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The valuation of contingent claims with short selling bans being imposed

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Abstract

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Keywords. Equal-risk pricing approach; Short selling ban; Hamilton-Jacobi-Bellman (HJB) equation; Non-monotonic payoff function.

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1 Introduction

During the Global Financial Crisis 2007-2009, most regulatory authorities around the world imposed restrictions or bans on short selling to reduce volatility of financial market and to limit the negative impacts of a downturn market (Beber & Pagano 2013). These interventions were implemented to restore the orderly financial markets and limit further drops of stock price. However, these regulations imposed on short selling also resulted in some new problems, one of which is the valuation of contingent claims in the market where short selling is partially restricted or completely banned. In the literature, how short selling restrictions affect stock price and the valuation of contingent claims have been studied extensively (Figlewski 1981, Diamond & Verrecchia 1987, Jones & Lamont 2002, Avellaneda & Lipkin 2009, Ma & Zhu 2017). In this paper, we only studies valuation of contingent claims in a financial market where short selling is completely banned.

According to the fundamental theorem of asset pricing (Shreve 2004), any contingent claim can be replicated perfectly by some self-financing hedging strategies in a complete market and, under a no-arbitrage condition, the price of the contingent claim must equal to the cost of constructing such a portfolio. However, such perfect hedging strategies are no longer available because the financial market has become incomplete due to the absence of short selling. In the literature, the valuation of contingent claims in an incomplete market has also been explored extensively and a large number of approaches and techniques have been proposed. Generally, the literature can be grouped into two categories.

Papers in the first category share a common feature that an equivalent martingale measure is chosen as the pricing measure according to some optimal criteria. Since there are many equivalent martingale measures in an incomplete market, the choice of pricing measure varies from person to person. Follmer & Schweizer (1991) first proposed a criterion to choose the minimal martingale measure. Then minimal entropy martingale measure was proposed by Frittelli (2000) to minimize the entropy difference between the objective proba-
bility measure and the risk-neutral measure. Similar concepts, such as the minimal distance martingale measure and minimax measure were also put forward by Goll & Rüschendorf (2001) and Bellini & Frittelli (2002), respectively. Each chosen pricing measure leads to a different price, which is “fair” according to the criteria they chose the measure. It is hard to justify which choice of these equivalent martingale measures is “correct”.

Papers in the second category include Karatzas & Kou (1996), Davis (1997), Rouge & El Karoui (2000), Musiela & Zariphopoulou (2004) and Hugonnier et al. (2005). The key idea of these papers is the so-called utility indifference pricing, which is characterized by an investor choosing a utility function according to his risk preference. Then two concepts of “price” are introduced. The utility indifference buying price \( p^b \) is the price at which the utility of the investor is indifferent between (1) paying nothing and not having the claim and (2) paying \( p^b \) now to receive the contingent claim at expire time (Henderson and Hobson, 2004). The utility indifference selling price is defined similarly. In finance literature, utility indifference price is also referred to as “private valuation”, which emphasizes the proposed price is for an individual with particular risk preference and not a transactional price (Detemple & Sundaresan 1999, Tepla 2000). In contrast to Black-Scholes price, utility indifference price is nonlinear due to the concavity of the utility function. In addition, it degenerates to the unique fair price when the market is complete.

For any contingent claim in an incomplete market, El Karoui & Quenez (1995) demonstrate that there will be a price interval within which the price must lead to arbitrage opportunities. The maximum price of this interval, called selling price, is the lowest price that allows the seller to hedge completely with an optimal hedging strategy. Similarly, the minimum price of this interval, called buying price, is the highest price that the buyer is willing to pay for a contingent claim. Both of these two concepts have been addressed in the literature on contingent claims hedging and pricing under transactions cost (Hodges 1989, Davis 1997, Constantinides & Zariphopoulou 1999, Munk 1999). Obviously, either selling price or buying price is a private price for seller or buyer because they just consider
to minimize unilateral risk. The buyer and seller have to negotiate and compromise with each other in order to reach an agreement on the transactional price.

Recently, Guo & Zhu (2017) proposed a completely new approach, referred to as the *equal-risk pricing approach*, which determines the valuation of contingent claims by simultaneously analyzing the risk exposure of both parties involved in the contract. It appears to be, but not the same as, the existing utility indifference price method as pointed out in Remark 3.2 of Guo & Zhu (2017). They aimed to find out an *equal-risk price* which distributes expected loss evenly between the two involved parties. Such a price is interpreted as a fair price that both parties are willing to accept during the negotiation if they intend to enter into a derivative contract. Equal-risk price is a transactional price and it must lie in the price interval consisting of selling price and buying price. Both the seller and buyer would face the same amount of risk when they accept such a price. The existence and uniqueness of such an equal-risk price has been established by Guo & Zhu (2017) and they also demonstrated that such an equal-risk price is consistent with the Black-Scholes price when short selling ban is removed. Although analytical pricing formulae have been obtained for European call and put options, the derivation heavily depends on the monotonicity of the payoff with respect to the stock price $S$, which has limited its application to general contingent claims.

The main contribution of this paper is that we have successfully established a unified PDE framework for such a new and efficient equal-risk pricing approach so that its range of application is significantly expanded. First of all, we derive analytical pricing formulae for European call and put options from our PDE framework, which demonstrates that it is consistent with the previous work of Guo & Zhu (2017). Furthermore, we apply our PDE approach to general contingent claims, such as a butterfly spread option, and produce its corresponding equal-risk price numerically, which could not be done with the analysis method proposed by Guo & Zhu (2017). After comparing its equal-risk price from this new pricing approach with Black-Scholes price, we numerically demonstrate how short selling
ban affects the valuation of contingent claims.

The paper is organized as follows. In Section 2, a financial market with short selling ban is introduced first and then the PDE framework is established to derive equal-risk price of general contingent claims. In Section 3, analytical pricing formulae are derived for European call and put options from our PDE framework. In Section 4, an ADI numerical scheme is introduced to solve the PDE system and two numerical experiments are conducted accordingly. Conclusions are provided in the last section.

2 A framework of equal-risk pricing approach

2.1 The financial market model

Consider a financial model on a complete probability space \((\Omega, \mathcal{F}, \mathbb{Q})\). Let \(\mathbb{F} = \{\mathcal{F}_t : t > 0\}\) be the filtration that represents the information flow available to market participants. For simplicity, we assume there are only two assets traded continuously in this market. One is a risk-free asset, the price of which satisfies the ordinary differential equation

\[ dP_t = rP_t dt, \tag{2.1} \]

where \(r\) is the risk-free interest rate. The other one is a risky asset with its price following the geometric Brownian motion as

\[ dS_t = rS_t dt + \sigma S_t dW_t, \tag{2.2} \]

where \(\sigma\) is the volatility of the underlying and \(W_t\) is a standard Brownian motion. Since this is the first paper to set up a unified PDE framework for equal-risk pricing approach, we assume that the stock price follows the geometric Brownian motion to illustrate how the short selling ban affects the valuation of contingent claims. Of course, some complicated
stock models, such as stochastic volatility and interest rate models, can also be adopted under our PDE framework in the future.

When short selling ban is removed, market is complete and there exists a unique equivalent martingale measure $Q$. The price of European contingent claims that expire at time $T$ with a payoff function $Z(S)$ can be easily calculated as $v = E_Q[e^{-rT}Z(S_T)]$. Such a price is accepted by both the seller and buyer of the claim since they are able to perfectly replicate the claim by corresponding self-financing trading strategies.

When short selling ban is imposed, market becomes incomplete as the perfect replication is no longer possible. In this case, a self-financing trading strategy is a progressively measurable non-negative process $\phi_t$, which represents the number of stock at time $t$. Given an initial wealth $v$, an investor, who adopts the trading strategy $\phi_t$, would hold $\phi_t$ shares of stock at time $t$ and leaves all the remaining on the risk-free bond account. Then the wealth process of such a portfolio, denoted as $v_t$, follows

$$dv_t = d(\phi_t S_t) + d(v_t - \phi_t S_t) = \phi_t dS_t + r(v_t - \phi_t S_t)dt = rv_t dt + \phi_t \sigma S_t dW_t, \quad (2.3)$$

where $\phi_t$ comes from the set of all self-financing, progressively measurable, non-negative and square integrable trading strategy

$$\Phi := \{\phi(t, \omega) : [0, T] \times \Omega \to R^+\mid E \int_0^T \phi^2(t, \omega)dt < \infty\}. \quad (2.4)$$

### 2.2 Equal-risk price for general contingent claims

As demonstrated above, short selling ban has made the market become incomplete even the stock price follows the geometric Brownian motion. Applying any unilateral utility-based arguments would lead to a price interval consisting of buying price as the lower bound and selling price as the upper bound. Any price that lies in this interval would make both the buyer and seller face risk whatever they do to hedge. Intuitively, a higher price of the
contingent claim increases the risk exposure of the buyer and decreases that of the seller; while a lower price has the opposite effect. Guo & Zhu (2017) proposed an idea to look for a price that lies in this interval and distributed the risk between the buyer and seller equally. To measure the risk exposure, they introduced the risk function as follows.

**Definition 1.** A function \( R : \mathbb{R} \to \mathbb{R} \) is called a *risk function* if it satisfies the following conditions:

1. \( R(x) \) is non-decreasing convex and has a finite lower bound \( LB \).
2. \( R(0) = 0 \) and \( R(x) > 0 \) for all \( x > 0 \).

**Remark 1.** It is easy to check that both \( R_1(x) = x^+ \) and \( R_2(x) = e^x - 1 \) are risk functions. The former is adopted by Guo & Zhu (2017), while the latter is the one we choose in this paper. Here we provide some reasons for our choice. From the view of mathematics, \( R_2(x) \) is a smooth function, while \( R_1(x) \) is not. In addition, \( R_1(x) \) maps all the negative \( x \) to zero. However, from the view of finance, a company that owns one million dollars should be more riskless than a company that has only one dollar. \( R_1(x) \) cannot tell the difference between these two companies, while \( R_2(x) \) can. Finally, the risk aversion of \( R_1(x) \) and \( R_2(x) \) are totally different, which implies that they represent different investors.

Suppose an investor has an opportunity to sell one unit of European contingent claim \( Z(S_T) \) at a transaction price \( v \). After receiving the payment, he would establish an hedging account with his initial wealth \( v \). \( Z(S_T) \) is a future liability for the seller, which represents the possible risk. The terminal wealth process of the hedging account \( v_T \) is an income that reduces the risk he takes at expire date \( T \). As a result, the risk exposure of the seller who sells a European contingent claim \( Z(S_T) \) at a price \( v \) with the current stock price \( S \) is defined as

\[
\rho^*(S; v; Z) = \inf_{\phi(\cdot) \in \Phi} E_Q^{S,v} R(Z(S_T) - v_T^{v,\phi(\cdot)}),
\]

where \( E_Q^{S,v} \) denotes the conditional expectation under the measure \( \mathbb{Q} \) with \( S_0 = S, v_0 = v \) and \( v_T^{v,\phi(\cdot)} \) is the solution of Equation (2.3) given trading strategy \( \phi(\cdot) \) and initial wealth \( v \).
How to calculate the minimum risk exposure for the seller is actually an optimal stochastic control problem with objective function (2.5) and dynamics $S_t$ and $v_t$ governed by Equations (2.2) and (2.3). According to the dynamic programming method (Yong and Zhou, 1999), the HJB equation governing the value function $F^s(t, S, v)$ is derived as

$$\begin{cases}
0 &= \frac{\partial F^s}{\partial t} + \inf_{\phi \geq 0} \mathcal{L}_1^\phi F^s, \\
F^s(T, S, v) &= R(Z(S) - v),
\end{cases}$$ \tag{2.6}
$$\mathcal{L}_1^\phi F = \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 F}{\partial S^2} + \phi S^2 \sigma^2 \frac{\partial^2 F}{\partial S \partial v} + \frac{1}{2} S^2 \sigma^2 \phi^2 \frac{\partial^2 F}{\partial v^2} + r S \frac{\partial F}{\partial S} + rv \frac{\partial F}{\partial v}. \tag{2.7}$$

The value function at $t = 0$ corresponds to the minimum risk exposure of the seller, i.e. $F^s(0, S, v) = \rho^s(S, v; Z)$.

Similarly, we come to the analysis of buyer’s risk exposure. Assume a buyer offers a price $v$ for a European contingent claim $Z(S_T)$ and his offer is accepted by a seller. The buyer has to borrow money $v$ right now to purchase the claim, which corresponds to the deterministic liability $ve^{rT}$ for the buyer at the expire date. Although the initial wealth is zero, the buyer would also establish a hedging account with a hedging strategy $\phi(\cdot)$, which comes from the admissible set $\Phi$ defined in Equation (2.4). Then the risk exposure of the buyer, who pays $v$ to purchase a European contingent claim $Z(S_T)$ with current stock price $S$, is defined as

$$\rho^b(S, v; Z) = \inf_{\phi(\cdot) \in \Phi} \mathbb{E}_Q^v S [R(v e^{rT} - v_0^{0, \phi}) - Z(S_T))] = \inf_{\phi(\cdot) \in \Phi} \mathbb{E}_Q^v S [R(v e^{rT} - \phi - Z(S_T))]. \tag{2.8}$$

To solve this optimal stochastic control problem associated with the buyer, another HJB equation governing the value function $F^b(t, S, v)$ is also established as

$$\begin{cases}
0 &= \frac{\partial F^b}{\partial t} + \inf_{\phi \geq 0} \mathcal{L}_2^\phi F^b, \\
F^b(T, S, v) &= R(v - Z(S)),
\end{cases}$$ \tag{2.9}
where

\[
\mathcal{L}_2^\phi F = \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 F}{\partial S^2} - \phi S^2 \sigma^2 \frac{\partial^2 F}{\partial S \partial \phi \partial v} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 F}{\partial v^2} + rS \frac{\partial F}{\partial S} + rv \frac{\partial F}{\partial v}.
\] (2.10)

**Remark 2.** It is pointed out that the differences between these two HJB equations (2.6) and (2.9) lie in the sign of the cross-derivative term and the terminal condition.

To make sure the optimal control problems (2.5) and (2.8) are well-defined, i.e. both of them have a finite infimum, we make an assumption about the set of admissible hedging strategy \( \Phi \).

**Assumption 1.** Given risk function \( R(x) \) and payoff function \( Z(S) \), there exists an admissible strategy \( \phi(\cdot) \) such that \( R(Z(S_T) - v_{T,\phi(\cdot)}) \) and \( R(v_{T,\phi(\cdot)} - Z(S_T)) \) is square integrable.

**Remark 3.** Obviously, when the payoff function \( Z(S) \) is bounded, such an assumption always holds. In other cases, we have to verify such an assumption to make sure that the corresponding optimal stochastic control problems are well-defined before solving them.

In fact, there exists a relation between the risk exposure of seller and buyer. From the view of finance, a buyer who purchases a European contingent claim \( Z(S) \) at a price \( v \) is equivalent to a seller who sells a contingent claim \(-Z(S)\) at the price of \(-v\) because they have the same cash flow. Therefore, they should face the same risk. Mathematically, it is expressed as

\[
\rho^b(S, v; Z) = \rho^s(S, -v; -Z).
\] (2.11)

Such a relation plays an important role in the rest of this paper.

Functions \( \rho^s(S, v; Z) \) and \( \rho^b(S, v; Z) \) represent the minimum risk exposure of the seller and buyer through selecting an optimal hedging strategy when the transactional price of claim is \( v \) and the underlying stock price is \( S \). The following lemma describes some properties of these functions.
Lemma 1. Assume that $Z, Z_1, Z_2$ are square integrable, $\mathcal{F}_T$-measurable random variables. The monotonicity and limits behavior of both risk functions $\rho^s(S, v; Z)$ and $\rho^b(S, v; Z)$ are described as follows:

1. If $Z_1 \leq Z_2$, then $\rho^s(S, v; Z_1) \leq \rho^s(S, v; Z_2)$ and $\rho^b(S, v; Z_1) \geq \rho^b(S, v; Z_2)$.

   If $v_1 \leq v_2$, then $\rho^s(S, v_1; Z) \geq \rho^s(S, v_2; Z)$ and $\rho^b(S, v_1; Z) \leq \rho^b(S, v_2; Z)$.

2. As $v$ tends toward $\infty$ or $-\infty$, the asymptotic behavior of them are

\[
\lim_{v \to \infty} \rho^s(S, v; Z) = \text{LB}, \quad \lim_{v \to -\infty} \rho^b(S, v; Z) = \infty,
\]

\[
\lim_{v \to -\infty} \rho^s(S, v; Z) = \infty, \quad \lim_{v \to -\infty} \rho^b(S, v; Z) = \text{LB}.
\]

Proof. We leave the proof of Lemma 1 in Appendix A.

Based on the risk exposure functions $\rho^s(S, v; Z)$ and $\rho^b(S, v; Z)$, the definition of equal-risk price for a European contingent claim is provided as follows.

Definition 2. Consider a European contingent claim with payoff function $Z(S_T)$. Then equal-risk price of this claim with current underlying stock price $S$, denoted by $\bar{v}(S)$, is a constant value which makes the seller and buyer face the same amount of risk, i.e.

\[
\rho^s(S, \bar{v}(S); Z) = \rho^b(S, \bar{v}(S); Z).
\]

In order to demonstrate that such an equal-risk price is well-defined, the following theorem states its existence and uniqueness.

Theorem 1. Consider a market where the stock follows the Black-Scholes model and short selling is banned. For a European contingent claim $Z(S_T)$, there exists a unique equal-risk price $\bar{v}(S)$ such that it satisfies the following equation,

\[
\rho^s(S, \bar{v}(S); Z) = \rho^b(S, \bar{v}(S); Z).
\]
Proof. The proof of this theorem is left in Appendix B.

In a brief summary, the equal-risk pricing approach consists of two steps. In the first step, we calculate the risk exposure of seller and buyer respectively through solving the corresponding stochastic optimal control problems. In the second step, equal-risk price is implied by Equation (2.12). Obviously, the first step is the significantly important and complicated. To solve the stochastic control problems associated with the seller and buyer, we have to deal with two nonlinear PDE systems (2.6) and (2.9). For some special contingent claims, the corresponding HJB equations can be solved analytically and the pricing formula for equal-risk price can be derived easily. However, for general claims, analytical solution of these HJB equations are unavailable and hence numerical scheme would be an alternative to solve them.

3 Equal-risk price of European call and put options

In this section, we first consider the equal-risk price of European call option. It is easy to verify that Assumption 1 always holds for payoff function $Z(S) = (S - K)^+$ and risk function $R(x) = x^+$. When risk function is of $R(x) = e^x - 1$, hedging strategy $\phi = \frac{\partial C_{BS}}{\partial S}$ makes $R(Z(S_T) - v^w_\phi)$ square integrable, which also makes Assumption 1 hold. After demonstrating that both problems are well-posed, we derive the risk exposure of seller and buyer in the following propositions.

**Proposition 1.** When contingent claim is a European call option with payoff $Z(S) = (S - K)^+$, the seller’s risk exposure is

$$\rho^s(S, v; Z) = R(e^{rT}[C_{BS}(S, K, r, \sigma, T) - v]),$$  

(3.1)

where $C_{BS}(S, K, r, \sigma, T)$ is the classic Black-Scholes formula for a European call option with the underlying price $S$, strike price $K$, risk-free interest rate $r$, volatility $\sigma$ and time
to expiration \( T - t \).

**Proof.** In order to derive the risk exposure of the seller, we would focus on the PDE system (2.6) with \( Z = (S - K)^+ \). Consider a trial solution to the PDE system (2.6) as

\[
F^*(t, S, v) = R(e^{r(T-t)}[C^{BS}(S, K, r, \sigma, T-t) - v]).
\]

(3.2)

Assuming that \( R(x) \) is twice differential, it follows from the chain rule that

\[
\begin{align*}
\frac{\partial F^*}{\partial t} &= R' e^{r(T-t)} \left( \frac{\partial C^{BS}}{\partial t} - rC^{BS} + rv \right), \\
\frac{\partial F^*}{\partial v} &= -R' e^{r(T-t)}, \\
\frac{\partial^2 F^*}{\partial S^2} &= R'' e^{2r(T-t)} \left( \frac{\partial C^{BS}}{\partial S} \right)^2 + R' e^{r(T-t)} \frac{\partial^2 C^{BS}}{\partial S^2}, \\
\frac{\partial^2 F^*}{\partial S \partial v} &= R'' e^{2r(T-t)} \frac{\partial C^{BS}}{\partial S} + R' e^{r(T-t)} \frac{\partial^2 C^{BS}}{\partial S^2}, \\
\frac{\partial^2 F^*}{\partial v^2} &= R'' e^{2r(T-t)} \left( \frac{\partial C^{BS}}{\partial S} \right)^2 + R' e^{r(T-t)} \frac{\partial^2 C^{BS}}{\partial S^2}.
\end{align*}
\]

Based on the convexity of function \( L_1 F \) with respect to \( \phi \), the optimal hedging strategy is

\[
\phi^* = \max \left\{ -\frac{\partial^2 F^*}{\partial S \partial v} (\frac{\partial^2 F^*}{\partial v^2})^{-1}, 0 \right\} = \max \left\{ \frac{\partial C^{BS}}{\partial S}, 0 \right\}
\]

(3.3)

The Black-Scholes Delta of a European call option \( \frac{\partial C^{BS}}{\partial S} \) is non-negative, which leads to \( \phi^* = \frac{\partial C^{BS}}{\partial S} \). After substituting \( \phi^* \) back into the HJB equation (2.6), we have

\[
\begin{align*}
\frac{\partial F^*}{\partial t} + \inf_{\phi \geq 0} \left\{ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F^*}{\partial S^2} + \phi S^2 \sigma^2 \frac{\partial^2 F^*}{\partial S \partial v} + \frac{1}{2} \sigma^2 S^2 \phi^2 \frac{\partial^2 F^*}{\partial v^2} + rS \frac{\partial F^*}{\partial S} + rv \frac{\partial F^*}{\partial v} \right\}
&= R' e^{r(T-t)} \left[ \frac{\partial C^{BS}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C^{BS}}{\partial S^2} + rS \frac{\partial C^{BS}}{\partial S} - rC^{BS} \right], \\
&= 0.
\end{align*}
\]

(3.4)

The last equation holds just because \( C^{BS} \) satisfies the Black-Scholes PDE. Consequently, the trial solution (3.2) is exactly the solution to the HJB equation (2.6). Therefore, the risk exposure of the seller is expressed as (3.1) because \( \rho^*(S, v; Z) = F^*(0, S, v) \).

**Remark 4.** The seller of a European call option would adopt the same optimal hedging strategy in the classic Black-Scholes model, i.e. \( \phi^* = \frac{\partial C^{BS}}{\partial S} \), which means that the ban
of short selling does not affect his hedging strategy. That is because such an optimal hedging strategy for European call options is non-negative when the payoff function is 
\[ Z(S_T) = (S_T - K)^+ \]
non-decreasing.

**Proposition 2.** When the contingent claim is a European call option with payoff 
\[ Z(S_T) = (S_T - K)^+ \]
the buyer’s risk exposure is

\[
\rho^b(S, v; Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R(ve^{rT} - (Se^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z} - K)^+)e^{-\frac{z^2}{2}}dz. \tag{3.5}
\]

**Proof.** We first claim that the optimal hedging strategy \( \phi^* \) for the buyer should be zero when 
\[ Z(S_T) = (S_T - K)^+ \]
i.e.

\[
\rho^b(S, v; Z) = E_Q R(v_T^{\phi^*} - Z). \tag{3.6}
\]

It suffices to demonstrate that 
\[ E_Q R(v_T^{\phi^*} - Z) \geq E_Q R(ve^{rT} - Z) \]
for any \( \phi(\cdot) \in \Phi \). According to the dynamics (2.3), we have 
\[ v_t^{\phi^*} = ve^{rt} - \sigma \int_0^t e^{r(t-u)} \phi_u S_u dW_u. \]
Since \( R(x) \) is a convex function, we have

\[
E_Q[R(v_t^{\phi^*} - Z) - R(ve^{rT} - Z)] \geq E_Q[-R'(ve^{rT} - Z)\sigma \int_0^T e^{r(T-u)} \phi_u S_u dW_u]. \tag{3.7}
\]

Following Lemma 3.2 in Guo & Zhu (2017), random variable \(-R'(ve^{rT} - Z(S_T))\) can be expressed as

\[
-R'(ve^{rT} - Z(S_T)) = -E_Q R'(ve^{rT} - Z(S_T)) + \int_0^T \psi_u S_u dW_u, \tag{3.8}
\]

where \( \psi(\cdot) \) is non-negative. Based on stochastic calculation, we have

\[
E_Q[-R'(ve^{rT} - Z)\sigma \int_0^T e^{r(T-u)} \phi_u S_u dW_u] = E_Q \int_0^T \sigma^2 e^{r(T-u)} \phi_u \psi_u S_u^2 du \geq 0, \tag{3.9}
\]

which completes the proof of our claim (3.6). Since the optimal trading strategy \( \phi^* \) is zero,
the HJB equation (2.9) becomes

\[
0 = \frac{\partial F^b}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F^b}{\partial S^2} + rS \frac{\partial F^b}{\partial S} + rv \frac{\partial F^b}{\partial v}.
\] (3.10)

By introducing time reversal \( \tau = T - t \) and function \( G(\tau, S, v) = F^b(t, S, v) \), we have

\[
\begin{align*}
\frac{\partial G}{\partial \tau} &= \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + rS \frac{\partial G}{\partial S} + rv \frac{\partial G}{\partial v}, \\
G(0, S, v) &= R(v - Z(S)).
\end{align*}
\] (3.11)

According to Feynman-Kac formula, the solution the such a linear PDE system can be written as a condition expectation

\[
G(\tau, S, v) = E^\nu,0 \cdot R(\nu^\tau - (S^\tau - K)^+)
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R(v e^{\tau x} - (S e^{(r - \frac{\sigma^2}{2})\tau + \sigma \sqrt{\tau} x} - K)^+) e^{-\frac{x^2}{2}} dx.
\] (3.12)

The risk exposure is expressed as (3.5) since \( \rho^b(S, v; Z) = F^b(0, S, v) = G(T, S, v) \).

**Remark 5.** It is noted that the optimal hedging strategy for the buyer of European call option is doing nothing when short selling is banned, which is totally different from the counterpart in the classic Black-Scholes model. The reason is that the optimal hedging strategy in the classic Black-Scholes model \( \phi^* = -\frac{\partial C^\text{BS}}{\partial S} \) is non-positive, which is infeasible due to the short selling ban.

After deriving the risk exposure of both the seller and buyer, the analytical pricing formula for European call options is provided in the following theorem.

**Theorem 2.** When short selling is banned in the Black-Scholes model, equal-risk price of European call options is produced as follows according to different risk functions.
1. When risk function is $R_1(x) = x^+$, equal-risk price $v$ is implied by

$$v = C^{BS}(S, K, r, \sigma, T) - [P^{BS}(S, K + ve^{rT}, r, \sigma, T) - P^{BS}(S, K, r, \sigma, T)], \quad (3.13)$$

where $P^{BS}(S, K, r, \sigma, T)$ is classic Black-Scholes formula for a European put option with underlying price $S$, strike price $K$, risk-free interest rate $r$, volatility $\sigma$ and time to expiration $T$.

2. When risk function is $R_2(x) = e^x - 1$, equal-risk price $v$ is explicitly expressed as

$$v = \frac{1}{2} \left\{ C^{BS}(S, K, r, \sigma, T) - e^{-rT} \ln \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(s^{(r-\frac{\sigma^2}{2})T+\sigma \sqrt{T}v-K}+\frac{v^2}{2}\right)} \, dv \right] \right\}. \quad (3.14)$$

**Proof.** The risk exposure of both seller and buyer have been derived in Propositions 1 and 2. According to Definition 2, equal-risk price of European call options is the root of Equation

$$\rho^a(S, v; (S - K)^+) = \rho^b(S, v; (S - K)^+). \quad (3.15)$$

When risk function is taken to be $R_1(x) = x^+$ and $R_2(x) = e^x - 1$, equal-risk price is derived easily as shown in Equations (3.13) and (3.14) after some simple calculations.

**Remark 6.** It is remarked that the analytical pricing formula is the same as the one provided by Guo & Zhu (2017) when the risk function is assume to $R_1(x) = x^+$, which demonstrates that our PDE approach is consistent with the analysis method from Guo & Zhu (2017). In addition, we produce an explicit and analytical pricing formula as Equation (3.14) when a new risk function $R_2(x) = e^x - 1$ is adopted. The significant difference between these two formula is that Equation (3.13) is not explicit and it has to be solved by root finding algorithm; while Equation (3.14) is explicit.

The pricing formula (3.13) has been interpreted in terms of the standard Black-Scholes prices and an adjustment term in Guo & Zhu (2017). In this paper, we mainly focus on the new explicit equal-risk price (3.14) when risk function is assumed to be $R_2(x) = e^x - 1$. 
To illustrate how the short selling ban affects the European call option price, we figure out the results computed from the equal-risk pricing formula (3.14) and those calculated from the classic Black-Scholes formula in Figure 1(a) with the parameters being set as

\[ K = 10, r = 0.05, T = 0.5, \sigma = 0.3. \] (3.16)

As shown in Figure 1(a), the absolute difference between equal-risk price and Black-Scholes price is significant for the large underlying price, which indicates that the short selling ban affects the value of European call option substantially. When the underlying stock price is low, no one wants to pay a much higher price for a call option with or without short selling ban. As a result, the option is relative low and the absolute difference between equal-risk price and Black-Scholes price is not very significant. In order to demonstrate the effect for the small underlying price, the relative difference between equal-risk price and Black-Scholes price is characterized by the percentage distance to Black-Scholes price defined by

\[ \frac{\text{Equal-risk price} - \text{Black-Scholes price}}{\text{Black-Scholes price}} \times 100\%, \] (3.17)

which is depicted in Figure 1(b). It is also observed that the relative difference is substantial although the absolute difference is not significantly large for small underlying price. From Figures 1(a) and 1(b), we draw a conclusion that the short selling ban would significantly decrease the value of European call option for both small and large underlying prices.

From Propositions 1 and 2, the optimal hedging in the classic Black-Scholes model is still available for the seller of European call options; while the counterpart is unavailable for the buyer due the short selling ban. If the transaction price of contingent claim is still set to be Black-Scholes price, the seller would face no risk; while the buyer could not eliminate the risk totally because the optimal hedging strategy is infeasible now. To transfer some risk from buyer to seller so that both of them face the same amount, equal-risk price should be lower than Black-Scholes price. The premium between these two prices is used...
(a) Two kinds of prices for European call option. (b) The percentage distance.

Figure 1: Comparisons between equal-risk price and Black-Scholes price.

to compensate the buyer because he takes too much risk due to the ban of short selling.

According to the relation (2.11) between the risk exposure of the buyer and seller, we derive equal-risk price for European put options as corollaries.

**Corollary 1.** When the contingent claim is a European put option with payoff \( Z(S) = (K - S)^+ \), the buyer’s risk exposure is

\[
\rho^b(S, v; Z) = R(e^{rT}[v - P^{BS}(S, K, r, \sigma, T)]).
\]  

(3.18)

**Proof.** Consider the seller’s risk exposure for a contingent claim \(-(K - S)\) first. To calculate \(\rho^s(S, v; -(K - S))\), we need to solve the corresponding HJB equation

\[
\begin{align*}
0 &= \frac{\partial F^s}{\partial t} + \inf_{\phi \geq 0} \mathcal{L}_1^\phi F^s, \\
F^s(T, S, v) &= R(-(K - S)^+ - v).
\end{align*}
\]  

(3.19)

With the same technique in Proposition 1, the solution can be produced as

\[
F^s(t, S, v) = R(e^{r(T-t)}[-P^{BS}(S, K, r, \sigma, T - t) - v]).
\]  

(3.20)
According to the relation (2.11), we have

\[ \rho^b(S, v; (K - S)^+) = \rho^s(S, -v; -(K - S)^+) = F^s(0, S, -v) = R(e^{rT}[v - P^BS(S, K, r, \sigma, T)]). \]

\[ \square \]

**Corollary 2.** When the contingent claim is a European put option with payoff \( Z(S) = (K - S)^+ \), the seller’s risk exposure is

\[ \rho^s(S, v; Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R\left( (K - S)e^{(r-\frac{x^2}{2})T+\sigma\sqrt{T}x}^+ - ve^{rT} \right) e^{-\frac{x^2}{2}} dx. \]  

(3.21)

**Proof.** Consider the buyer’s risk exposure for a contingent claim \(-(K - S)^+\) first. To compute \( \rho^b(S, v; -(K - S)) \), we goes to the HJB equation

\[
\begin{cases}
0 = \frac{\partial F^b}{\partial t} + \inf_{\phi \geq 0} \mathcal{L}_2^\phi F^b, \\
F^b(T, S, v) = R(v + (K - S)^+). 
\end{cases}
\]  

(3.22)

Similar to Proposition 2, the solution to such a PDE system is

\[ F^b(t, S, v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R(ve^{r(T-t)} + (K - S)e^{(r-\frac{x^2}{2})(T-t)+\sigma\sqrt{T-t}x}^+)e^{-\frac{x^2}{2}} dx. \]  

(3.23)

From relation (2.11), the seller’s risk exposure of European put options is

\[ \rho^s(S, v; (K - S)^+) = \rho^b(S, -v; -(K - S)^+) = F^b(0, S, -v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R((K - S)e^{(r-\frac{x^2}{2})T+\sigma\sqrt{T}x}^+ - ve^{rT})e^{-\frac{x^2}{2}} dx. \]  

\[ \square \]

**Corollary 3.** When short selling is banned in the Black-Scholes model, equal-risk price of European put options is derived according to different risk functions.
1. When risk function is \( R_1(x) = x^+ \), equal-risk price is implied by

\[
v = P^{BS}(S, K, r, T, \sigma) + P^{BS}(S, K - ve^{rT}, r, T, \sigma).
\] (3.24)

2. When risk function is \( R_2(x) = e^x - 1 \), equal-risk price is explicitly expressed as

\[
v = \frac{1}{2} \left\{ P^{BS}(S, K, r, T, \sigma) + e^{-rT} \ln \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(K-Se^{rT})^2+\sigma^2T} dx \right\}.
\] (3.25)

**Proof.** The proof is similar to Theorem 1. \( \square \)

Again, when risk function is taken of \( R_1(x) = x^+ \), equal-risk price (3.24) for European put options is the same with that produced by Guo & Zhu (2017). With the same parameter as stated in (3.16), the comparisons between equal-risk price (3.25) and Black-Scholes price for European put options are plotted in Figure 2(a) to demonstrate the effect of short selling ban on the European put option price. From Figure 2(a), the absolute difference between two prices is significant when the underlying price is not very large. The percentage distance of equal-risk price to Black-Scholes price is depicted in Figure 2(b), which indicates that the relative difference is significantly large even though the absolute difference is small for large underlying price. From both Figures 2(a) and 2(b), we come to a conclusion that
equal-risk price of a European put option is higher than Black-Scholes price. In other words, the short selling ban has pumped up the European put option price substantially. Compared with the classic Black-Scholes model, the buyer would pay more to purchase a European put option when short selling is banned. Such a premium compensates the seller of the European put option for he cannot short the underlying stock to hedge his risk due to the short selling ban.

As a brief summary of this section, we have analytically produced equal-risk price of European call and put options because the PDE systems (2.6) and (2.9) can be solved analytically, which is also consistent with the results of Guo & Zhu (2017). However, equal-risk price is still hard to produce analytically when the payoff function of contingent claim is not monotonic, such as the butterfly spread option. In a complete market, a butterfly spread option can be replicated by a linear combination of European call and put options. As a result, its price is actually also a linear combination of the price of the corresponding European call and put options. However, as pointed out by Guo & Zhu (2017), such a replication method does not work any longer when short selling ban is imposed. We apply a numerical scheme to produce equal-risk price of a butterfly spread option in the next section.

4 Numerical scheme for the PDE system

In this section, we provide a numerical scheme to solve the HJB equations (2.6) and (2.9). Mathematically, they are of the same type and we only take the former as an example when we demonstrate our numerical scheme. The other one can also be numerically solved similarly. In the following, the numerical discretization is implemented first and then two experiments are conducted accordingly.
4.1 Discretization

In order to solve the HJB equation (2.6) effectively, we introduce time reversal \( \tau = T - t \) to change the terminal value problem to be an initial value problem as

\[
\begin{aligned}
F_{\tau} &= \inf_{\phi \geq 0} \left\{ \frac{1}{2} S^2 \sigma^2 F_{ss} + \phi S^2 \sigma^2 F_{sv} + \frac{1}{2} S^2 \sigma^2 \phi^2 F_{vv} + rSF_s + rvF_v \right\}, \\
F_{s}(0, S, v) &= R(Z(S) - v), (\tau, S, v) \in \Omega := [0, T] \times [0, \infty) \times \mathbb{R}.
\end{aligned}
\] (4.1)

To implement the numerical scheme, we truncate the unbounded domain into a finite one:

\[ \bar{\Omega} = [0, T] \times [0, S_{\text{max}}] \times [-v_{\text{max}}, v_{\text{max}}]. \]

It is noted that there is only a terminal condition in the PDE system (4.1). In order to establish the properly-closed PDE system, some boundary conditions are needed. In this subsection, we focus on the numerical scheme and assume that some Dirichlet boundary conditions have been imposed properly. The details of how to impose these boundary conditions according to the financial reasoning are left in the next subsection. Of course, such a truncation would introduce some errors. As pointed out by Barles et al (1995), we can expect these errors incurred by imposing approximate boundary to be arbitrarily small by extending the computational domain.

The discretization is performed by placing a set of uniformly distributed grids in the computation domain \( \bar{\Omega} \) as

\[
\begin{aligned}
S_i &= (i - 1) \cdot \Delta S, i = 1, \ldots, N_1, \\
v_j &= (j - 1) \cdot \Delta v, j = 1, \ldots, N_2, \\
\tau_l &= (l - 1) \cdot \Delta \tau, l = 1, \ldots, M,
\end{aligned}
\]

where \( N_1, N_2 \) and \( M \) are the number of grids in the \( S, v \) and \( \tau \) directions and the step sizes are correspondingly \( \Delta S = \frac{S_{\text{max}}}{N_1 - 1}, \Delta v = \frac{v_{\text{max}}}{N_2 - 1}, \) and \( \Delta \tau = \frac{T}{M - 1}. \) The value of the
unknown function $F^s(\tau, S, v)$ at a grid point thus is thus denoted by $F^n_{i,j} = F^s(\tau_n, S_i, v_j)$.

We first adopt an explicit scheme to approximate the unknown function $\phi$ as follows:

$$\phi^n_{i,j} := \phi(\tau_n, S_i, v_j) = \max\{-\frac{\Delta v}{4\Delta S} \frac{F^n_{i+1,j+1} + F^n_{i-1,j-1} - F^n_{i+1,j-1} - F^n_{i-1,j+1}}{F^n_{i,j} + F^n_{i,j-1}}, 0\}, \quad (4.2)$$

and then apply an implicit scheme for the unknown function $F$

$$\frac{F^{n+1}_{i,j} - F^n_{i,j}}{\Delta \tau} = \mathcal{L}_3(\phi^n_{i,j})F^{n+1}_{i,j}, \quad (4.3)$$

where

$$\mathcal{L}_3(\phi)F = a F_{SS} + \rho F_{Sv} + b F_{vv} + c F_S + d F_v, \quad (4.4)$$

with $a = \frac{1}{2}\sigma^2 S^2$, $b = \frac{1}{2}\phi^2 \sigma^2 S^2$, $\rho = \phi \sigma^2 S^2$, $c = r S$, $d = r v$.

The alternative direction implicit (ADI) scheme is then applied to discretize the linear operator $\mathcal{L}_3$. In the first step, only the derivatives with respect to $S$ are evaluated in terms of unknown values $F^{2n+1}$, while the other derivatives are replaced in terms of known values of $F^{2n}$. The difference equation obtained in the first step is implicit in the $S$-direction and explicit in $v$-direction. The procedure is then repeated at next step with the difference equation implicit in the $v$-direction and explicit in the $S$-direction. The cross derivative is always treated explicitly. Thus, we have two difference equations:

$$\frac{F^{2n+1}_{i,j} - F^{2n}_{i,j}}{\Delta \tau} = a_i \frac{F^{2n+1}_{i+1,j} - 2F^{2n}_{i,j} + F^{2n}_{i-1,j}}{\Delta S^2} + b_i \frac{F^{2n+1}_{i,j+1} - 2F^{2n}_{i,j} + F^{2n}_{i,j-1}}{\Delta v^2} + c_i \frac{F^{2n+1}_{i,j+1} - F^{2n+1}_{i,j-1}}{2\Delta v}, \quad (4.5)$$

$$\frac{F^{2n+2}_{i,j} - F^{2n+1}_{i,j}}{\Delta \tau} = b_i \frac{F^{2n+2}_{i,j} - 2F^{2n+1}_{i,j} + F^{2n}_{i,j}}{\Delta v^2} + d_i \frac{F^{2n+2}_{i,j+1} - 2F^{2n+1}_{i,j} + F^{2n}_{i,j-1}}{2\Delta v} + c_i \frac{F^{2n+2}_{i,j+1} - F^{2n+2}_{i,j-1}}{2\Delta S} + a_i \frac{F^{2n+1}_{i+1,j+1} - 2F^{2n+1}_{i,j} + F^{2n}_{i-1,j-1}}{4\Delta S \Delta v} + \rho_{i,j} \frac{F^{2n+1}_{i+1,j+1} - F^{2n+1}_{i-1,j-1}}{4\Delta S \Delta v} \quad (4.6)$$
The unknown functions $F_{i,j}^n$ and $\phi_{i,j}^n$ are both derived by solving these difference equations.

After solving the PDE system (2.6) and (2.9), the risk exposure of both the seller and buyer are produced numerically on the grids. To find out equal-risk price for the contingent claims, we have to find the root of equation (2.12) numerically, which is similar to determining the optimal exercise price from the values of American put option through the free-boundary condition. We demonstrate how to produce equal-risk price numerically in the following.

Given a current underlying stock price $S$, it is assumed to be located between two grid points $S_i$ and $S_{i+1}$. When the offer price $v$ is larger than equal-risk price $v(S_i)$, the seller would take less risk for he gets more compensation, i.e.

$$\rho^s(S_i, v; Z) < \rho^b(S_i, v; Z), \quad v > v(S_i).$$

On the other hand, when the offer price $v$ is smaller than equal-risk price $v(S_i)$, the buyer takes less risk because he pays less, i.e.

$$\rho^s(S_i, v; Z) > \rho^b(S_i, v; Z), \quad v < v(S_i).$$

Consequently, equal-risk price of the claim $Z$ with current price $S_i$ is produced as

$$v(S_i) = \max_j \{ v_j, j = 1, \cdots, N_2 \mid \rho^s(S_i, v_j; Z) > \rho^b(S_i, v_j; Z) \}.$$  

Similarly, equal-risk price of the claim $Z$ with current price $S_{i+1}$ is obtained as

$$v(S_{i+1}) = \max_j \{ v_j, j = 1, \cdots, N_2 \mid \rho^s(S_{i+1}, v_j; Z) > \rho^b(S_{i+1}, v_j; Z) \}.$$  

Consequently, equal-risk price of the contingent claim $Z$ with current price $S$ is approxi-
mated by

\[ v(S) = \frac{v(S_i) + v(S_{i+1})}{2}. \]  

(4.11)

4.2 Numerical experiments

In this subsection, two numerical experiments are conducted to illustrate the performance and convergence of our numerical scheme provided above. Both of these experiments were carried out with Matlab 2016a on an Intel(R) Xeon (R) CPU and risk function is assumed to be \( R_2(x) = e^x - 1. \)

4.2.1 Experiment 1: European call option

In the first experiment, the contingent claim is European call option, of which the value functions \( F^s(t, S, v) \) and \( F^b(t, S, v) \) have been obtained analytically according to Propositions 1 and 2. The analytical solutions are considered as the benchmark to illustrate the performance of our numerical scheme. Before implementing our numerical scheme, we need to provide the proper boundary conditions for the PDE systems (2.6) and (2.9).

First of all, we consider the boundary condition on \( S = 0. \) The stock price stays at zero once it hits zero for it follows geometric Brownian motion. As a result, the European call option is worthless at the expire date. The seller of such a claim faces no liability; while the buyer gets nothing. In addition, the hedging strategies for both seller and buyer must be \( \phi^* = 0 \) because they could not invest on a stock whose price is zero. Therefore, the boundary conditions at \( S = 0 \) are

\[
\begin{align*}
F^s(t, 0, v) &= R(-ve^{r(T-t)}), \\
F^b(t, 0, v) &= R(ve^{r(T-t)}).
\end{align*}
\]  

(4.12)

On the other hand, \( S \to \infty \) implies \( S_T \to \infty \), which indicates that the European call option is priceless. The buyer of such a claim would have an infinite income at the expire
date. The boundary condition for the buyer at \( S \to \infty \) is imposed as

\[
\lim_{S \to \infty} F^b(t, S, v) = \lim_{S \to \infty} \inf_{\phi(\cdot) \in \Phi} E_Q^{S,v}[R(v_T^{\phi(\cdot)} - (S_T - K)^+)] = \lim_{S \to \infty} R(-S) = -1. \tag{4.13}
\]

Such a bounded Dirichlet boundary condition is approximated by

\[
F^b(t, S_{\text{max}}, v) = -1. \tag{4.14}
\]

As for the seller, we have

\[
\lim_{S \to \infty} F^s(t, S, v) = \lim_{S \to \infty} \inf_{\phi(\cdot) \in \Phi} E_Q^{S,v}[R((S_T - K)^+ - v_T^{\phi(\cdot)})] = \infty. \tag{4.15}
\]

When the value function approaches infinity on the boundary, we have to do growth order analysis so that such a boundary condition can be imposed on the truncated boundary. For any admissible hedging strategy \( \phi \), by applying the Jensen’s inequality to the risk function \( R(x) \), we have

\[
E_Q^{S,v}[R(Z(S_T) - v_T^{\phi(\cdot)})] \geq R(E_Q^{S,v}[Z(S_T) - v_T^{\phi(\cdot)}]) = R(e^{r(T-t)}(C_{BS}(S, K, r, \sigma, T - t) - v)). \tag{4.16}
\]

Consequently, the asymptotic behavior of the value function \( F^s(t, S, v) \) is described as

\[
\lim_{S \to \infty} F^s(t, S, v) \geq \lim_{S \to \infty} R(C_{BS}(S, K, r, \sigma, T - t)e^{r(T-t)} - ve^{r(T-t)}) \to \infty \quad \text{for } t \in [0, T],
\]

which means that the growth order of \( F^s(t, S, v) \) with respect to \( S \) is higher than that of the right hand for any \( t \). On the other hand, at the specific time \( t = T \), it follows that

\[
\lim_{S \to \infty} F^s(T, S, v) = \lim_{S \to \infty} R((S - K)^+ - v), \tag{4.18}
\]
which implies that the growth order of $F(t, S, v)$ is the same as the right hand of the above equation at $t = T$. In order to make sure the boundary condition at $S \to \infty$ is consistent with the terminal condition at the corner point, the boundary condition on $S = S_{\text{max}}$ is

$$F^s(t, S_{\text{max}}, v) = R((S_{\text{max}} - K)^+ - ve^{r(T-t)}). \quad (4.19)$$

**Remark 7.** When the value function is bounded as Equation (4.13), it can be directly imposed on the truncated boundary as Equation (4.14). When the value function approaches infinity on the boundary, such as Equation (4.17), we should do growth order analysis first and then impose an approximate boundary condition as Equation (4.19) to make sure that it is consistent with the terminal condition. In the rest of this paper, such steps would be repeated. Without demonstrating details again, we would directly provide the truncated boundary conditions.

Following Lemma 1, the boundary conditions along the $v$ direction are

$$\begin{align*}
\lim_{v \to \infty} F^b(t, S, v) &= \infty, \\
\lim_{v \to \infty} F^s(t, S, v) &= -1, \\
\lim_{v \to -\infty} F^b(t, S, v) &= -1, \\
\lim_{v \to -\infty} F^s(t, S, v) &= \infty.
\end{align*} \quad (4.20)$$

which are approximated by

$$\begin{align*}
F^b(t, S, v_{\text{max}}) &= R(v_{\text{max}}e^{r(T-t)} - (S - K)^+), \\
F^s(t, S, v_{\text{max}}) &= -1, \\
F^b(t, S, -v_{\text{max}}) &= -1, \\
F^s(t, S, -v_{\text{max}}) &= R((S - K)^+ + v_{\text{max}}e^{r(T-t)}).
\end{align*} \quad (4.21)$$

After providing these proper boundary conditions for the value functions $F^s(t, S, v)$ and $F^b(t, S, v)$, we now implement our numerical scheme. The parameters used in the this
experiment are listed in Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( K )</th>
<th>( T )</th>
<th>( r )</th>
<th>( \sigma )</th>
<th>( S_{\text{max}} )</th>
<th>( v_{\text{max}} )</th>
<th>( v_0 )</th>
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<td>5</td>
<td>2</td>
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Table 1: Parameters.

Given \( \tau = T \) and \( v = v_0 \), the values of \( F^s(\tau, S, v) \) and \( F^b(t, S, v) \) are computed at different values of \( S \) and then listed in Tables 2 and 3. To determine the numerical rates of convergence, we choose a sequence of meshes by successively halving the mesh parameters. The analytical solutions (3.1) and (3.5) obtained in Propositions 1 and 2 are considered as a benchmark when we report the \( l_2 \) error. The ratio column of Tables 2 and 3 is the ratio of successive \( l_2 \) error as the grid is refined by a factor of two.

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<th>((N_1, N_2, M))</th>
<th>( S = 4 )</th>
<th>( S = 4.5 )</th>
<th>( S = 5 )</th>
<th>( S = 5.5 )</th>
<th>( S = 6 )</th>
<th>( l_2 ) error</th>
<th>ratio</th>
</tr>
</thead>
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<td>-0.8399</td>
<td>-0.7981</td>
<td>-0.7216</td>
<td>-0.5889</td>
<td>0.0452</td>
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<td>-0.8592</td>
<td>-0.8362</td>
<td>-0.7892</td>
<td>-0.7023</td>
<td>-0.5492</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: The values of \( F^s(T, S, v_0) \) with different meshes.

<table>
<thead>
<tr>
<th>((N_1, N_2, M))</th>
<th>( S = 4 )</th>
<th>( S = 4.5 )</th>
<th>( S = 5 )</th>
<th>( S = 5.5 )</th>
<th>( S = 6 )</th>
<th>( l_2 ) error</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(21,21,160)</td>
<td>6.3860</td>
<td>5.7689</td>
<td>4.8250</td>
<td>3.7099</td>
<td>2.6162</td>
<td>0.1635</td>
<td></td>
</tr>
<tr>
<td>(41,41,320)</td>
<td>6.3423</td>
<td>5.6985</td>
<td>4.7540</td>
<td>3.6598</td>
<td>2.5889</td>
<td>0.0403</td>
<td>4.1</td>
</tr>
<tr>
<td>(81,81,640)</td>
<td>6.3307</td>
<td>5.6812</td>
<td>4.7369</td>
<td>3.6475</td>
<td>2.5819</td>
<td>0.0099</td>
<td>4.1</td>
</tr>
<tr>
<td>(161,161,1280)</td>
<td>6.3302</td>
<td>5.6791</td>
<td>4.7348</td>
<td>3.6465</td>
<td>2.5820</td>
<td>0.0071</td>
<td>1.4</td>
</tr>
<tr>
<td>Benchmark (3.5)</td>
<td>6.3268</td>
<td>5.6755</td>
<td>4.7313</td>
<td>3.6435</td>
<td>2.5800</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: The values of \( F^b(T, S, v_0) \) with different meshes.

From Tables 2 and 3, it is observed that the successive \( l_2 \) error is approaching to zero as the grid spacing is diminished, which show that our numerical results are in good agreement with the benchmark solution. Therefore, we choose the numerical results calculated on the grid \((161, 161, 1280)\) to produce equal-risk price numerically.

In Figure 3, we demonstrate how risk exposure functions \( F^s(T, S, v) \) and \( F^b(T, S, v) \) changes as \( v \) varies with \( S = 5 \). As expected, the seller’s risk exposure is increasing;
while the buyer’s risk exposure is decreasing as \( v \) goes toward infinity. Equal-risk price of European call options with current price \( S = 5 \) corresponds to the offer price \( v \) that makes \( \rho^b(S, v; Z) = \rho^s(S, v; Z) \), which is numerically solved according to formula (4.11).

We repeat the above steps again and again with different values of \( S \). Equal-risk price of European call options is plotted in Figure 4(a) as the underlying stock price varies from 0 to 10, compared with the results calculated from analytical pricing formula (3.14). The absolute error between them is plotted in Figure 4(b). From Figures 4(a) and 4(b), our numerical equal-risk price is in a good agreement with the analytical pricing formula except near the boundary \( S = S_{\text{max}} \). This error is actually incurred by truncating the domain.
and imposing an approximate boundary condition there. As pointed out by Barles et al (1995), by extending the computational domain, it is possible to make the near-field error arbitrarily small. The first experiment demonstrates that our method to produce equal-risk price by solving HJB equation numerically is consistent with analytical pricing formula, which provides us more confidence to apply it to the general contingent claim.

4.2.2 Experiment 2: Butterfly spread option

The second experiment we conduct is to derive equal-risk price for a butterfly spread option, of which the payoff function is defined as

\[ Z(S) = (S - K_1)^+ - 2(S - \frac{K_1 + K_2}{2})^+ + (S - K_2)^+. \]  

(4.22)

Figure 5 provides a diagram of such a payoff function. Obviously, it is non-monotonic and non-smooth. Guo & Zhu (2017) could not provide its corresponding equal-risk price according to their analysis methods. Obviously, the payoff function is bounded in this case and therefore Assumption 1 always holds for any risk function. As a result, the optimal control problems (2.5) and (2.8) are both well-defined and have a finite infimum. We now apply our numerical scheme to solve the corresponding HJB equations first and then derive its equal-risk price numerically.

![Figure 5: Payoff of a butterfly option with $K_1 = 4, K_2 = 6$.](image-url)
Before implementing the scheme, we also need to specify the boundary conditions according to the financial reasoning. Similar to the analysis in the first experiment, the stock price would stay at zero (or infinity) once it hits zeros (or infinity) at any time \( t \) because it follows geometric Brownian motion. According to the payoff function of the butterfly spread option, it becomes worthless at both \( S = 0 \) and \( S \to \infty \). The seller of the claim faces no liability and he has no motivation to hedge. Consequently, he would invest his initial wealth on the risk-free account and obtains the profits \( ve^{r(T-t)} \) at time \( T \). Consequently, the boundary conditions at \( S = 0 \) and \( S \to \infty \) are imposed as

\[
\begin{align*}
F_s(t, 0, v) &= R(-ve^{r(T-t)}), \\
\lim_{S \to \infty} F_s(t, S, v) &= R(-ve^{r(T-t)}). 
\end{align*}
\] (4.23)

According to the same financial reasoning, on the boundaries \( S = 0 \) and \( S \to \infty \), the buyer pays \( v \) at time \( t \) for a worthless contingent claim and has no motivation to hedge. At the expire date, the buyer only faces a deterministic liability \( ve^{r(T-t)} \) and we impose the boundary conditions as

\[
\begin{align*}
F_b(t, 0, v) &= R(ve^{r(T-t)}), \\
\lim_{S \to \infty} F_b(t, S, v) &= R(ve^{r(T-t)}). 
\end{align*}
\] (4.24)

The boundary condition along the \( v \) direction are also implied by Lemma 1, i.e

\[
\begin{align*}
\lim_{v \to \infty} F_s(t, S, v) &= -1, \\
\lim_{v \to \infty} F_b(t, S, v) &= \infty, \\
\lim_{v \to -\infty} F_s(t, S, v) &= \infty, \\
\lim_{v \to -\infty} F_b(t, S, v) &= -1.
\end{align*}
\] (4.25)
which are approximated by

\[
\begin{align*}
F^s(t, S, v_{\text{max}}) &= -1, \\
F^b(t, S, v_{\text{max}}) &= R(v_{\text{max}}e^{r(T-t)} - Z(S)), \\
F^s(t, S, -v_{\text{max}}) &= R(Z(S) + v_{\text{max}}e^{r(T-t)}), \\
F^b(t, S, v_{\text{max}}) &= -1,
\end{align*}
\]

(4.26)

to make sure they are consistent with the terminal condition.

After all the boundary conditions are provided properly, we apply our numerical scheme to numerically solve the PDE system associated with the butterfly spread option. The parameters in the second experiment are listed in Table 4

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$T$</th>
<th>$r$</th>
<th>$\sigma$</th>
<th>$S_{\text{max}}$</th>
<th>$v_{\text{max}}$</th>
<th>$v_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>values</td>
<td>4</td>
<td>6</td>
<td>0.5</td>
<td>0.05</td>
<td>0.3</td>
<td>10</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4: Parameters.

When the contingent claim is a butterfly spread option, we do not have an analytical solution in hand and we choose the results computed on the uniform mesh with 321 × 321 × 2560 nodes as a benchmark solution. The numerical results of the value functions $F^s(T, S, v_0)$ and $F^b(T, S, v_0)$ calculated on different meshes are reported in Tables 5 and 6.

<table>
<thead>
<tr>
<th>$(N_x, N_y, N_T)$</th>
<th>$S = 4$</th>
<th>$S = 4.5$</th>
<th>$S = 5$</th>
<th>$S = 5.5$</th>
<th>$S = 6$</th>
<th>$l_2$ error</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(11,11,40)</td>
<td>-0.5654</td>
<td>-0.4601</td>
<td>-0.3767</td>
<td>-0.4398</td>
<td>-0.4925</td>
<td>0.1183</td>
<td></td>
</tr>
<tr>
<td>(21,21,80)</td>
<td>-0.5429</td>
<td>-0.4816</td>
<td>-0.4548</td>
<td>-0.4715</td>
<td>-0.5106</td>
<td>0.0294</td>
<td>4.0</td>
</tr>
<tr>
<td>(41,41,160)</td>
<td>-0.5445</td>
<td>-0.4919</td>
<td>-0.4696</td>
<td>-0.4832</td>
<td>-0.5172</td>
<td>0.0068</td>
<td>4.3</td>
</tr>
<tr>
<td>(81,81,320)</td>
<td>-0.5451</td>
<td>-0.4944</td>
<td>-0.4729</td>
<td>-0.4859</td>
<td>-0.5189</td>
<td>0.0015</td>
<td>4.7</td>
</tr>
<tr>
<td>(321, 321, 2560)</td>
<td>-0.5453</td>
<td>-0.4951</td>
<td>-0.4739</td>
<td>-0.4867</td>
<td>-0.5194</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5: The values of $F^s(T, S, v_0)$ on different meshes

The $l_2$ error reported in Tables 5 and 6 indicates the numerical results have converged and they can be used to produce equal-risk price for the butterfly spread option by solving Equation (2.12). Given $S = 5$, we plot the risk exposure of both the seller and buyer in Figure 6. The equal-risk price for the butterfly spread option with current price $S = 5$
Table 6: The values of $F^b(T, S, v_0)$ on different meshes

<table>
<thead>
<tr>
<th>$(N_x, N_y, N_T)$</th>
<th>$S = 4$</th>
<th>$S = 4.5$</th>
<th>$S = 5$</th>
<th>$S = 5.5$</th>
<th>$S = 6$</th>
<th>$l_2$ error</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(11, 11, 40)$</td>
<td>-0.5480</td>
<td>-0.4491</td>
<td>-0.3658</td>
<td>-0.4112</td>
<td>-0.4385</td>
<td>0.1368</td>
<td></td>
</tr>
<tr>
<td>$(21, 21, 80)$</td>
<td>-0.5427</td>
<td>-0.4807</td>
<td>-0.4508</td>
<td>-0.4568</td>
<td>-0.4669</td>
<td>0.0324</td>
<td>4.2</td>
</tr>
<tr>
<td>$(41, 41, 160)$</td>
<td>-0.5444</td>
<td>-0.4914</td>
<td>-0.4665</td>
<td>-0.4704</td>
<td>-0.4763</td>
<td>0.0071</td>
<td>4.5</td>
</tr>
<tr>
<td>$(81, 81, 320)$</td>
<td>-0.5451</td>
<td>-0.4939</td>
<td>-0.4701</td>
<td>-0.4736</td>
<td>-0.4786</td>
<td>0.0013</td>
<td>5.6</td>
</tr>
<tr>
<td>$(321, 321, 2560)$</td>
<td>-0.5452</td>
<td>-0.4946</td>
<td>-0.4710</td>
<td>-0.4742</td>
<td>-0.4786</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

should be the offer price that makes the risk exposure of seller and buyer equal, which can be numerically solved by formula (4.11).

When short selling is allowed and the market is complete, a butterfly spread option can be replicated by three European call options as shown in Equation (4.22). Its Black-Scholes price is a linear combination of three call options

$$v = C^{BS}(S, K_1, r, \sigma, T) - 2C^{BS}(S, \frac{K_1 + K_2}{2}, r, \sigma, T) + C^{BS}(S, K_2, r, \sigma, T).$$

(4.27)

Such a Black-Scholes price is taken as the benchmark solution to illustrate how short selling ban affects the price of the butterfly spread option. Equal-risk price calculated through our PDE method and numerical results calculated from the formula (4.27) are plotted in Figure 7(a) and the percentage distance to Black-Scholes price is depicted in Figure 7(b).

Unlike the cases of European call (put) options where short selling decreases (increases)
(a) Two kinds of price for the butterfly spread option.

(b) The percentage distance.

Figure 7: Comparisons between equal-risk price and Black-Scholes price.

the option price for all the underlying stock prices, it is observed from Figure 7(a) that equal-risk price is higher than Black-Scholes price when $S > 5$; while it is lower than Black-Scholes price on the other side. Figure 7(b) shows that the relative difference between equal-risk price and Black-Scholes price is significant even though the absolute difference is tiny, which demonstrates that short selling ban indeed affects the price of the butterfly spread option. The effect of short selling ban depends on the current underlying stock price, which is totally different from the cases we considered before. To explain the effect of short selling ban, we come to its payoff function displayed in Figure 5. Locally, the payoff function is monotonically increasing with the underlying stock price when $S < 5$. In this region, the short selling ban pushes down the option price as it does in the case of European call option. On the other side, the short selling ban imposes an opposite effects.

Finally, we consider how the hedging strategy is affected by the ban of short selling. Take the seller of this claim as an example. The optimal hedging strategy for the seller is numerically calculated from the PDE system (4.1). To make comparisons, the corresponding optimal hedging strategy in the Black-Scholes model without short selling ban is
\[ \phi^{BS} = \frac{\partial C^{BS}(S, K_1, r, \sigma, T)}{\partial S} - 2 \frac{\partial C^{BS}(S, \frac{K_1+K_2}{2}, r, \sigma, T)}{\partial S} + \frac{\partial C^{BS}(S, K_2, r, \sigma, T)}{\partial S}. \] (4.28)

The numerical results calculated from the PDE system and the formula (4.28) are plotted in Figure 8(a) with \( v = 0.5 \).

![Optimal hedging strategy for the seller](image)

(a) Optimal hedging strategy for the seller.

![The absolute difference](image)

(b) The absolute difference

Figure 8: Comparison between the optimal hedging strategy for the seller of a butterfly spread option.

It is observed from Figure 8(a) that the optimal hedging strategy takes both positive and negative values as the underlying stock price varies when short selling is allowed. After imposing the short selling ban, the negative part becomes zero and the positive negative part becomes larger as the absolute difference between them is plotted in Figure 8(b) when \( S < 5 \).

5 Conclusions

This paper has discussed the valuation of contingent claims when short selling ban is imposed. We take the classical Black-Scholes model as an example to demonstrate how the short selling ban affects the valuation of contingent claims. The similar approach can be easily extended to other models. Following the equal-risk pricing approach provided by
Guo & Zhu (2017), we extend their work to general cases by establishing a unified PDE framework. When contingent claim is European call or put option, the corresponding PDE systems have been analytically solved so that equal-risk pricing formula is derived analytically, which is consistent with the previous work in Guo & Zhu (2017). In addition, our PDE approach has been successfully applied to deal with the case where the payoff function is non-monotonic, such as the butterfly spread option. According to the numerical results, the effects of short selling ban are discussed through comparisons between equal-risk price and Black-Scholes price. Generally, short selling bans would decrease the price of European call option; while it has an opposite effect on European put options. As for the butterfly spread option, short selling ban draws down the option price when payoff function is increasing with underlying stock price; while it pushes up the option price when the payoff function is decreasing.

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Appendix A  The proof of Lemma 1

1. If $Z_1 \leq Z_2$, the following inequality always holds because of the monotonicity of $R(x)$ for any admissible hedging strategy $\phi(\cdot)$,

$$R(Z_1(S_T) - v^v_{T,\phi^v(\cdot)}) \leq R(Z_2(S_T) - v^v_{T,\phi^v(\cdot)}).$$ (A.1)
Taking expectation and infimum on both sides leads to

$$\rho^*(S, v; Z_1) \leq \rho^*(S, v; Z_2).$$

(A.2)

When $v_1 \leq v_2$, for any admissible hedging strategy $\phi(\cdot)$ the inequality becomes

$$R(Z(S_T) - v_T^{v_1, \phi(\cdot)}) \geq R(Z(S_T) - v_T^{v_2, \phi(\cdot)}),$$

(A.3)

which results in

$$\rho^*(S, v_1; Z) \geq \rho^*(S, v_2; Z).$$

(A.4)

By relation (2.11), the monotonicity of $\rho^b(S, v; Z)$ is characterized as

$$\rho^b(S, v; Z_1) = \rho^*(S, -v; -Z_1) \geq \rho^*(S, -v; -Z_2) = \rho^b(S, v; Z_2)$$

$$\rho^b(S, v_1; Z) = \rho^*(S, -v_1; -Z) \leq \rho^*(S, -v_2; -Z) = \rho^b(S, v_2; Z).$$

2. Choosing any admissible strategy $\phi$ satisfying Assumption 1, we obtain

$$\lim_{v \to \infty} \rho^*(S, v; Z) \leq \lim_{v \to \infty} E_Q R(Z(S_T) - v_T^{\phi(\cdot)}) = LB.$$  

(A.5)

Due to the fact that $R(Z(S_T) - v_T^{\phi(\cdot)}) \geq LB$ always holds for any $\phi(\cdot) \in \Phi$, we have

$$\lim_{v \to \infty} \rho^*(S, v; Z) \geq LB.$$  

(A.6)

Combing Equations (A.5) and (A.6) together, we have $\lim_{v \to \infty} \rho^*(S, v; Z) = LB$.

For any $\phi(\cdot) \in \Phi$, we apply Jensen’s inequality to risk function $R(x)$ and obtain

$$E_Q^{v, S}[R(v_T^{v, -\phi(\cdot)} - Z(S_T))] \geq R(ve^{rT} - E_Q Z(S_T)).$$

(A.7)
Taking infimum and limits on both sides results in

$$\lim_{v \to \infty} \rho^b(S, v; Z) = \lim_{v \to \infty} \inf_{\phi(\cdot) \in \Phi} E_Q^{v, S} [R(v \phi(\cdot) - Z(S_T))] \geq \lim_{v \to \infty} R^e v - E_Q Z(S_T) = \infty.$$ 

Following the relation (2.11), it is easy to derive that

$$\lim_{v \to \infty} \rho^s(S, v; Z) = \lim_{v \to \infty} \rho^b(S, -v; -Z) = \lim_{v \to \infty} \rho^b(S, v; -Z) = \infty$$

$$\lim_{v \to \infty} \rho^b(S, v; Z) = \lim_{v \to \infty} \rho^s(S, -v; -Z) = \lim_{v \to \infty} \rho^s(S, v; -Z) = LB,$$

which completes the proof.

**Appendix B  The proof of Theorem 1**

Given the current underlying price $S$ and the European contingent claim $Z$, we construct a map:

$$H(v) := \rho^b(S, v; Z) - \rho^s(S, v; Z). \quad (B.1)$$

According to Lemma 1, such a map $H(v)$ is continuous and non-decreasing. On one hand, we have

$$\lim_{v \to -\infty} H(v) = \lim_{v \to -\infty} [\rho^b(S, v; Z) - \rho^s(S, v; Z)] = -\infty. \quad (B.2)$$

On the other hand, as $v$ tends toward infinity, we obtain

$$\lim_{v \to \infty} H(v) = \lim_{v \to \infty} [\rho^b(S, v; Z) - \rho^s(S, v; Z)] = \infty. \quad (B.3)$$

Hence, we conclude that there exists at least one solution to $H(v) = 0$ on $(-\infty, \infty)$.

To demonstrate the uniqueness of the solution, we first assume that the equation $H(v) = 0$ has two different solutions $v_1 > v_2$. According to the monotonicity described in Lemma
1, we have

$$
\rho^b(S, v_1; Z) \geq \rho^b(S, v_2; Z) = \rho^*(S, v_2; Z) \geq \rho^*(S, v_1; Z) = \rho^b(S, v_1; Z),
$$

(B.4)

which implies that $\rho^b(S, v_1; Z) = \rho^b(S, v_2; Z)$. Again, according to the monotonicity and convexity of $\rho^b(S, v; Z)$ with respect to $v$, we come to a conclusion that $\rho^b(S, v; Z)$ is constant for $v \leq v_1$. It follows that

$$
\rho^*(S, v_1; Z) = \rho^b(S, v_2; Z) = \lim_{v \to -\infty} \rho^b(S, v; Z) = LB
$$

(B.5)

By Jensen’s inequality, we have

$$
\begin{cases}
R(v_1 e^{rT} - E_Q Z(S_T)) \leq \rho^*(S, v_1; Z) = LB \leq 0, \\
R(E_Q Z(S_T) - v_2 e^{rT}) \leq \rho^b(S, v_2; Z) = LB \leq 0.
\end{cases}
$$

(B.6)

The above equations implies that both $v_1 e^{rT} - E_Q Z(S_T)$ and $E_Q Z(S_T) - v_2 e^{rT}$ are non-positive because that $R(x) \geq 0$ for any $x \geq 0$. However, this conclusion contradicts the fact that

$$
v_1 e^{rT} - E_Q Z(S_T) + E_Q Z(S_T) - v_2 e^{rT} = (v_1 - v_2) e^{rT} > 0.
$$

(B.7)

Therefore, the solution must be unique.

References


in an intertemporal economy with proportional transaction costs and general preferences’, *Finance and Stochastics* 3(3), 345–369.


