 \right).
$$

Then

$$
\text{Cov} [\ln S_u, \ln S_v] = \left( \sigma^2_S + \frac{\sigma^2 \lambda^2}{2} \right) \min (u, v)
$$

$$+
\frac{\sigma^2 \lambda^2}{2} \left( 2e^{-\lambda u} + 2e^{-\lambda v} - e^{-\lambda |v-u|} - e^{-\lambda (v+u)} - 1 \right).
$$

Using

$$
\text{Var} [X_t] = \text{Var} [m_1 (t)] + \text{Var} [m_2 (t)] - 2\text{Cov} [m_1 (t), m_2 (t)]
$$

and tending t to $\infty$ Equation (29) follows. Since the processes $W^S$ and $\mu$ are supposed to be independent, there holds:

$$
\text{Cov} [X_t, \mu_t] = \text{Cov} [m_1 (t), \mu_t] - \text{Cov} [m_2 (t), \mu_t].
$$

Moreover

$$
\text{Cov} [m_i (t), \mu_t] = \frac{1}{L_i} \int_{t-L_i}^{t} \text{Cov} [\ln S_u, \mu_t] \, du,
$$

and

$$
\text{Cov} [\ln S_u, \mu_t] = \int_{0}^{u} \text{Cov} [\mu_s, \mu_t] \, ds,
$$

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then
\[ \text{Cov} \left[ m_i(t), \mu_t \right] = g(t, L_i), \]
where the function \( g \) is defined in Equation (31). Equation (30) follows

The use of Lemma 3.2 gives:
\[
\mathbb{E} \left[ \ln \left( \frac{Q_T}{Q_0} \right) \right] = -\frac{\gamma^2 \sigma_S^2}{2} (T - L_2) + \alpha \Phi \left( \frac{m(L_1, \sigma_S)}{\sqrt{\text{Var}[X_t]}} \right) \int_{L_2}^{T} \text{Cov} \left[ X_t, \mu_t \right] \sqrt{\text{Var}[X_t]} dt
\]

\[ - \frac{(\alpha^2 + 2\alpha \gamma) \sigma_S^2}{2} (T - L_2) \Phi \left( \frac{m(L_1, \sigma_S)}{\sqrt{\text{Var}[X_t]}} \right). \]

Moreover, a straightforward computation shows that:
\[
\lim_{T \to \infty} \int_{L_2}^{T} \frac{\text{Cov}[X_t, \mu_t]}{\sqrt{\text{Var}[X_t]}} dt = \frac{\sigma^2}{2\lambda^3 L_1 L_2 \sqrt{\text{Var}[X_t]}} \left( L_2 \left( 1 - e^{-\lambda L_1} \right) - L_1 \left( 1 - e^{-\lambda L_2} \right) \right),
\]

the result of Theorem 3.1 follows.

3.4. Strategy based on a single moving average. Suppose that \( L_1 = 0 \) and \( L_2 = L \). In this case, the fraction of wealth invested by the agent in the risky asset becomes:
\[ \theta_1^t = \gamma + \alpha \mathbf{1}_{S_t \geq G(t, L)}, \]
where \( G \) is the geometric moving average defined in Equation (21), and the self financing portfolio \( Q^1 \) becomes:
\[
\frac{dQ^1_t}{Q^1_t} = \frac{\theta^t_1 dS_t}{S_t}, \quad (32)
\]
\[ Q^1_0 = x. \quad (33) \]

This particular case corresponds to the allocation profile studied in Zhu and Zhou (2009), under the additional assumptions that the two Wiener processes \( W^S \) and \( W^\mu \) are uncorrelated and that the trend mean reverts to 0. In this context, we provide the asymptotic expected logarithmic return of the trading strategy (already in Zhu and Zhou (2009)):
Theorem 3.3. Consider the portfolio given by Equation (32). In this case:

\[
\lim_{T \to \infty} E \left[ \frac{\ln \left( \frac{Q^1_T}{Q^0_0} \right)}{T} \right] = -\frac{\gamma^2 \sigma^2 S}{2} - \frac{(\alpha^2 + 2\alpha \gamma) \sigma^2 S^2}{2} \Phi \left( \frac{m^1_{(L,\sigma S)}}{\sqrt{s^1_{(L,\lambda,\sigma,\sigma S)}}} \right) + \frac{\alpha \sigma^2 \mu^1_{(L,\sigma S)}}{2\lambda^2} \Phi' \left( \frac{m^1_{(L,\sigma S)}}{\sqrt{s^1_{(L,\lambda,\sigma,\sigma S)}}} \right),
\]

where \( \Phi \) is the cumulative distribution function of the standard normal variable and:

\[
m^1_{(L,\sigma S)} = m^1_{(0,\sigma S)} = -\frac{\sigma^2 S}{4} L,
\]

\[
s^1_{(L,\lambda,\sigma,\sigma S)} = s^1_{(0,\lambda,\sigma,\sigma S)} = \frac{\left( \frac{\sigma^2 S}{\lambda^2} + \sigma^2 S \right) L}{3} - \frac{\sigma^2 \mu^1_{(L,\sigma S)}}{2\lambda^2} \left( 1 - \frac{2(1 - e^{-\lambda L}(1 + \lambda L))}{\lambda^2 L^2} \right),
\]

and the functions \( s \) and \( m \) are introduced in Theorem 3.1.

Proof. This result is a consequence of Theorem 3.1. Indeed, tending \( L_1 \) to 0 and using \( L_2 = L \), the result follows. \( \square \)

4. Simulations

In this section, numerical simulations and empirical tests based on real data are performed. The aim of these tests is to compare the robustness of the optimal strategy under parameters misspecification with that of strategies based on moving average crossover. First, the best durations of the Kalman filter and of the optimal strategy under parameters misspecification are illustrated over several trend regimes. Then, the asymptotic expected logarithmic returns of the moving average crossover strategy (see Section 3) with \( (L_1, L_2) = (5 \text{ days}, 252 \text{ days}) \), and that of the optimal strategy with a duration \( \tau = 252 \text{ days} \), are analyzed. With this configuration, the stability of their respective performances are compared over several market regimes and models, including a Heston stochastic volatility model. Finally, backtests of the two strategies are performed using real data, confirming the theoretical findings.

4.1. Optimal durations. In this subsection, we consider the model (1)-(2)
4.1.1. Well-specified Kalman filter. In these simulations, we consider a signal-to-noise ratio inferior to 1. This assumption corresponds to a trend standard deviation smaller than the volatility of the risky asset. Using $\tau^* = \frac{1}{\lambda \beta}$ and $\beta = \sqrt{1 + \frac{2 \text{SNR}}{\lambda}}$, The figures 1 and 2 represent the optimal Kalman filter duration $\tau^*$ as a function of the trend mean reversion speed $\lambda$ and of the signal-to-noise ratio. This duration is a decreasing function of these parameters. Indeed, if the variation of the trend process is low and if the measurement noise is high compared to the trend standard deviation, the filtering window must be long. Moreover, we observe that for a trend mean reversion speed smaller than 1 (which corresponds to a slow trend process), the duration $\tau^*$ is larger than 0.5 years and can reach 10 years. If the trend mean reversion speed is larger than 1, this duration is smaller than 1 year.

**Figure 1.** Optimal duration (in years) of the Kalman filter with $\lambda \in [0.1, 1]$

**Figure 2.** Optimal duration (in years) of the Kalman filter with $\lambda \in [1, 10]$

4.1.2. Best filtering window for the optimal strategy under parameters misspecification. Under parameters misspecification, we can also define an optimal duration using the strategy introduced in Section 2 and Proposition 5. This duration is the one maximizing the asymptotic
expected logarithmic return of the optimal strategy under parameters misspecification. This optimal window exists if and only if \( \frac{\text{SNR}}{\lambda} > \frac{2m_r}{2-m_r} \). Assuming that \( m_r = 1 \), the condition becomes \( \frac{\text{SNR}}{\lambda} > 2 \). Figures 3 and 4 represent this duration \( \tau_{\text{opt}}(m_r = 1) \) as a function of the trend mean reversion speed \( \lambda \) with respectively \( \text{SNR} = 1 \) and \( \text{SNR} = 0.5 \). This duration has a behaviour similar to that of the optimal Kalman filter, except when the trend mean reversion speed \( \lambda \) tends to \( \frac{\text{SNR}}{2} \). Indeed, if \( \lambda = \frac{\text{SNR}}{2} \), the condition \( \frac{\text{SNR}}{\lambda} > 2 \) is not satisfied and the optimal duration becomes infinite.
Optimal duration with $m=1$ and SNR=0.5

Figure 4. Optimal duration (in years) of the misspecified filter with $m = 1$ and SNR= 0.5

4.2. Robustness of the optimal strategy and of the moving average crossover strategy.

4.2.1. Stability of the performances over several theoretical regimes under constant spot volatility. In this subsection, we consider the model (1)-(2). Moreover, we assume that a year contains 252 days and that the risky asset volatility is equal to $\sigma_S = 30\%$. We consider two trading strategies. The first one is the optimal strategy (introduced in Section 2) with a duration $\tau \epsilon = 252$ days ($= 1$ year) and leverage values $m \epsilon = 1$ or $m \epsilon = 0.5$. The second strategy is the moving average crossover strategy (introduced in section 3) with $(L_1, L_2) = (5 \text{ days}, 252 \text{ days})$ and the following allocation:

$$\theta_t = -1 + 2 \mathbb{1}_{G(t,L_1) > G(t,L_2)},$$

where $G$ is the geometric moving average defined in Equation (21). Then, if the short geometric average is larger (respectively smaller) than the long geometric average, we buy (respectively sell) the risky asset. In order to compare the stability of these two strategies, the asymptotic expected logarithmic returns found in Theorems 2.3 and 3.1 are used.

Figures 5, 6, 7, 8, 9, 10, 11 and 12 show the performances of these strategies after 100 years as a function of the trend volatility $\sigma_\mu$ respectively with $\lambda = 1, 2, 3$ and 4 and $m_\epsilon = 1$ and 0.5. For $m_\epsilon = 1$, the optimal strategy can sometimes perform better (for example with $\lambda = 1$ and $\sigma_\mu = 90\%$ ), but it can also entail higher losses than the crossover strategy (for example with $\lambda = 4$ and $\sigma_\mu = 10\%$). For $m_\epsilon = 0.5$, a similar conclusion can be drawn.
In other words, these tests show that the theoretical performance of the crossover strategy is more robust than that of the optimal strategy.

\[ \text{Figure 5.} \text{ The expected logarithmic returns of the optimal strategy (with } \tau = 252 \text{ days and } m = 1) \text{ and of the crossover strategy (} L_1 = 5 \text{ days and } L_2 = 252 \text{ days) as functions of } \sigma_\mu \text{ with } \lambda = 1, \sigma_S = 30\% \text{ and } T = 100 \text{ years} \]

\[ \text{Figure 6.} \text{ The expected logarithmic returns of the optimal strategy (with } \tau = 252 \text{ days and } m = 1) \text{ and of the crossover strategy (} L_1 = 5 \text{ days and } L_2 = 252 \text{ days) as functions of } \sigma_\mu \text{ with } \lambda = 2, \sigma_S = 30\% \text{ and } T = 100 \text{ years} \]
Figure 7. The expected logarithmic returns of the optimal strategy (with $\tau_c = 252$ days and $m_c = 1$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) as functions of $\sigma_\mu$ with $\lambda = 3$, $\sigma_S = 30\%$ and $T = 100$ years.

Figure 8. The expected logarithmic returns of the optimal strategy (with $\tau_c = 252$ days and $m_c = 1$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) as functions of $\sigma_\mu$ with $\lambda = 4$, $\sigma_S = 30\%$ and $T = 100$ years.
The expected logarithmic returns of the optimal strategy (with $\tau = 252$ days and $m = 0.5$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) as functions of $\sigma_{\mu}$ with $\lambda = 1$, $\sigma_S = 30\%$ and $T = 100$ years

Figure 9.

The expected logarithmic returns of the optimal strategy (with $\tau = 252$ days and $m = 0.5$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) as functions of $\sigma_{\mu}$ with $\lambda = 2$, $\sigma_S = 30\%$ and $T = 100$ years

Figure 10.
**Figure 11.** The expected logarithmic returns of the optimal strategy (with $\tau = 252$ days and $m = 0.5$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) as functions of $\sigma_\mu$ with $\lambda = 3$, $\sigma_S = 30\%$ and $T = 100$ years.

**Figure 12.** The expected logarithmic returns of the optimal strategy (with $\tau = 252$ days and $m = 0.5$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) as functions of $\sigma_\mu$ with $\lambda = 4$, $\sigma_S = 30\%$ and $T = 100$ years.

4.2.2. *Stability of the performances over several theoretical regimes under Heston’s stochastic volatility model.* The aim of this subsection is
to check whether the crossover strategy is more robust than the optimal trading strategy under Heston’s stochastic volatility model (see Heston (1993) or Mikhailov and Nögel (2003) for details). To this end, consider a financial market living on a stochastic basis \((\Omega, \mathcal{G}, \mathbf{G}, \mathbb{P})\), where \(\mathbf{G} = \{\mathcal{G}_t, t \geq 0\}\) is the natural filtration associated to a three-dimensional Wiener process \((W^S, W^\mu, W^V)\), and \(\mathbb{P}\) is the objective probability measure. The dynamics of the risky asset \(S\) is given by

\[
\frac{dS_t}{S_t} = \mu_t dt + \sqrt{V_t} dW^S_t, \\
\frac{d\mu_t}{\mu_t} = -\lambda \mu_t dt + \sigma \mu dW^\mu_t, \\
\frac{dV_t}{V_t} = \alpha \left(V_{\infty} - V_t\right) dt + \epsilon \sqrt{V_t} dW^V_t
\]

with \(\mu_0 = 0\), \(V_0 > 0\), \(d\langle W^S, W^\mu \rangle_t = 0\), and \(d\langle W^S, W^V \rangle_t = \rho dt\). We also assume that \((\lambda, \sigma \mu) \in \mathbb{R}_+^* \times \mathbb{R}_+^*\) and that the Feller condition \(2kV_{\infty} > \epsilon\) holds - in this case, the variance \(V\) cannot reach 0 and remains always positive, see Cox et al. (1985) for details.

Denote by \(\mathbf{G}^S = \{\mathcal{G}_t^S\}\) be the natural filtration associated to the price process \(S\). The process \(V\) is \(\mathbf{G}^S\)-adapted. Now, assume that the agent aims at maximizing her expected logarithmic wealth (on an admissible domain \(\mathcal{A}\) representing all the \(\mathbf{G}^S\)-progressive and measurable processes). In this case, her optimal portfolio is given by (see Bjork et al. (2010)):

\[
\frac{dP_t}{P_t} = \frac{E \left[ \mu_t | \mathcal{G}_t^S \right]}{V_t} dS_t
\]

Let \(\delta\) be a discrete time step, and denote by the subscript \(k\) the value of a process at time \(t_k = k\delta\). Using the scheme that produces the smallest discretization bias for the variance process (see Lord et al. (2010) for details), the discrete time model is:

\[
y_{k+1} = \frac{S_{k+1} - S_k}{\delta S_k} = \mu_{k+1} + u_{k+1}, \\
\mu_{k+1} = e^{-\lambda \delta} \mu_k + v_k, \\
V_{k+1} = V_k + \alpha \left(V_{\infty} - V_k\right) \delta + \epsilon \sqrt{V_k} z_k
\]

where \(x^+ = \max(0, x)\), \(u_{k+1} \sim \mathcal{N}\left(0, \frac{\sigma^2}{2\delta}\right)\), \(v_k \sim \mathcal{N}\left(0, \frac{\sigma^2}{2\lambda} \left(1 - e^{-2\lambda \delta}\right)\right)\) and \(z_k \sim \mathcal{N}(0, \delta)\).

Monte Carlo simulations are now performed to check if the crossover strategy is more robust than the optimal trading strategy under Heston’s stochastic volatility model. To this end, consider the discrete model (37)-(38)-(39) and assume the following parameter values: \(\alpha = 4\) (quarterly mean-reversion of the variance process), \(\epsilon = 5\%\), \(V_\infty = V_0 = \ldots\)
(0.3)^2 (initial and asymptotic spot volatility equal to 30%), ρ = −60%.
Moreover, fix an investment horizon equal to 50 years and δ = 1/252.
With this setup and various trend regimes, we simulate M paths of the
risky asset over 50 years and implement two strategies:

(1) The discrete time version of the optimal strategy presented
above. Since the process V is G^S-adapted, V_k is observable at
time t_k and the conditional expectation of the trend is analytically tractable
with the non-stationary discrete time Kalman filter (see Kalman et al. (1962)). The agent thinks that the
parameters are equal to λ^a = 1 and σ^a = 90% when she uses
the Kalman filter.

(2) The moving average crossover strategy (introduced in section
3) with (L_1, L_2) = (5 days, 252 days) and allocation θ_k = −1 +
2 1_{G^d(k,L_1)>G^d(k,L_2)}, where G^d(k,L) is the discrete geometric moving
average computed on the last L values of S.

Figures 13 and 14 show the expected performances of both strategies as
a function of the trend volatility σ_μ, with M = 10000 and λ respectively
equal to 1 and 2. These results confirm that the performance of the
crossover strategy is less sensitive to a change in the trend regime than
the optimal strategy in the presence of parameters misspecification.
As for Figures 15, 16, 17 and 18 they show the empirical distribution
of the logarithmic return of these strategies in various configurations.
One can observe that, even with a good calibration, the logarithmic return of the crossover strategy has a smaller dispersion less than the
logarithmic return of the optimal strategy: the crossover strategy is
more robust than the optimal strategy.
Figure 13. The expected logarithmic returns of the optimal strategy (with $\lambda^a = 1$ and $\sigma^a_{\mu} = 90\%$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) as functions of $\sigma_{\mu}$ with $M = 10000$, $\lambda = 1$, $\alpha = 4$, $\epsilon = 5\%$, $V_\infty = V_0 = 0.3^2$, $\rho = -60\%$ and $T = 50$ years.

Figure 14. The expected logarithmic returns of the optimal strategy (with $\lambda^a = 1$ and $\sigma^a_{\mu} = 90\%$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) as functions of $\sigma_{\mu}$ with $M = 10000$, $\lambda = 2$, $\alpha = 4$, $\epsilon = 5\%$, $V_\infty = V_0 = 0.3^2$, $\rho = -60\%$ and $T = 50$ years.
Figure 15. Empirical distribution of the logarithmic return of the optimal strategy (with $\lambda^a = 1$ and $\sigma_{\mu}^a = 90\%$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) with $M = 10000$, $\sigma_{\mu} = 90\%$, $\lambda = 1$, $\alpha = 4$, $\epsilon = 5\%$, $V_\infty = V_0 = 0.3^2$, $\rho = -60\%$ and $T = 50$ years.

Figure 16. Empirical distribution of the expected logarithmic return of the optimal strategy (with $\lambda^a = 1$ and $\sigma_{\mu}^a = 90\%$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) with $M = 10000$, $\sigma_{\mu} = 10\%$, $\lambda = 1$, $\alpha = 4$, $\epsilon = 5\%$, $V_\infty = V_0 = 0.3^2$, $\rho = -60\%$ and $T = 50$ years.
Figure 17. Empirical distribution of the expected logarithmic return of the optimal strategy (with $\lambda^a = 1$ and $\sigma^a_{\mu} = 90\%$) and of the crossover strategy ($L_1 = 5$ days and $L_2 = 252$ days) with $M = 10000$, $\sigma_{\mu} = 90\%$, $\lambda = 2$, $\alpha = 4$, $\epsilon = 5\%$, $V_{\infty} = V_0 = 0.3^2$, $\rho = -60\%$ and $T = 50$ years.

Figure 18. Empirical distribution of the expected logarithmic return of the optimal strategy (with $\lambda^a = 1$ and $\sigma^a_{\mu} = 90\%$) and of the cross average strategy ($L_1 = 5$ days and $L_2 = 252$ days) with $M = 10000$, $\sigma_{\mu} = 10\%$, $\lambda = 2$, $\alpha = 4$, $\epsilon = 5\%$, $V_{\infty} = V_0 = 0.3^2$, $\rho = -60\%$ and $T = 50$ years.
4.2.3. Tests on real data. Finally, the two strategies are compared using real data.

The performance of a strategy is measured by its annualised Sharpe ratio indicator (see Sharpe (1966)) based on daily returns. For the optimal strategy, we assume that \( \tau = 252 \) business days, that \( m = 0.1 \) (it has no impact on the Sharpe ratio indicator), and that the volatility \( \sigma_S \) is computed over all the data and available since the beginning of the backtest. The moving average crossover strategy is based on the same assumptions as in the previous section (a window of \( x \) days is replaced by a window of \( x \) business days).

The trading universe is composed of nine stock indices (the SP 500 Index, the Dow Jones Industrial average Index, the Nasdaq Index, the Euro Stoxx 50 Index, the Cac 40 Index, the Dax Index, the Nikkei 225 Index, the Ftse 100 Index and the Asx 200 Index) and nine foreign exchange rates (EUR/CNY, EUR/USD, EUR/JPY, EUR/GBP, EUR/CHF, EUR/MYR, EUR/BRL, EUR/AUD and EUR/ZAR). The period considered is from 12/22/1999 to 2/1/2015.

To perform the backtest, we assume that each of these assets is tradable daily at its closing price, and transaction costs are neglected. Figure 19 gives the Sharpe ratio of the two strategies for each of the 18 underlyings. We observe that, even when over-fitting the volatility for the optimal strategy, the moving average crossover strategy outperforms the optimal strategy except for the EUR/BRL.
Figure 19. Sharpe ratio of the optimal strategy (with $\tau = 252$ bd) and of the crossover strategy ($L_1 = 5$ bd and $L_2 = 252$ bd) on real data from 12/22/1999 to 2/1/2015
5. Conclusion

This work compares the optimal portfolio allocation with a moving average crossover strategy in a model-based, unobserved mean-reverting diffusion, in the presence of parameters misspecification.

After deriving, for both strategies, formulae for the asymptotic expectation of the logarithmic returns as a function of the model parameters and studying their sensitivity to parameters misspecification, we analyzed their respective performances, clearly showing that a strategy based on moving average crossover is more robust than the optimal trading strategy. This model-based conclusion has been confirmed by empirical tests on real data.


