Time Changes, Lévy Jumps and Asset Returns
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Abstract
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Keywords: Lévy jumps, time changes, tempered stable law, time series, option pricing.

JEL classification: C5, G12
1 Introduction

Over the past decade time-changed Lévy models have emerged as an important class of models for asset returns. Lévy jumps provide the most flexible modelling tool for tail risk. Time changes reproduce the widely documented return stochastic volatility. Moreover, return probability densities can be obtained via a single Fourier inversion, which makes the models amenable to time series analysis and derivative pricing. Surprisingly, time-changed Lévy models have been studied so far mainly from a theoretical standpoint. With the notable exception of Bates (2012), there is nearly no time series analysis of time-changed Lévy models in the literature.

This paper studies a general class of time-changed Lévy models, and conducts an extensive time series and option pricing analysis. One modelling innovation is to introduce a rigorous and simple filtering procedure, which makes the models easy to implement in real data applications. To conduct derivative pricing and study risk premia, we derive the pricing kernel in analytic form. Absence of arbitrage turns out to require a particular drift specification for time-changed Lévy models not appeared in the literature. Finally, we provide two empirical analyses of 16 time-changed Lévy models based on index time series returns and option data.

Lévy processes are characterized by independent increments over non-overlapping time periods. Modelling asset returns as Lévy increments can generate easily non-normal returns, but cannot reproduce return stochastic volatility. Stochastic volatility can be captured with stochastic time changes, that is by stochastically changing the clock on which the Lévy process is run. Intuitively, the original clock can be regarded as calendar time, the
stochastically changed clock as business time. Figure 1 conveys this intuition. In business
time asset returns are driven by independent Lévy increments (lower panel of Figure 1). In
calendar time asset returns exhibit stochastic volatility (upper panel of Figure 1). Stretching
and compressing the business time produces low and high return stochastic volatility.

One difficulty with time-changed Lévy models is that the time change is not observable.
This problem may have restrained model applications and it is fundamentally equivalent
to not observing the instantaneous return stochastic volatility in continuous time models.
A number of approaches have been proposed in the literature, but nearly all of them rely
on complex filtering procedures or are computationally demanding. Bates (2006) provides
an in-depth discussion of this point. To overcome this problem, we introduce a rigorous
and simple filtering procedure for time-changed Lévy models. The procedure is to use an
observable and easy to compute variable, such as GARCH return variance, as a proxy for
the unobservable time change. From a modelling perspective the time change is then ob-
served with measurement error. Attaching some distribution to the measurement error, we
can embed the proxy into the characteristic function of time-changed Lévy processes. The
resulting characteristic function of asset returns is conditioned on observable quantities only.
After one Fourier inversion, we are able to recover the transition density of time-changed
Lévy processes and estimate the models using maximum likelihood. We label the estimation
procedure Filtered Maximum Likelihood.

To capture the evidence on the leverage effect, i.e., the often negative correlation between
asset returns and volatility changes, we model the drift of asset returns as a specific function
of the time change. Figure 2 illustrates the effectiveness of our approach to capture the
leverage effect in an option pricing context. In that context, the primary reason for leverage
is to capture skewed option implied volatilities. The figure shows 30-day implied volatilities of SPX options on August 31, 2015. It also shows model-based implied volatilities generated by two time-changed Lévy models calibrated to option data. The only difference between the two models is the specification of the return drift. The drift is zero in one model, while it is a specific function of the time change in the other model. The first model can only generate a symmetric implied volatility smile, while the latter model is able to reproduce the observed skewed implied volatilities.

We construct time-changed Lévy processes by applying two time changes to a Brownian motion with a specific drift, and using our filtering approach for the unobservable time change. Our time-changed Lévy processes consist of four nested processes. Despite the apparent model complexity, because the processes are all independent from each others, our models retain a high degree of analytical tractability. The key is to work with four nested characteristic functions that all have analytical forms, and to derive the characteristic function of asset returns conditioning on observable variables only.

We consider 16 time-changed Lévy models. We obtain 12 models by combining three Lévy subordinators (listed in Table 1) to generate jump returns; one-factor or two-factor processes to generate return stochastic volatility; with or without our filtering approach for the unobservable time change. All these models are of infinite activity. We also consider four finite activity models in which the Lévy subordinator follows a Poisson process, combined with one-factor or two-factor processes, and with or without filtering.

We begin the empirical analysis of time-changed Lévy models by fitting the models to daily index market returns from 1926 to 2015. Our main findings from the time series analysis can be summarized as follows. First, infinite activity Lévy models substantially outperform
finite activity models and GARCH models. The latter models are used as a benchmark, and known to provide a good fit to market returns. The high performance of infinite activity Lévy models mainly comes from the flexibility of the Lévy measure. Second, our filtering approach for the time change uniformly improves the fitting of market returns, irrespective of the Lévy model used. Third, among the 16 time-changed Lévy models, the best performing model features a tempered stable subordinator, a two-factor volatility process, and it is estimated using our filtering approach. The tempered stable subordinator and our filtering approach are as important as the two-factor model driving return volatility to achieve the high fitting accuracy.

Next, we investigate the option pricing performance of the proposed time-changed Lévy models. To do so we derive a necessary and sufficient condition to pin down a risk-neutral measure. This no-arbitrage condition imposes a drift condition that is novel in the literature, and requires the time change to enter the return drift in a particular way. One implication of this result is that if time-changed Lévy models do not feature the correct drift, to preserve no arbitrage the Lévy process must be identically zero under the risk-neutral measure, undermining the Lévy feature of the models.

To carry out the empirical option pricing analysis we consider weekly cross-sections of options from 1996 to 2015. Various empirical findings emerge from this analysis. First, the time-changed Lévy model based on a tempered stable subordinator and two-factor volatility process substantially outperforms many competing finite and infinite models. Its root mean square error is 21.5% lower than a one-factor variance gamma model.

This paper contributes to two large streams of literature. There exists an extensive research on fitting time series models to asset returns. Prominent examples include Andersen
et al. (2002), Eraker et al. (2003), and Bates (2006). In this vast literature there is virtually no time series study of time-changed Lévy models. Bates (2012) is a notable exception. He fits various time-changed Lévy models to market returns to quantify crash risks. We develop a novel class of time-changed Lévy models, introduce a different estimation method, and analyze empirically time series and option pricing performance of our models.

Another large stream of literature develops option pricing models. Examples include Pan (2002), Christoffersen et al. (2008), and Bardgett et al. (2018). Many of the proposed models can be regarded as applications of the affine jump diffusion framework of Duffie et al. (2000). While this framework is an important theoretical advance, a limitation arises due to the exclusive use of compound Poisson processes to model jumps. To allow for general jump structures and stochastic volatility, some studies develop option pricing models based on time-changed Lévy processes. Carr et al. (2003) (CGMY) study a number of processes, including time-changed Variance Gamma (Madan and Seneta, 1990) and Normal Inverse Gaussian (Barndorff-Nielsen and Shephard, 2001) processes. Carr and Wu (2004) The work above focuses on the theoretical analysis of option pricing models. Huang and Wu (2004) provide a specification analysis of one-factor time-changed Lévy models by calibrating several models to option data. Ornthanalai (2014) uses index returns and option data to analyze the equity risk premium in infinite activity Lévy models. We derive the change of measure analytically, and investigate empirically the option pricing performance of time-changed Lévy models. We note that our filtering method is in a similar spirit as Corsi et al. (2013). They use the realized volatility computed from high-frequency data as a proxy of the unobservable volatility driving their conditionally Gaussian discrete-time models. We use a readily available proxy of the time change, and for-
mally embed this proxy in the characteristic function of time-changed Lévy models. Finally, none of the papers above investigates time series fitting and option pricing performance of time-changed Lévy models in a unified setting.

1.1 Background: Lévy processes

This section provides a quick review of Lévy processes. Carr and Wu (2004) and Bates (2012), among others, provide more extensive reviews.

A Lévy process $L_t$ is a stochastic process with independent and identical distributed (i.i.d.) increments over non-overlapping time intervals of equal lengths. Lévy processes in finance are typically used to model log asset prices, with the Brownian motion in the Black–Scholes model being a well-known example. For notational simplicity, we consider pure jump Lévy processes, which have no diffusive component and zero drift.

Pure jump Lévy processes are characterized by their Lévy density $k(x)$, which gives the intensity of jumps of size $x$. Pure jump processes are finite activity if $\int k(x) dx < \infty$ and infinite activity otherwise. Integrals are on the real line excluding zero. Finite activity means that there is a nonzero probability that no jumps will occur over a time interval. The compound Poisson jump process of Merton (1976) is a classic example of such processes. Infinite activity means that the Lévy process can exhibit an infinite number of jumps within any finite time interval. Within the infinite activity category, Lévy processes are finite variation if $\int |x| k(x) dx < \infty$ and infinite variation otherwise. An infinite variation process has sample paths of infinite length, a property shared by the Brownian motion. All Lévy processes must have $\int \min(1, x^2) k(x) dx < \infty$ to be well defined, but their variance $\int x^2 k(x) dx$ can be infinite, with the stable distribution being an example. To sum up,
depending on the specification of \( k(x) \), Lévy processes can exhibit a great deal of flexibility in modelling jumps.

Alternatively, Lévy processes can be characterized by their characteristic function

\[
\Phi(u; L_t) = \mathbb{E}[\exp(iuL_t)] = \exp(t\psi(u))
\]

where \( i = \sqrt{-1} \), \( u \) is real and \( \psi(u) \) is the cumulant exponent of \( L_t \).\(^1\) The Lévy–Khintchine formula relates the jump intensity \( k(x) \) and the cumulant exponent \( \psi(u) \). For example, when \( L_t \) has finite variation, \( \psi(u) = \int (\exp(iux) - 1)k(x) \, dx \). Therefore, modelling the Lévy density \( k(x) \) is equivalent to modelling the characteristic function \( \Phi(u; L_t) \), and vice versa. Because (1) is exponentially linear in \( t \), \( L_t \) is said to be infinitely divisible.

Differentiating \( n \) times the cumulant generating function \( \log(\Phi(u; L_t)) = t\psi(u) \), times \( 1/i^n \), gives the \( n \)-th central moment of \( L_t \), provided it exists. Any finite variance Lévy process has variance \( \mathbb{V}[L_t] = t\psi''(0) 1/i^2 \), where \( \psi''(0) \) is the second derivative of \( \psi \) with respect to \( u \) evaluated at zero. Therefore, randomizing time \( t \) is fundamentally equivalent to randomizing variance \( \mathbb{V}[L_t] \), which is the basic insight of stochastic time changes. Time-changed Lévy models randomize time to generate stochastic volatility.

\section{Setup}

This section introduces the general class of time-changed Lévy models and discusses how to make the models operational.

\(^1\)Wu (2006) introduce the terminology of cumulant exponent. Carr et al. (2003) call \( \psi \) the log characteristic function at unit time, and Bertoin (1996) call \( -\psi \) the characteristic exponent.
2.1 Model Specification

The price of the financial asset (e.g., index) under consideration evolves in continuous time and is given by

\[ S_t = S_0 e^{(r-\delta)t + X_t} \]

where \( r \) and \( \delta \) are constants that represent, respectively, the risk free rate of interest and the dividend yield of the asset and

\[ X_t = \log \left( \frac{e^{\delta t} S_t}{S_0} \right) - rt \]

models excess log-returns, under the usual assumption that dividends are continuously re-invested. To reduce notational complexity, we use interchangeably \( X_t \) or \( X(t) \), and likewise for any other process that we encounter, and we omit subscripts when no confusion arises.

We assume that the log-return \( X_t \) is a time-changed Lévy process given by

\[ X_t = \beta Y_t + \gamma s(Y_t) + \sigma W(s(Y_t)) \quad (2) \]

where \( \beta \), \( \gamma \) and \( \sigma \) are constants, \( W_t \) is a standard \( \mathbb{P} \)-Brownian motion in the usual filtration, \( s_t \) and \( Y_t \) are two \( \mathbb{P} \)-subordinators,\(^2\) and \( \mathbb{P} \) denotes the physical measure.

The drift specification \( \beta Y_t + \gamma s(Y_t) \) of \( X_t \) is novel for time-changed Lévy models. As will become clear later, this drift specification serves two important purposes, namely capturing leverage effects and ensuring absence of arbitrage.

In (2), the two subordinators \( s_t \) and \( Y_t \) generate stochastic time changes. The process \( s_t \) is a \( \mathbb{P} \)-Lévy subordinator with characteristic exponent

\[ \psi_{\mathbb{P}}(u) = \frac{1}{t} \log \mathbb{E}^{\mathbb{P}}[e^{iu s_t}] = i u m_{\mathbb{P}} + \int_0^{\infty} (e^{ix} - 1) \Pi_{\mathbb{P}}(dx) \quad (3) \]

\(^2\)A subordinator is a positive, nondecreasing, right-continuous, with left limit process and initial value zero.
for some constant $m_P \in \mathbb{R}_+$ and some measure $\Pi_P(dx)$ on $(0, \infty)$. Since

$$E^P[s_{t}/t] = \psi_P'(0)/i = m_P + \int_0^\infty x\Pi_P(dx)$$

we can guarantee that the process has $\mathbb{P}$-mean $E^P[s_t] = t$ for all $t \geq 0$ by requiring that

$$m_P = 1 - \int_0^\infty x\Pi_P(dx) \geq 0.$$ 

The process $Y_t$ is also a $\mathbb{P}$-subordinator that we rewrite as

$$Y_t = \int_0^t y_s \, ds$$

for some nonnegative process $y_t$, which can be interpreted as the rate of time change, and it is independent from $s_t$ and $W_t$, and has unconditional $\mathbb{P}$-mean $E^P[y_t] = 1$.

The two subordinators $s_t$ and $Y_t$ generate two time changes. The first time change $t \mapsto s_t$ generates return jumps and allows for a flexible jump structure, e.g., large or infinitely many small jumps. The second time change $t \mapsto Y_t$ generates persistent return stochastic volatility. Randomness of $y_t$ induces stochastic volatility. Mean reversion of $y_t$ induces volatility persistence. Importantly, $s_t$, $Y_t$ and $W_t$ are all independent of each others, making the model highly tractable for estimation and option pricing purposes. Because $E^P[s_t] = E^P[Y_t] = E^P[s(Y_t)] = t$, time changes are an unbiased reflection of calendar time $t$. Absent any time change, i.e., $s_t = Y_t = t$, the log-return $X_t$ would be a Brownian motion with drift. In that case, Model (2) would produce random returns, but neither jumps nor stochastic volatility.

Model (2) encompasses several models used in the literature, and it is fully specified by choosing $s_t$ and $y_t$. Varying $s_t$ and $y_t$ in (2) can generate a wide range of processes, which underscores the high flexibility of time-changed Lévy models. For example, setting $\beta = 0$,
Y_t = t, and s_t to a Gamma process with mean rate t, Model (2) reduces to the Variance Gamma model (Madan and Seneta, 1990). As another example, under the same restrictions as above but setting s_t to an inverse Gaussian process, Model (2) reduces to the Normal Inverse Gaussian model (Barndorff-Nielsen, 1997).

We construct the subordinator s_t as the positive part of a Lévy measure. To ensure significant flexibility in modelling jump activity, we consider the Lévy measure of tempered stable processes, which is given by

$$\Pi(dx) = c \frac{\exp(-\nu x)}{x^{1+\alpha}} dx, \quad x \in (0, \infty)$$

where c, ν > 0 and α ∈ [0, 1). The parameter α is the so-called tail exponent and controls the decay of the Lévy measure. As discussed in Carr et al. (2003), Bates (2012) and others, the parameter α is key for controlling the jump activity. If α approaches one, the tail distribution resembles the tail of a Gaussian distribution. If α is less than one, s_t has infinite activity. When α = 0 or α = 1/2, (5) specializes to the Lévy measure of Gamma (Madan and Seneta, 1990) and Inverse Gaussian (Barndorff-Nielsen and Shephard, 2001) processes, respectively. The function exp (−νx) is the so-called tempering function. The parameter ν mainly affects large jumps, and ensures the existence of moment returns. Non-tempered stable processes are characterized by ν = 0, and not all their moments exist. Rachev et al. (2011), among others, provide a discussion of tempered stable processes. The parameter c controls the scale of the process and it is such that \(E^P[s_t] = t\).

In our time series and option pricing analysis, we use three Lévy subordinators, i.e., Gamma, Inverse Gaussian, and Tempered Stable. The latter subordinator nests the first two. These subordinators are among the most widely studied subordinators in the the-
oretical literature on Lévy processes. Table 1 summarizes the specifications of the three subordinators. Column 3 reports the value of the scaling parameter $c$, as a function of the other parameters in the Lévy measure (5), to ensure that $E^p[s_t] = t$. Column 5 reports the cumulant exponent of $s_t$.

Insert Table 1 about here

To generate return stochastic volatility in (2) we randomize time through the rate of time change $y_t$ in (4). We consider two models for $y_t$, namely one-factor and two-factor Heston models. One-factor Heston model (SV1) is given by

$$dy_t = \kappa_{P,y}(1 - y_t) \, dt + \sigma_{P,y} \sqrt{y_t} \, dW_{t}^{P,y}$$

(6)

and two-factor Heston model (SV2) is given by

$$dy_t = \kappa_{P,y}(m_t - y_t) \, dt + \sigma_{P,y} \sqrt{y_t} \, dW_{t}^{P,y}$$

$$dm_t = \kappa_{P,m}(1 - m_t) \, dt + \sigma_{P,m} \sqrt{m_t} \, dW_{t}^{P,m}$$

(7)

where $\kappa_{P,y}$, $\kappa_{P,m}$, $\sigma_{P,y}$, and $\sigma_{P,m}$ are nonnegative constant, and $W_{t}^{P,y}$ and $W_{t}^{P,m}$ are independent Brownian motions. Because of its analytical tractability, the SV1 model is often used for modelling return stochastic volatility. However, several studies provide evidence that two factors (one fast moving, one slow moving) are necessary to capture return volatility dynamics; see Andersen et al. (2002), Alizadeh et al. (2002), Engle and Rangel (2008), and Corradi et al. (2013). The SV2 model features two factors and it is a popular extension of the SV1 model; see Aït-Sahalia et al. (2017) and references therein.

In contrast to classic stochastic volatility models, we do not use SV1 and SV2 to model directly the instantaneous volatility. Instead, we use those models to generate the time
change $t \mapsto Y_t$. There is however a tight link between $y_t$ in (6) or (7) and the instantaneous variance of log-returns. Appendix A shows that the instantaneous variance $v_t$ is given by

$$v_t = \lim_{\Delta \to 0} \frac{V_t[X_{t+\Delta} - X_t]}{\Delta} = (\sigma^2 + \gamma^2 V[s_1]) y_t$$

(8)

where $V_t$ is the time-$t$ conditional variance. Thus, modelling the rate of time change $y_t$ is equivalent to modelling the instantaneous variance $v_t$, which motivates our choice of SV1 and SV2 for $y_t$.

Having determined the instantaneous variance of log-returns, we can now discuss the leverage effect in Model (2). The leverage effect refers to the observed phenomenon that asset returns and volatility changes are often negatively correlated.\textsuperscript{3} We define the leverage effect as the contemporaneous conditional covariance between the asset return and the instantaneous variance change, over a finite time horizon. Appendix A shows that

$$\text{Cov}_0[X_t - X_0, v_t - v_0] = (\sigma^2 + \gamma^2 V[s_1]) \text{Cov}_0[X_t, y_t]$$

$$= (\beta + \gamma)(\sigma^2 + \gamma^2 V[s_1]) \text{Cov}_0[Y_t, y_t]$$

where $\text{Cov}_0[Y_t, y_t]$ is positive for SV1 and SV2 models. Thus, the covariance between asset returns and variance changes is negative whenever $\beta + \gamma < 0$. Intuitively, in Model (2) return volatility increases when $Y_t$ increases. Because of the drift specification $\beta Y_t + \gamma s(Y_t)$, an increase of $Y_t$ can produce a negative log-return when $\beta + \gamma < 0$, capturing the leverage

\textsuperscript{3}The name “leverage effect” was introduced by Black (1976), who suggested that a large negative return increases the financial and operating leverage, and rises equity return volatility; see also Christie (1982). Alternative economic interpretations based on risk premia and volatility feedback effects have been suggested. For example, an anticipated increase in volatility commands a higher rate of return from the asset, which is produced by a fall in the asset price; e.g., French et al. (1987), and Campbell and Hentschel (1992). Bekaert and Wu (2000) provide a discussion of the leverage effect. Aït-Sahalia et al. (2013) estimate the magnitude of the effect using intraday data. We use the name leverage effect as it is commonly used by researchers when referring to the negative correlation between asset returns and volatility changes.
One important feature of Model (2) is that the conditional characteristic function of $X_t$ can be computed in a relatively straightforward way by noting that

$$
\Phi_P(u; X_t) = \mathbb{E}_0^P [\exp(iuX_t)] = \mathbb{E}_0^P \left[ \exp \left( iu\beta Y_t + i \left( \frac{u\gamma + iu^2\sigma^2}{2} \right) s(Y_t) \right) \right]
$$

(9)

$$
= \mathbb{E}_0^P \left[ \exp \left( i \left( \frac{u\beta - i\psi(u\gamma + iu^2\sigma^2)}{2} \right) Y_t \right) \right]
$$

$$
= \mathbb{E}_0^P \left[ \exp \left( iq_P(u) \int_0^t y_s ds \right) \right]
$$

where the second equality follows by subconditioning on the whole path of the processes $Y_t$ and $s(Y_t)$; the third equality follows from the second equality and (3) by subconditioning on the whole path of the process $Y_t$; the fourth equality follows from (4) after setting

$$
q_P(u) = u\beta - i\psi(u\gamma + iu^2\sigma^2)
$$

(10)

where the function $\psi$ is determined by the specification of the Lévy subordinator $s_t$ in Table 1.

The last expression in (9) is reminiscent of a bond-type pricing formula, with the bond priced at time 0, maturing at time $t$, paying no coupon, and $y_s$ playing the role of the instantaneous interest rate. To compute this expression we use indeed tools developed in the bond literature. Because SV1 and SV2 are affine models (Duffie et al., 2000), the characteristic function of $X_t$ is exponentially affine in the state variables. Specifically, for the SV2 model

$$
\Phi_P(u; X_t) = \exp \left( A_P(t; iq_P(u)) + B_P(t; iq_P(u)) y_0 + C_P(t; iq_P(u)) m_0 \right)
$$

(11)

\footnote{When $\beta + \gamma < 0$, the conditional expected excess log-return in (2) is negative, $\mathbb{E}_0[X_t] = (\beta + \gamma)\mathbb{E}_0[Y_t] < 0$, but it does not preclude a positive conditional expected excess return $\mathbb{E}_0[e^{X_t}] > 1$, as is the case according to our model estimates.}
where the functions $A_P(\tau; iq)$, $B_P(\tau; iq)$ and $C_P(\tau; iq)$ solve the system of first order differential equations

\begin{align*}
A_P'(t; iq) &= \kappa_{P,m}C_P(t; iq) \\
B_P'(t; iq) &= iq - \kappa_{P,y}B_P(t; iq) + \frac{1}{2}\sigma_{P,y}^2B_P(t; iq)^2 \\
C_P'(t; iq) &= \kappa_{P,y}B_P(t; iq) - \kappa_{P,m}C_P(t; iq) + \frac{1}{2}\sigma_{P,m}^2C_P(t; iq)^2
\end{align*}

subject to the boundary condition $A_P(0; iq) = B_P(0; iq) = C_P(0; iq) = 0$, and $'$ denotes the first derivative. Because there is no analytic solution for $A_P(\tau; iq)$, $B_P(\tau; iq)$ and $C_P(\tau; iq)$ (Grasselli and Tebaldi, 2008), we develop an efficient method to compute those functions in Appendix B. Our method delivers the unknown functions as power series solutions of the above system of first order differential equations. One notable feature of the method is that the coefficients of the power series can be computed recursively, significantly lowering the computation time.

As an illustrative example of (2), we now discuss the Variance Gamma SV1 model. In this model, $s_t$ is a Gamma subordinator (Table 1) and $y_t$ follows the SV1 process in (6). The time-0 conditional expected excess log-return is given by

$$E_0[X_t] = (\beta + \gamma)E_0[Y_t] = (\beta + \gamma)(y_0 \omega_1(t) + 1 - \omega_1(t)) t$$

where $\omega_1(t) = (1 - \exp(-\kappa_{P,y}t))/\kappa_{P,y}$. At first order, $\omega_1(t) = 1 + o(t)$, where $o(t)$ is some function approaching zero at a faster rate than $t$. The conditional variance is

$$V_0[X_t] = (\sigma^2 + \gamma^2/\nu)E_0[Y_t] + (\beta + \gamma)^2(\omega_2(t)y_0 + \omega_3(t))$$

where $\omega_2(t)$ and $\omega_3(t)$ are lengthy $o(t)$ functions. Plugging $E_0[Y_t] = y_0 (1 + o(t))$ in (13) and using $\mathbb{V}[s_1] = \nu$ from Table 1, we have that the time-0 instantaneous variance in (8) is
given by \( v_0 = (\sigma^2 + \gamma^2 \nu) y_0 \). Furthermore, in contrast to models based on SV2 processes, the characteristic function (9) can be computed explicitly

\[
\Phi_P(u; X_t) = \exp \left( A_P(t; iq_P(u)) + B_P(t; iq_P(u)) y_0 \right)
\]  

(14)

where the functions \( A_P(t; iq) \) and \( B_P(t; iq) \) solve the system of first order differential equations

\[
A_P'(t; iq) = \kappa_P y B_P(t; iq)
\]

\[
B_P'(t; iq) = iq - \kappa_P y B_P(t; iq) + \frac{1}{2} \sigma_P^2 y B_P(t; iq)^2
\]

subject to the boundary condition \( A_P(0; iq) = B_P(0; iq) = 0 \), with explicit solution given by

\[
A_P(t; iq) = -\frac{\kappa_P y}{\sigma_P^2 y} \left[ 2 \log \left( \frac{2\eta - (\eta - \kappa_P y)(1 - e^{-\eta t})}{2\eta} \right) + (\eta - \kappa_P y) t \right]
\]

\[
B_P(t; iq) = \frac{2iq(1 - e^{-\eta t})}{2\eta - (\eta - \kappa_P y)(1 - e^{-\eta t})}
\]

where \( \eta = \sqrt{\kappa_P^2 y - 2iq\sigma_P^2 y} \). Finally, when \( s_t \) is a Gamma subordinator, its cumulant exponent \( \psi_P(u) \) is given in Table 1 and the function \( q_P(u) \) defined in (10), which enters the functions \( A_P(t; iq) \) and \( B_P(t; iq) \) above, is given by

\[
q_P(u) = u\beta + i\nu \log \left( 1 - i\left( u\gamma + iu^2\sigma_P^2/2 \right)/\nu \right).
\]

(15)

Unfortunately, despite the analytical tractability of the class of models in (2), these models are not operational. The characteristic function of \( X_t \) in (11) or (14) depends on the rate of time change driven by \( y \) and possibly by \( m \), which are both unobservable. Furthermore, the model is spelled out in continuous time while in practice asset and derivative prices are observed at discrete times. The next section further develops the setup to make the models operational.
2.2 Making Time-changed Lévy Models Operational

Let $\Delta > 0$ denote a fixed time step, for example one day. The discrete time dynamics of the excess log-return $X_{(k+1)\Delta} - X_{k\Delta}$, $k = 0, 1, 2, \ldots$, can be derived from (2) by noting that

$$X_{(k+1)\Delta} - X_{k\Delta} \overset{D}{=} \beta Y_{k\Delta:(k+1)\Delta} + \gamma s(Y_{k\Delta:(k+1)\Delta}) + \sigma W(s(Y_{k\Delta:(k+1)\Delta}))$$  \hspace{1cm} (16)

where $\overset{D}{=}$ means equal in distribution, and $Y_{k\Delta:(k+1)\Delta} = \int_{k\Delta}^{(k+1)\Delta} y_s \, ds$. Intuitively, at each discrete date $k\Delta$, $k = 0, 1, 2, \ldots$, it is as if the process $Y$ accumulating the stochastic time change would restart at zero, and the accumulated stochastic time change over the time interval $[k\Delta, (k + 1)\Delta]$ would impact the excess log-return over that interval. In (16) we use the so-called infinite divisibility of a Lévy process, namely that $s_{(k+1)\Delta} \overset{D}{=} s_{k\Delta} + s_{\Delta}$. In our applications, $\Delta$ is one day when estimating models using daily log-returns, and $\Delta$ is the option time to maturity when pricing an option. We note that (16) can be extended easily to the case in which $\Delta$ is a non-negative random variable, accommodating random arrival of observed asset prices as is the case for example for intraday data.

The characteristic function of $X_{(k+1)\Delta} - X_{k\Delta}$, when $y$ follows the SV1 model, is given by

$$\Phi_P(u; X_{(k+1)\Delta} - X_{k\Delta}) = \exp \left( A_P(\Delta; iq_P(u)) + B_P(\Delta; iq_P(u)) y_{k\Delta} \right)$$  \hspace{1cm} (17)

and similarly when $y$ follows the SV2 model. These characteristic functions are not operational because they depend on $y_{k\Delta}$ and $m_{k\Delta}$, which are not observable. For simplicity we discuss how to operationalize (16) when $y$ follows SV1. The case of SV2 can be treated in a similar way.

To make (16) operational, we assume that there exists an observable proxy $\tilde{y}_k$ of $y_{k\Delta}$. The proxy needs to exist only at the discrete dates $k\Delta$, $k = 0, 1, 2, \ldots$, and it is related to the
latent $y_{k\Delta}$ as $\tilde{y}_k = y_{k\Delta}/\epsilon_k$, where $\epsilon_k$ models a nonnegative observation error. In other words, we interpret $\tilde{y}_k$ as a signal regarding the level of the latent process $y$ at date $k\Delta$. Given that $y$ controls the return stochastic variance, the observable proxy $\tilde{y}$ will be some measure of the time varying return variability. In our application we will use GARCH variances, as we will discuss in Section 3.3. Formally, the observed filtration, collecting observable variables, is discrete and given by $F_s^o = O_{[t/\Delta]}$, where $[x]$ denotes the integer part of a number $x \in \mathbb{R}_+$, and $O_n$ is the $\sigma$-algebra generated by the random variables $\{(X_{k\Delta}, \tilde{y}_{k\Delta}) : k = 0, 1, 2, \ldots, n\}$. For tractability we assume that

$$\mathbb{P}\{\epsilon_k \in A \} | O_k = \mathbb{P}\{\epsilon_k \in A \} | \tilde{y}_k = \int_A f(x, \tilde{y}_k) \, dx, \quad A \subseteq \mathbb{R}_+ \quad (18)$$

for some density function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$.

Julien: Need to explain why things would not be tractable without this assumption

Using (18) and substituting $y_{k\Delta} = \tilde{y}_k \epsilon_k$ in (17) we arrive at

$$\Phi_\varphi(u; X_{(k+1)\Delta} - X_{k\Delta}) = \mathbb{E}_{O_k} \left[ \exp \left( A_\varphi(\Delta; iq_\varphi(u)) + B_\varphi(\Delta; iq_\varphi(u)) y_{k\Delta} \right) \right]$$

$$= \exp \left( A_\varphi(\Delta; iq_\varphi(u)) \right) \int_0^\infty \exp \left( B_\varphi(\Delta; iq_\varphi(u)) \tilde{y}_k \epsilon \right) f(\epsilon, \tilde{y}_k) \, d\epsilon \quad (19)$$

where the integral term above is computed by simply plugging $-iB_\varphi(\Delta; iq_\varphi(u)) \tilde{y}_k$ in the characteristic function of $\epsilon$ conditional on $O_k$. That is, specifying some density for the observation error $\epsilon_k$, with known characteristic function, (19) is readily available in closed form.

Using (19) a large class of time-changed Lévy models can be made operational. The characteristic function of $X_{(k+1)\Delta} - X_{k\Delta}$ is now only a function of observable quantities, and unknown parameters to be estimated. Because using the proxy $\tilde{y}$ allows us to filter out the unobservable $y$, we label the estimation and pricing methods based on (19) as “filtering.” If
y follows the SV2 model we assume that two proxies $\tilde{y}$ and $\tilde{m}$ of $y$ and $m$ are available, and both their observation errors satisfy the Markovian-type property in (18).

To make $\tilde{y}_k$ an unbiased proxy of $y_{k}\Delta$, we set $\mathbb{E}_{\mathcal{O}_k}[1/\epsilon_k] = 1$. In our empirical application below we assume that $1/\epsilon_k$ follows an Inverse Gamma $IG(\delta, \delta - 1)$ distribution with shape and scale parameters $\delta$ and $(\delta - 1)$, respectively. Therefore, $\epsilon_k$ follows a Gamma $G(\delta, 1/(\delta - 1))$ distribution with characteristic function $\Phi(u; \epsilon_k)$, and the integral term in (19) is given by $\Phi(-iB_2(\Delta, iq(\Delta)) \tilde{y}_k; \epsilon_k) = (1 - B_2(\Delta, iq(\Delta)) \tilde{y}_k/(\delta - 1))^{-\delta}$.

We label the time-changed Lévy models using three blocks of letters $B_1B_2B_3$. The first block $B_1$ is $F$ or empty depending on whether the model uses or not the filtering method (19). The second block $B_2$ is VG, NIG or NTS, when $s_t$ is the Gamma, Normal Inverse Gaussian, or Normal Tempered Stable subordinator, respectively, in Table 1. The third block $B_3$ is SV1 or SV2 when the return stochastic volatility is modelled as the one-factor (6) or two-factor (7) process, respectively. For example, VGSV1 is the Variance Gamma model with the one-factor process driving the stochastic volatility discussed in (12)–(15), and FVGSV1 is the filtered version of it using (19).

3 Time Series Analysis of Time-Changed Lévy Models

This section describes the data, the estimation method, and the time series performance of time-changed Lévy models.

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5We also experimented with the Inverse Gaussian distribution and empirical results were largely unchanged.
3.1 Data

For the time series analysis we use the daily log-returns of the Standard & Poor’s 500 index (S&P 500). Data are downloaded from the Center for Research in Security Prices and consist of 23,513 daily log-returns from December 1, 1926 to December 31, 2015. The in-sample tests use 15,675 observations, until November 28, 1984, that is about 2/3 of the full sample. The remaining observations are used for out-of-sample tests. Several episodes of financial market turmoil, such as the Black Monday (1987) or the Global financial crisis (2007–2008), belong to the out-of-sample period. It will be interesting to assess the flexibility of time-changed Lévy models in capturing large negative returns during those periods, given that the models will not be estimated using those returns.

3.2 Model Estimation

We estimate the time-changed Lévy models by maximum likelihood in conjunction with the filtering approach in (19). Because the probability density of the log-return in (2) is not available in closed-form, we first compute the characteristic function and then recover the probability density via Fourier inversion of the characteristic function.

There are various approaches for recovering probability densities from characteristic functions, with the Fast Fourier transform (FFT) or fractional FFT being popular methods. However, the probability density of Lévy processes has typically large kurtosis and the numerical errors of classic Fourier inversion can be potentially large. This issue is even more severe for probability densities over short time horizons, like one day, as in our time series estimates. The problem is the slow decay of the real part of the characteristic function, which
is known as the Gibbs phenomenon. To achieve high accuracy in the computation of the probability density and overcome the Gibbs phenomenon, we use the COS method of Fang and Oosterlee (2008) and enrich the method with an exponential damping. Appendix C presents our method.

3.3 Empirical Findings

We estimate 16 time-changed Lévy models. We construct 12 models by combining the three Lévy subordinators \( s_t \) in Table 1; the one-factor (SV1) or two-factor (SV2) processes for the return stochastic volatility; with or without filtering of the stochastic time change as in (19). All these models are of infinite activity. We also consider four finite activity models in which \( s_t \) follows a Poisson process, combined with SV1 or SV2 processes and with or without filtering. For the finite activity models, the block \( B_2 \) of letter in the model labels is given by \( P \); see end of Section 2.2 for the explanation of model labels.

In SV1 models with filtering, we set the observable proxy \( \tilde{y}_k \) to the GARCH variance from a GJR GARCH model (Glosten et al., 1993). In SV2 models with filtering, we set \( \tilde{y}_k \) and \( \tilde{m}_k \) to the short-run and long-run GARCH variance components, respectively, from a two-component GARCH model (Christoffersen et al., 2008). We estimate both GARCH models by maximum likelihood, assuming Student-\( t \) innovations and using S&P 500 daily log-returns. Then, we compute and scale the GARCH variances such that \( \hat{y}_k \)'s have sample mean one, in agreement with the filtering approach (19). In models without filtering, the (theoretically) unobservable processes \( y_t \) and \( m_t \) are set directly equal to the scaled GARCH variances, i.e., observation errors are assumed to be absent.

The GJR and the two-component \( t \)-GARCH are popular models to recover daily log-
return volatilities. Besides using these models to obtain the proxies of latent variables, we also use them as a benchmark for the time series performance of time-changed Lévy models.

Tables 2 and 3 present the estimation results for time-changed Lévy models, when the return stochastic volatility is driven by a one-factor (SV1) or two-factor (SV2) process, respectively. Several findings emerge from the model estimates. First, going from finite to infinite activity models, the fitting of index returns sharply improves. Vuong (1989) Likelihood Ratio (LR) tests soundly reject the null hypothesis that FPSV1 or FPSV2 are equivalent to any infinite activity model. Second, using the filtering approach in (19) significantly improves model fitting, across all time-changed Lévy models. LR tests soundly reject null hypotheses such as models FVGSV1 and VGSV2 are equivalent, for one- and two-factor and for any subordinator process. Third, the best performing model is FNTSSV2. By far this model has the largest log-likelihood and also (unreported) lowest Akaike information criterion (AIC), across all models. Fourth, the benchmark GJR and two-component $t$-GARCH models substantially underperform any infinite activity model. The log-likelihood values of the two GARCH models are only, 52,568.58 and 52,608.24, respectively. The GARCH models still outperform finite activity models. Fifth, estimates of the tempering parameter $\nu$ in the Lévy measure (5) are largely away from zero, casting evidence against stable infinite variation subordinators. Finally, estimates of the tail parameter $\alpha$ in (5) point to a tempered stable subordinator for the index returns. Estimates of $\alpha$ range between 0.69 and 0.85, and are statistically away from both 1/2 and 1 (see Table 1). This finding is in line with Bates (2012), who uses different time-changed Lévy models but reports similar estimates of tail
A closer inspection of the best performing model, FNTSSV2, allows us to gauge the sources of its performance. Taking the classic VGSV1 as a baseline time-changed Lévy model, we can decompose the log-likelihood increment from VGSV1 to FNTSSV2 as follows

\[ \text{FNTSSV2} - \text{VGSV1} = (\text{FNTSSV2} - \text{NTSSV2}) + (\text{NTSSV2} - \text{NTSSV1}) + (\text{NTSSV1} - \text{VGSV1}) \]

with the above equality being in log-likelihood units. The total log-likelihood increment from VGSV1 to FNTSSV2 can be broken down as follows: 23.8% of the increment is due to the filtering approach in (19) (comparing FNTSSV2 with NTSSV2); 34.4% due to adding one volatility factor (comparing NTSSV2 with NTSSV1); 41.8% due to a flexible jump specification (comparing NTSSV1 with VGSV1). While modeling volatility as a two-factor process is quite important, using our filtering approach (19) and a flexible jump specification largely contribute to the accurate model fitting, too.

Figure 3 shows the out-of-sample, normal probability plots for the normalized returns for a select set of Lévy models and the benchmark $t$-GARCH models. Comparing Panels (a)–(b) with (c)–(d) shows the benefit of going from one-factor to two-factor volatility process for infinite activity VG processes. Major improvements are visible in fitting the left tail of the index return distribution. Comparing Panels (a)–(c) with (b)–(d) shows that using the filtering approach (19) uniformly improves the fitting of the whole return distribution. Finally, Panels (e)–(f) show the substantially inferior performance of the benchmark $t$-GARCH models relative to infinite activity Lévy models.

To summarize, infinite activity Lévy models fit market returns substantially better than

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6His estimates of tail parameters range between 0.93 and 0.97, and in his notation are $Y_n$ and $Y_p$, divided by 2, to map the parameter range to $[0, 1]$. 

22
finite activity and GARCH models. Our filtering approach (19) generally improves the fitting of the whole return distribution, irrespective of the time-changed Lévy model. Among the 16 time-changed Lévy models, the best performing model features a tempered stable subordinator, two-factor volatility process, and it is estimated using our filtering approach (19).

4 Option Pricing: Theoretical Analysis

This section derives the change of measure, the risk-neutral characteristic function, and the option pricing formula for the general class of time-changed Lévy models in (2).

4.1 Risk-neutralization

The discrete-time market at dates $k\Delta, k = 0, 1, 2, \ldots$, that consists in the risky asset and a riskless asset with return $r\Delta$ is intrinsically incomplete. Therefore, the set of risk-neutral measures consists in infinitely many elements. To narrow down this set and pin down the pricing measure implicit in option prices we restrict ourselves to equivalent probability measures $Q$ with the following properties:

1) There exists a risk premium $\theta$ such that

$$W_t^Q = W_t^P + \theta t$$

is a standard Brownian motion under $Q$.

2) The process $s_t$ remains a Lévy process under $Q$ with characteristic exponent given by

$$\psi_Q(u) = \frac{1}{t} \log \mathbb{E}^Q[e^{iux}] = iu m_Q + \int_0^\infty (e^{iux} - 1) \Pi_Q(dx)$$

for some constant $m_Q \in \mathbb{R}_+$ and some measure $\Pi_Q(dx)$ on $\mathbb{R}_+$.  

23
Using these restrictions and arguments similar to those of Section 2 shows that the martingale condition
\[ E^Q_{O_k} \left[ e^{-(r-\delta)\Delta \frac{S_{(k+1)\Delta}}{S_{k\Delta}}} \right] = 1 \]
can be equivalently stated as
\[ 1 = E^Q_{O_k} \left[ \exp \left( X_{(k+1)\Delta} - X_{k\Delta} \right) \right] = E^Q_{O_k} \left[ \exp \left( (\beta - \phi_Q(\sigma \theta - \sigma^2/2 - \gamma))(Y_{(k+1)\Delta} - Y_{k\Delta}) \right) \right] \]
where \( \phi_Q(\lambda) = -\psi_Q(i\lambda) \) is the Laplace exponent of the non-decreasing Lévy process \( s_t \) under \( Q \). Since \( Y_t \) is a nondecreasing process we conclude that under (20) and (21) a necessary and sufficient condition for \( Q \) to be a risk-neutral measure is that
\[ \beta = \phi_Q(\sigma \theta - \sigma^2/2 - \gamma). \quad (22) \]
Given time series estimates for \( (\beta, \gamma, \sigma) \), this condition places a joint constraint on the Brownian risk premium \( \theta \), the risk-neutral drift \( m_Q \) and the Levy measure \( \Pi_Q(dx) \). It also motivates our specification of the return drift in (2). In fact, if \( \beta = 0 \), as in traditional time-changed Lévy models, then \( s_t \) must be identically zero under \( Q \), because \( \phi_Q \) in (22) is identically zero.

Julien: Discuss implications for “traditional” time-changed Lévy models, with \( \beta = 0 \)?

Importantly, (22) shows that, beyond the requirement of equivalence, the risk neutralization of our time-changed Lévy models in (2) does not impose any restriction on the risk-premia that influence the distribution of the process \( y_t \) or the distribution of the observation errors \( (\epsilon_k)_{k=0}^{\infty} \).

Julien (?): Discuss leverage under \( Q \)

24
4.2 The $\mathbb{Q}$-Characteristic Function

To compute an option price we need to derive the risk-neutral characteristic function

$$
\Phi_{\mathbb{Q}}(u; X_{n\Delta} - X_{k\Delta}) = \mathbb{E}_{\mathbb{Q}_k}^\mathbb{Q} \left[ \exp \left( iu(X_{n\Delta} - X_{k\Delta}) \right) \right]
$$

for $n > k$, where $n\Delta - k\Delta$ is the time to maturity of the option. The same arguments as in the computation of the $\mathbb{P}$-characteristic function show that

$$
\Phi_{\mathbb{Q}}(u; X_{n\Delta} - X_{k\Delta}) = \mathbb{E}_{\mathbb{Q}_k}^\mathbb{Q} \left[ \exp \left( iq_{\mathbb{Q}}(u) \int_{k\Delta}^{n\Delta} y_s \, ds \right) \right]
$$

(23)

with

$$
q_{\mathbb{Q}}(u) = u\beta - i\psi_{\mathbb{Q}} \left( u(\gamma - \sigma\theta) + iu^2\sigma^2 \right) / 2.
$$

As in Section 2.2, for tractability reasons we assume that the observation error $\epsilon_k$, embedded in observable proxies of the latent variables $y_{k\Delta}$ and $m_{k\Delta}$, under $\mathbb{Q}$ satisfies the same Markovian-type properties as under $\mathbb{P}$, that is

$$
\mathbb{Q}\left[\{\epsilon_k \in A\} | O_k\right] = \mathbb{Q}\left[\{\epsilon_k \in A\} | \tilde{y}_k\right] = \int_A f_{\mathbb{Q}}(x, \tilde{y}_k) \, dx, \quad A \subseteq \mathbb{R}_+^2
$$

for some density function $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$.

For the volatility risk premia, we use the most general form of risk premia that preserves the square-root feature of $y_t$ when changing measure from $\mathbb{P}$ to $\mathbb{Q}$; see Cheridito et al. (2007).

For the SV1 model (6)

$$
dW_{t}^{\mathbb{P},y} = dW_{t}^{\mathbb{Q},y} + a_y \sqrt{y_t} \, dt + \frac{b_y}{\sqrt{y_t}} \, dt
$$

(24)

where $a_y$ and $b_y$ control the risk premia, and $W_{t}^{\mathbb{Q},y}$ is a $\mathbb{Q}$-Brownian motion. The SV1 model under $\mathbb{Q}$ is therefore

$$
dy_t = \kappa_{\mathbb{Q},y}(\theta_{\mathbb{Q},y} - y_t) \, dt + \sigma_{\mathbb{P},y} \sqrt{y_t} \, dW_{t}^{\mathbb{Q},y}
$$
where $\kappa_{Q,y} = \kappa_{P,y} - \sigma_{P,y} a_y$ and $\theta_{Q,y} = (\kappa_{P,y} + \sigma_{P,y} b_y)/(\kappa_{P,y} - \sigma_{P,y} a_y)$. The risk premia can alter the speed of mean reversion $\kappa_{Q,y}$ and the long run mean $\theta_{Q,y}$ of $y_t$ under $Q$. Notice that $\theta_{Q,y}$ is not necessarily one. The constraints to be imposed on $\kappa_{Q,y}$ and $\theta_{Q,y}$ depend on those that are imposed on $\kappa_{P,y}$ and $\sigma_{P,y}$, and vice versa. That is, if the Feller condition holds (fails) under either $P$ or $Q$ then equivalence requires that it also holds (fails) under the other.

For the SV2 model (7) the risk premia are such that

$$dW^P_t = dW^Q_t + a_y \sqrt{y_t} dt + b_y \sqrt{y_t} dt$$

$$dW^P_m = dW^Q_m + a_m \sqrt{m_t} dt + b_m \sqrt{m_t} dt$$

where $W^Q_m$ is a $Q$-Brownian motion. The SV2 model under $Q$ is

$$dy_t = \kappa_{Q,y} (\theta_{Q,y,c} + \theta_{Q,y,d} m_t - y_t) dt + \sigma_{P,y} \sqrt{y_t} dW^Q_t$$

$$dm_t = \kappa_{Q,m} (\theta_{Q,m} - m_t) dt + \sigma_{P,m} \sqrt{m_t} dW^Q_m$$

where $\kappa_{Q,m} = \kappa_{P,m} - \sigma_{P,m} a_m$,

$$\theta_{Q,m} = \frac{\kappa_{P,m} + \sigma_{P,m} b_m}{\kappa_{P,m} - \sigma_{P,m} a_m}, \quad \theta_{Q,y,c} = \frac{\sigma_{P,y} b_y}{\kappa_{P,y} - \sigma_{P,y} a_y}, \quad \theta_{Q,y,d} = \frac{\kappa_{P,y}}{\kappa_{P,y} - \sigma_{P,y} a_y}.$$

As usual when pricing options via Fourier inversion we rewrite the time-$k\Delta$ forward price of a call option expiring at time $T = n\Delta$, with $n > k$, and strike price $K$, as

$$E^Q_{\Omega_k} \left[ \max(S_{n\Delta} - K, 0) \right] = S_{k\Delta} E^Q_{\Omega_{k}} \left[ \max(e^{\log(S_{n\Delta}/S_{k\Delta})} - K/S_{k\Delta}, 0) \right].$$

Then, using the COS method of Fang and Oosterlee (2008), we Fourier invert the $Q$-characteristic function of $\log(S_{n\Delta}/S_{k\Delta}) = (r - \delta)(n-k)\Delta + X_{n\Delta} - X_{k\Delta}$ given by $\Phi_{Q}(u; X_{n\Delta} - X_{k\Delta})$ in (23).
5 Option Pricing: Empirical Results

This section describes the option data and the empirical pricing performance of the proposed time-changed Lévy models.

5.1 Data

To evaluate the option pricing performance of time-changed Lévy models, we use the European options on the S&P 500 index (symbol: SPX). SPX options are among the most actively traded options in the world and have been investigated in a number of empirical studies, including Christoffersen et al. (2008), Corsi et al. (2013), and Ornthanalai (2014).

We consider closing prices of out-of-the-money (OTM) put and call SPX options for each Wednesday from January 1, 1996 to August 31, 2015. It is well-known that OTM options are more actively traded than in-the-money options. All option data are downloaded from OptionMetrics. We apply the usual screening criteria on raw option contracts; see Barone-Adesi et al. (2008). An option price is computed as the average of bid and ask prices. Options with time to maturity less than 10 days or more than 360 days, prices less than $0.05, or zero trading volume are discarded. We also compute the implied volatility of each option, and eliminate those options whose implied volatility differs by more than 5% from the implied volatility reported in OptionMetrics. The final dataset contains 543,597 option contracts, with the number of call and put options being 178,906 (32.91%) and 364,691 (67.09%), respectively. The large number of put options reflects the increased demand of those options during and after the 2007–2009 financial crisis.

The risk free rate for each option maturity and each day in our sample is calculated as
follows. Using the term structure of zero-coupon default-free interest rates, the risk free rate for each given maturity is obtained by linearly interpolating the two interest rates whose maturities straddle the maturity of each given option.

We define moneyness, \( m \), as the ratio of the strike price over the index price, i.e., \( K/S \). A put option is said to be deep OTM if \( m < 0.95 \), or OTM if \( 0.95 \leq m < 1 \). A call option is said to be OTM if \( 1 \leq m < 1.10 \), or deep OTM if \( 1.10 \leq m \). An option contract with time to maturity \( T \) has short maturity (\( 10 \leq T < 80 \)), intermediate maturity (\( 80 \leq T < 180 \)), or long maturity (\( 180 \leq T \)).

Table 4 reports descriptive statistics for the 543,597 option contracts. The number of deep OTM put (call) options is 304,801 (74,675), which corresponds to 56.07% (13.74%) of the option contracts (Panel A). Short and long maturity options account for 54.93% and 20.89%, respectively, of the total sample. The average put (call) price ranges from $3.14 ($0.97) for short maturity, deep OTM options to $73.47 ($55.32) for long maturity, OTM options (Panel B). Call option prices tend be more volatile than put options in relative terms (Panel C). For each maturity bucket, the implied volatility smile across moneyness is evident (Panel D). When the time to maturity increases, the implied volatility smile tend to become flatter, but the standard deviation of implied volatilities remain high especially for deep OTM puts (Panel E).

![Insert Table 4 about here](image)

### 5.2 Option Pricing Performance

In this section, we compare the option pricing performance of the 16 time-changed Lévy processes discussed at the beginning of Section 3.3. We split the option data into an in-sample
part, which includes 2/3 of the full sample (1996 to mid-2008), and an out-of-sample part, which includes the remaining 1/3 of the sample (mid-2008 to 2015). Previous option pricing applications of time-changed Lévy models, such as Huang and Wu (2004), have focused on the VGSV1 model. We use this model as a benchmark.

For each model, given its time series parameter estimates (Tables 2 and 3), we use in-sample option data to calibrate the risk neutral and risk premium parameters, enforcing the no-arbitrage condition (22). Specifically, for each model, we take $\beta, \sigma$ and $\gamma$ from time series estimates, we calibrate the jump parameter $\nu_Q^Q$, and $\alpha_Q^Q$ for NTS models, and $a_y, b_y$ in (24) and $a_y, b_y, a_m, b_m$ in (25) for SV1 and SV2 models, respectively, and we set $\theta$ such that the no-arbitrage condition (22) holds.\footnote{Equation (22) can be solved for $\theta$ analytically. For example, when $s_t$ is a Gamma subordinator, $\theta = (\exp(\beta/\nu^Q) - 1 + \gamma/\nu^Q + \sigma^2/(2\nu^Q))\nu^Q/\sigma$.} To restrict the number of calibrated parameters, we assume no risk premium for observation errors $\epsilon_k$ and impose $\mathbb{E}_Q[s_t] = t$. As usual, we achieve the parameter calibration by minimizing the sum of squared pricing errors, between observed and model-based option prices, across all Wednesdays in the in-sample period, and treat this minimization as a non-linear least squares problem to obtain standard errors for the calibrated parameters.

Table 5 shows the calibrated parameters for the infinite activity models. To save space and given their inferior time series performance, we do not report the calibrated parameters of finite activity models.

Discuss risk neutral and risk premium parameters

Two findings emerge from Table 5 regarding jump and volatility risk premia. First,
the Lévy measure under $\mathbb{Q}$ is substantially different than under $\mathbb{P}$. Both $\nu^\mathbb{Q}$ and $\alpha^\mathbb{Q}$ are systematically lower than their counterparts under $\mathbb{P}$. A low $\nu^\mathbb{Q}$ means that the tempering function decays slowly and large jumps are more likely to occur under $\mathbb{Q}$ than under $\mathbb{P}$. A low $\alpha^\mathbb{Q}$ indicates high jump activity near zero. For example, in VGSV1 we need to quantify this second, risk premia for SV1 and SV2 exhibit non-linearities. We need to discuss this.

In line with the literature (Corsi et al., 2013), we evaluate the option pricing performance using the root mean square error (RMSE), which is given by

$$RMSE = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\tilde{P}_{i}^{\text{mkt}} - \tilde{P}_{i}^{\text{mod}})^2}$$

where $N$ is the number of options, $\tilde{P}_{i}^{\text{mkt}} = P_{i}^{\text{mkt}}/S$ and $\tilde{P}_{i}^{\text{mod}} = P_{i}^{\text{mod}}/S$ are the observed and model-based option prices divided by the level of the S&P 500 index.

Table 5 reports the RMSE of each model relative to the RMSE of the VGSV1 model, which serves as a benchmark. Various findings arise from this table. First, FNTSSV2 outperforms all competing models. In-sample, FNTSSV2 has a RMSE 21.5% lower relative to VGSV1, with a similar performance out-of-sample. Second, models with filtering always outperform models without filtering, sometimes by a large extent, although observation errors carry no risk premium. For example, FVGSV1 has a RMSE 10% lower relative to VGSV1. Third, depending on the Lévy subordinator, going from one-factor to two-factor models, the pricing performance improves from 2% to 16% in-sample and from 1% to 14% out-of-sample data.

Tables 6 and 7 break down the in-sample and out-of-sample, respectively, pricing performance of selected models across moneyness and maturities. To save space we consider four models. Panel A reports the RMSE of the benchmark VGSV1. The pricing performance of
VGSV1 is quite uniform across moneyness but tends to deteriorate with the time to maturity. Relying only on one factor for the stochastic volatility does not appear to be enough to reproduce the term structure of option prices. The ratios of RMSE of PSV1 to VGSV1 (Panel B) are nearly always larger than one, and underscore the importance of moving from a finite activity to an infinite activity model. Notably, the ratios of RMSE of FVHSV1 to VGSV1 (Panel C) are nearly always smaller than one both in- and out-of-sample, and highlight the benefits of using our filtering approach. Finally, the ratios of FNTSSV2 to VGSV1 (Panel D) are substantially smaller than one in the vast majority of cases, and confirm the importance of using a flexible Lévy measure and a two-factor process for stochastic volatility when pricing options.
Table 1: Specification of the Lévy subordinators.

<table>
<thead>
<tr>
<th>$s_t$</th>
<th>$\alpha$</th>
<th>$c$</th>
<th>$\mathbb{V}[s_t]$</th>
<th>$t \psi(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gamma</td>
<td>0</td>
<td>$\nu$</td>
<td>$t/\nu$</td>
<td>$-tc \log(1 - iu/\nu)$</td>
</tr>
<tr>
<td>Inverse Gaussian</td>
<td>$\frac{1}{2}$</td>
<td>$\sqrt{\nu/\pi}$</td>
<td>$\frac{t}{2\nu}$</td>
<td>$-tc \sqrt{\pi} \left( \sqrt{\nu - iu} - \sqrt{\nu} \right)$</td>
</tr>
<tr>
<td>Tempered Stable</td>
<td>$(0, 1)$</td>
<td>$-\frac{\nu^{1-\alpha}}{\alpha \Gamma(-\alpha)}$</td>
<td>$(1 - \alpha) t \nu$</td>
<td>$tc \Gamma(-\alpha) \left( (\nu - iu)^\alpha - \nu^\alpha \right)$</td>
</tr>
</tbody>
</table>

Note: Lévy measure of $s_t$ is in (5). $\mathbb{V}[s_t]$ is the variance of $s_t$, $t \psi(u) = \log(\mathbb{E}[e^{iu s_t}])$ is the cumulant exponent of $s_t$, $\Gamma(\cdot)$ is the Gamma function, and $i = \sqrt{-1}$.
Table 2: Parameter estimates for time-changed Lévy models, one-factor stochastic volatility.

Time-changed Lévy models (2) are obtained from a finite activity Poisson subordinator (Model label P) and the three infinite activity Lévy subordinators in Table 1 (Model labels VG, NIG, NTS), combined with one-factor (SV1) process (6) for the stochastic return volatility, with (F) or without filtering approach for the stochastic time change as in (19). With filtering the observation error for the latent $y$ follows an $IG(\delta_y, \delta_y - 1)$. Model labels are described at the end of Section 2.2. Standard errors are in parenthesis. lnL is Log-likelihood. Bm is Brownian motion. Data are daily S&P 500 log-returns over 1926–1984.

<table>
<thead>
<tr>
<th>Model</th>
<th>lnL</th>
<th>$\nu$</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\kappa_{F,y}$</th>
<th>$\sigma_{F,y}$</th>
<th>$\delta_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PSV1</td>
<td>41,943.82</td>
<td>-0.223</td>
<td>0.027</td>
<td>0.251</td>
<td>1.418</td>
<td>0.940</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FPSV1</td>
<td>42,213.69</td>
<td>-0.515</td>
<td>0.240</td>
<td>0.547</td>
<td>2.027</td>
<td>1.324</td>
<td>49.61</td>
<td>(0.001)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>VGSV1</td>
<td>52,835.50</td>
<td>108.69</td>
<td>-0.101</td>
<td>0.189</td>
<td>1.586</td>
<td>0.122</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FVGSV1</td>
<td>52,849.42</td>
<td>121.33</td>
<td>0.090</td>
<td>0.195</td>
<td>2.327</td>
<td>0.463</td>
<td>10.06</td>
<td>(0.000)</td>
<td>(0.000)</td>
</tr>
<tr>
<td>NIGSV1</td>
<td>52,837.24</td>
<td>41.84</td>
<td>-0.133</td>
<td>0.193</td>
<td>7.699</td>
<td>0.193</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FNIGSV1</td>
<td>52,851.12</td>
<td>30.87</td>
<td>0.092</td>
<td>0.217</td>
<td>1.299</td>
<td>0.060</td>
<td>22.53</td>
<td>(0.000)</td>
<td>(0.000)</td>
</tr>
<tr>
<td>NTSSV1</td>
<td>52,847.26</td>
<td>53.94</td>
<td>0.850</td>
<td>0.209</td>
<td>1.104</td>
<td>0.180</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FNTSSV1</td>
<td>52,855.80</td>
<td>50.40</td>
<td>0.895</td>
<td>-0.107</td>
<td>0.191</td>
<td>1.980</td>
<td>0.287</td>
<td>8.69</td>
<td></td>
</tr>
</tbody>
</table>
Table 3: Parameter estimates for time-changed Lévy models, two-factor stochastic volatility.

See the notes to Table 2 for parameter definitions and sample data. Time-changed Lévy models (2) are based on two-factor (SV2) process (7) for the stochastic return volatility. With filtering the observation error for the latent \( y \) follows an \( IG(\delta_y, \delta_y - 1) \) and for the latent \( m \) follows an \( IG(\delta_m, \delta_m - 1) \).

<table>
<thead>
<tr>
<th>Model</th>
<th>lnL</th>
<th>Jump</th>
<th>Drift</th>
<th>Bm</th>
<th>Stochastic Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \nu )</td>
<td>( \alpha )</td>
<td>( \gamma )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>PSV2</td>
<td>51,468.88</td>
<td>-0.261</td>
<td>0.171</td>
<td>0.180</td>
<td>1.083</td>
</tr>
<tr>
<td>FPSV2</td>
<td>51,583.12</td>
<td>-0.221</td>
<td>0.190</td>
<td>0.179</td>
<td>1.843</td>
</tr>
<tr>
<td>VGSV2</td>
<td>52,847.91</td>
<td>107.52</td>
<td>-0.130</td>
<td>0.109</td>
<td>0.211</td>
</tr>
<tr>
<td>FVGSV2</td>
<td>52,855.33</td>
<td>105.93</td>
<td>-0.110</td>
<td>0.099</td>
<td>0.184</td>
</tr>
<tr>
<td>NIGSV2</td>
<td>52,852.61</td>
<td>34.12</td>
<td>-0.187</td>
<td>0.179</td>
<td>0.212</td>
</tr>
<tr>
<td>FNIGSV2</td>
<td>52,859.04</td>
<td>44.51</td>
<td>-0.163</td>
<td>0.156</td>
<td>0.194</td>
</tr>
<tr>
<td>NTSSV2</td>
<td>52,857.00</td>
<td>50.04</td>
<td>0.860</td>
<td>-0.145</td>
<td>0.137</td>
</tr>
<tr>
<td>FNTSSV2</td>
<td>52,863.72</td>
<td>45.97</td>
<td>0.825</td>
<td>-0.142</td>
<td>0.135</td>
</tr>
</tbody>
</table>
Table 4: S&P 500 index option data.

Out-of-the-money put and call options on the S&P 500 index from January 1, 1996 until August 31, 2015. Moneyness $m$ is defined as the ratio of the strike price over the index price, i.e., $K/S$. Time to maturity $T$ is in days.

<table>
<thead>
<tr>
<th>Panel</th>
<th>$10 \leq T &lt; 80$</th>
<th>$80 \leq T &lt; 180$</th>
<th>$180 \leq T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Number of option contracts</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K/S &lt; 0.95$</td>
<td>167,363</td>
<td>74,207</td>
<td>63,231</td>
</tr>
<tr>
<td>$0.95 \leq K/S &lt; 1$</td>
<td>39,575</td>
<td>12,238</td>
<td>8,077</td>
</tr>
<tr>
<td>$1 \leq K/S &lt; 1.10$</td>
<td>66,292</td>
<td>22,479</td>
<td>15,460</td>
</tr>
<tr>
<td>$1.10 \leq K/S$</td>
<td>25,357</td>
<td>22,504</td>
<td>26,814</td>
</tr>
<tr>
<td>Panel B: Average option prices</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K/S &lt; 0.95$</td>
<td>3.140</td>
<td>8.404</td>
<td>19.007</td>
</tr>
<tr>
<td>$0.95 \leq K/S &lt; 1$</td>
<td>19.169</td>
<td>42.518</td>
<td>73.468</td>
</tr>
<tr>
<td>$1 \leq K/S &lt; 1.10$</td>
<td>9.287</td>
<td>24.950</td>
<td>55.317</td>
</tr>
<tr>
<td>$1.10 \leq K/S$</td>
<td>0.974</td>
<td>2.834</td>
<td>10.396</td>
</tr>
<tr>
<td>Panel C: Standard deviation option prices</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K/S &lt; 0.95$</td>
<td>4.992</td>
<td>10.731</td>
<td>20.822</td>
</tr>
<tr>
<td>$0.95 \leq K/S &lt; 1$</td>
<td>12.549</td>
<td>15.985</td>
<td>25.669</td>
</tr>
<tr>
<td>$1 \leq K/S &lt; 1.10$</td>
<td>11.186</td>
<td>18.329</td>
<td>27.163</td>
</tr>
<tr>
<td>$1.10 \leq K/S$</td>
<td>2.581</td>
<td>5.501</td>
<td>13.936</td>
</tr>
<tr>
<td>Panel D: Average implied volatility</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K/S &lt; 0.95$</td>
<td>0.337</td>
<td>0.335</td>
<td>0.313</td>
</tr>
<tr>
<td>$0.95 \leq K/S &lt; 1$</td>
<td>0.180</td>
<td>0.189</td>
<td>0.200</td>
</tr>
<tr>
<td>$1 \leq K/S &lt; 1.10$</td>
<td>0.144</td>
<td>0.158</td>
<td>0.177</td>
</tr>
<tr>
<td>$1.10 \leq K/S$</td>
<td>0.250</td>
<td>0.194</td>
<td>0.180</td>
</tr>
<tr>
<td>Panel E: Standard deviation implied volatility</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K/S &lt; 0.95$</td>
<td>0.144</td>
<td>0.150</td>
<td>0.117</td>
</tr>
<tr>
<td>$0.95 \leq K/S &lt; 1$</td>
<td>0.062</td>
<td>0.057</td>
<td>0.053</td>
</tr>
<tr>
<td>$1 \leq K/S &lt; 1.10$</td>
<td>0.062</td>
<td>0.058</td>
<td>0.051</td>
</tr>
<tr>
<td>$1.10 \leq K/S$</td>
<td>0.126</td>
<td>0.079</td>
<td>0.060</td>
</tr>
</tbody>
</table>
Table 5: Calibrated parameters and option pricing performance of time-changed Lévy models.

Risk neutral and risk premium parameters are calibrated to in-sample option data as described in Section 5.2. Risk-neutral parameters $\nu_Q$ and $\alpha_Q$ determine the Lévy jump measure (Table 1); volatility risk-premiius $a_y, b_y$ for SV1 models, and $a_y, b_y, a_m, b_m$ for SV2 models are defined in (24) and (25), respectively; Brownian risk premium $\theta$ is defined in (20). Model labels are described at the end of Section 2.2. Model pricing performance is measured as the root mean square pricing error (RMSE). IS (OoS) is the in-sample (out-of-sample) RMSE of corresponding model divided by the RMSE of VGSV1 (benchmark) model. For VGSV1 IS and OoS in square brackets are the RMSE. In-sample option data spans 1996 to mid-2008, out-of-sample option data spans mid-2008 to 2015.

<table>
<thead>
<tr>
<th>Model</th>
<th>IS</th>
<th>$\nu^Q$</th>
<th>$\alpha^Q$</th>
<th>$a_y$</th>
<th>$b_y$</th>
<th>$a_m$</th>
<th>$b_m$</th>
<th>$\theta$</th>
<th>OoS</th>
</tr>
</thead>
<tbody>
<tr>
<td>VGSV1</td>
<td>0.009</td>
<td>13.88</td>
<td>3.230</td>
<td>0.913</td>
<td>(0.208)</td>
<td>(1.240)</td>
<td>(0.000)</td>
<td>0.038</td>
<td>0.005</td>
</tr>
<tr>
<td>FVGSV1</td>
<td>0.898</td>
<td>10.98</td>
<td>1.781</td>
<td>0.819</td>
<td>(0.514)</td>
<td>(2.009)</td>
<td>(0.004)</td>
<td>0.017</td>
<td>0.872</td>
</tr>
<tr>
<td>NIGSV1</td>
<td>0.911</td>
<td>5.102</td>
<td>11.828</td>
<td>1.094</td>
<td>(0.094)</td>
<td>(1.438)</td>
<td>(0.075)</td>
<td>0.070</td>
<td>0.879</td>
</tr>
<tr>
<td>FNIGSV1</td>
<td>0.880</td>
<td>3.571</td>
<td>7.683</td>
<td>1.035</td>
<td>(0.082)</td>
<td>(0.893)</td>
<td>(0.006)</td>
<td>0.039</td>
<td>0.871</td>
</tr>
<tr>
<td>NTSSV1</td>
<td>0.875</td>
<td>4.250</td>
<td>2.653</td>
<td>0.239</td>
<td>(0.088)</td>
<td>(0.741)</td>
<td>(0.000)</td>
<td>0.020</td>
<td>0.866</td>
</tr>
<tr>
<td>FNTSSV1</td>
<td>0.864</td>
<td>3.711</td>
<td>3.574</td>
<td>0.209</td>
<td>(0.072)</td>
<td>(0.221)</td>
<td>(0.000)</td>
<td>0.024</td>
<td>0.853</td>
</tr>
</tbody>
</table>

Panel B:

<table>
<thead>
<tr>
<th>Model</th>
<th>in-sample</th>
<th>$\nu^Q$</th>
<th>$\alpha^Q$</th>
<th>$a_y$</th>
<th>$b_y$</th>
<th>$a_m$</th>
<th>$b_m$</th>
<th>$\theta$</th>
<th>OoS</th>
</tr>
</thead>
<tbody>
<tr>
<td>VGSV2</td>
<td>0.862</td>
<td>10.97</td>
<td>3.540</td>
<td>0.294</td>
<td>(0.339)</td>
<td>(0.009)</td>
<td>(0.007)</td>
<td>0.009</td>
<td>0.846</td>
</tr>
<tr>
<td>FVGSV2</td>
<td>0.852</td>
<td>12.33</td>
<td>2.904</td>
<td>0.303</td>
<td>(1.229)</td>
<td>(0.085)</td>
<td>(0.009)</td>
<td>0.009</td>
<td>0.835</td>
</tr>
<tr>
<td>NIGSV2</td>
<td>0.855</td>
<td>7.042</td>
<td>4.053</td>
<td>0.311</td>
<td>(0.079)</td>
<td>(0.111)</td>
<td>(0.005)</td>
<td>0.009</td>
<td>0.855</td>
</tr>
<tr>
<td>FNIGSV2</td>
<td>0.840</td>
<td>8.196</td>
<td>5.621</td>
<td>0.449</td>
<td>(1.505)</td>
<td>(0.748)</td>
<td>(0.006)</td>
<td>0.009</td>
<td>0.830</td>
</tr>
<tr>
<td>NTSSV2</td>
<td>0.801</td>
<td>3.814</td>
<td>3.289</td>
<td>0.106</td>
<td>(0.041)</td>
<td>(0.005)</td>
<td>(0.000)</td>
<td>0.009</td>
<td>0.832</td>
</tr>
<tr>
<td>FNTSSV2</td>
<td>0.785</td>
<td>3.709</td>
<td>4.347</td>
<td>0.268</td>
<td>(0.183)</td>
<td>(0.041)</td>
<td>(0.078)</td>
<td>0.009</td>
<td>0.804</td>
</tr>
</tbody>
</table>
Table 6: Option pricing performance across moneyness and maturities in-sample.

Panel A: price root mean square errors of VGSV1 across moneyness and maturity of out-of-the-money options on the S&P 500 index. Panel B: ratio of root mean square errors of PSV1 to VGSV1. Panel C: ratio of root mean square errors of FVGSV1 to VGSV1. Panel D: ratio of root mean square errors of FNTSSV2 to VGSV1. Maturity T is in days. Moneyness is K/S, where K and S are the strike and underlying price, respectively. Model parameters are from Tables 2, 3 and 5. In-sample period is from January 1, 1996 to January 16, 2013.

<table>
<thead>
<tr>
<th>Panel A: VGSV1</th>
<th>T &lt; 30</th>
<th>30 ≤ T &lt; 80</th>
<th>80 ≤ T &lt; 180</th>
<th>180 ≤ T &lt; 250</th>
<th>250 ≤ T</th>
</tr>
</thead>
<tbody>
<tr>
<td>K/S &lt; 0.95</td>
<td>0.004</td>
<td>0.008</td>
<td>0.014</td>
<td>0.013</td>
<td>0.018</td>
</tr>
<tr>
<td>0.95 ≤ K/S &lt; 0.975</td>
<td>0.004</td>
<td>0.007</td>
<td>0.014</td>
<td>0.019</td>
<td>0.017</td>
</tr>
<tr>
<td>0.975 ≤ K/S &lt; 1</td>
<td>0.004</td>
<td>0.007</td>
<td>0.013</td>
<td>0.019</td>
<td>0.017</td>
</tr>
<tr>
<td>1 ≤ K/S &lt; 1.025</td>
<td>0.004</td>
<td>0.007</td>
<td>0.011</td>
<td>0.015</td>
<td>0.017</td>
</tr>
<tr>
<td>1.025 ≤ K/S &lt; 1.05</td>
<td>0.003</td>
<td>0.008</td>
<td>0.011</td>
<td>0.016</td>
<td>0.018</td>
</tr>
<tr>
<td>1.05 ≤ K/S &lt; 1.10</td>
<td>0.004</td>
<td>0.008</td>
<td>0.012</td>
<td>0.014</td>
<td>0.019</td>
</tr>
<tr>
<td>1.10 ≤ K/S</td>
<td>0.001</td>
<td>0.008</td>
<td>0.015</td>
<td>0.014</td>
<td>0.019</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: PSV1/VGSV1</th>
<th>T &lt; 30</th>
<th>30 ≤ T &lt; 80</th>
<th>80 ≤ T &lt; 180</th>
<th>180 ≤ T &lt; 250</th>
<th>250 ≤ T</th>
</tr>
</thead>
<tbody>
<tr>
<td>K/S &lt; 0.95</td>
<td>1.104</td>
<td>1.109</td>
<td>1.110</td>
<td>1.089</td>
<td>0.990</td>
</tr>
<tr>
<td>0.95 ≤ K/S &lt; 0.975</td>
<td>1.099</td>
<td>1.089</td>
<td>1.084</td>
<td>1.063</td>
<td>1.008</td>
</tr>
<tr>
<td>0.975 ≤ K/S &lt; 1</td>
<td>1.073</td>
<td>1.085</td>
<td>1.068</td>
<td>1.060</td>
<td>1.025</td>
</tr>
<tr>
<td>1 ≤ K/S &lt; 1.025</td>
<td>1.058</td>
<td>1.096</td>
<td>1.060</td>
<td>1.029</td>
<td>1.014</td>
</tr>
<tr>
<td>1.025 ≤ K/S &lt; 1.05</td>
<td>1.086</td>
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<td>1.077</td>
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<tr>
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<td>0.791</td>
<td>0.798</td>
<td>0.803</td>
<td>0.815</td>
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<td>0.782</td>
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<td>0.976</td>
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<td>0.975 ≤ K/S &lt; 1</td>
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<td>0.762</td>
<td>0.788</td>
<td>0.781</td>
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Table 7: Option pricing performance across moneyness and maturities out-of-sample.

Price root mean square errors of selected models across moneyness and maturity. See the notes to Table 6 for model, maturity and moneyness definitions. The out-of-sample period is from January 23, 2013 until August 26, 2015.

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<td>$180 \leq T &lt; 250$</td>
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<td>0.004</td>
<td>0.008</td>
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<td>0.009</td>
<td>0.012</td>
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<td>0.004</td>
<td>0.008</td>
<td>0.009</td>
<td>0.010</td>
</tr>
<tr>
<td>$1 \leq K/S &lt; 1.025$</td>
<td>0.001</td>
<td>0.004</td>
<td>0.007</td>
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<td>0.011</td>
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<tr>
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<td>0.004</td>
<td>0.008</td>
<td>0.008</td>
<td>0.012</td>
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<td>$K/S &lt; 0.95$</td>
<td>1.090</td>
<td>1.108</td>
<td>1.116</td>
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<td>$180 \leq T &lt; 250$</td>
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<tr>
<td>$K/S &lt; 0.95$</td>
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<td>0.973</td>
<td>0.980</td>
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<td>$0.95 \leq K/S &lt; 0.975$</td>
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<tr>
<td>$0.975 \leq K/S &lt; 1$</td>
<td>0.950</td>
<td>0.941</td>
<td>0.932</td>
<td>0.945</td>
<td>0.981</td>
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<td>0.915</td>
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<td>$1.025 \leq K/S &lt; 1.05$</td>
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<td>$180 \leq T &lt; 250$</td>
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<td>$K/S &lt; 0.95$</td>
<td>0.802</td>
<td>0.811</td>
<td>0.805</td>
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<td>0.795</td>
<td>1.009</td>
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<td>$0.975 \leq K/S &lt; 1$</td>
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Figure 1: Simulated daily log-returns in calendar time and business time. Time-changed Lévy model is the Variance Gamma model (Madan and Seneta, 1990) with Heston process as business time rate.

Figure 2: Implied volatilities of 30-day SPX options on August 31, 2015. Superimposed are the model-based implied volatilities of a Variance Gamma model (Madan and Seneta, 1990) with no leverage ($\beta = \gamma = 0$), and a modified Variance Gamma model with leverage as in (2).
Figure 3: Out-of-sample, normal probability plots for the normalized returns for different Lévy models (2), and benchmark GARCH models. Out-of-sample data are daily S&P 500 log-returns over 1984–2015. Lévy model labels are described at the end of Section 2.2. The diagonal lines are the theoretical quantiles conditional upon correct specification, whereas the data (+) are the empirical quantiles.
A  The Leverage Effect

To facilitate readability of lengthy expressions we use a light notation, omitting subscripts whenever possible. The SV2 family of models in (7) is defined by the system of stochastic differential equations

\[ dY_t = y_t \, dt \]
\[ dy_t = \kappa_y (m_t - y_t) \, dt + \sigma_y \sqrt{y_t} \, dW_t^y \]
\[ dm_t = \kappa_m (1 - m_t) \, dt + \sigma_m \sqrt{m_t} \, dW_t^m \]

where \( W^y \) and \( W^m \) are independent Brownian motions.

Define the instantaneous variance of the return process by

\[ v_t \equiv \lim_{\Delta \to 0} \frac{\text{Var}[X_{t+\Delta} - X_t | \mathcal{F}_t]}{\Delta} \]

where \( \mathcal{F}_t \) is the time-\( t \) information set. Irrespective of the model used for the process \( y_t \) in (4) we have that

\[ \text{Var}[X_{t+\Delta} - X_t | \mathcal{F}_t] = (\beta + \gamma)^2 \text{Var}[Y_{t+\Delta} - Y_t | \mathcal{F}_t] + (\sigma^2 + \gamma^2 \text{Var}[s_1]) \, \text{Var}[Y_{t+\Delta} - Y_t | \mathcal{F}_t] \]

and it immediately follows that

\[ v_t = (\sigma^2 + \gamma^2 \text{Var}[s_1]) \, y_t. \]

To prove this relation we start by observing that

\[ \text{Var}[X_{t+\Delta} - X_t | \mathcal{F}_t] = \beta^2 \text{Var}[Y_{t+\Delta} - Y_t | \mathcal{F}_t] + \gamma^2 \text{Var}[s_{Y_{t+\Delta}} - s_{Y_t} | \mathcal{F}_t] \]
\[ + \sigma^2 \text{Var}[W(s_{Y_{t+\Delta}}) - W(s_{Y_t}) | \mathcal{F}_t] \]
\[ + 2\sigma \beta \text{Cov}[Y_{t+\Delta} - Y_t, W(s_{Y_{t+\Delta}}) - W(s_{Y_t}) | \mathcal{F}_t] \]
\[ + 2\gamma \beta \text{Cov}[Y_{t+\Delta} - Y_t, s_{Y_{t+\Delta}} - s_{Y_t} | \mathcal{F}_t] \]
\[ + 2\gamma \sigma \text{Cov}[s_{Y_{t+\Delta}} - s_{Y_t}, W(s_{Y_{t+\Delta}}) - W(s_{Y_t}) | \mathcal{F}_t]. \]

Then, using the properties of the Lévy process \( s_t \) and the fact that \( W_t \) is a Brownian motion we deduce that

\[ \text{Var}[s_{Y_{t+\Delta}} - s_{Y_t} | \mathcal{F}_t] = \text{Var}[Y_{t+\Delta} - Y_t | \mathcal{F}_t] + \text{Var}[s_1] \, \text{Var}[Y_{t+\Delta} - Y_t | \mathcal{F}_t] \]
\[ \text{Var}[W(s_{Y_{t+\Delta}}) - W(s_{Y_t}) | \mathcal{F}_t] = \mathbb{E}[s_{Y_{t+\Delta}} - s_{Y_t} | \mathcal{F}_t] = \mathbb{E}[Y_{t+\Delta} - Y_t | \mathcal{F}_t] \]
\[ \text{Cov}[Y_{t+\Delta} - Y_t, W(s_{Y_{t+\Delta}}) - W(s_{Y_t}) | \mathcal{F}_t] = 0 \]
\[ \text{Cov}[Y_{t+\Delta} - Y_t, s_{Y_{t+\Delta}} - s_{Y_t} | \mathcal{F}_t] = \text{Var}[Y_{t+\Delta} - Y_t | \mathcal{F}_t] \]
\[ \text{Cov}[s_{Y_{t+\Delta}} - s_{Y_t}, W(s_{Y_{t+\Delta}}) - W(s_{Y_t}) | \mathcal{F}_t] = 0 \]

and the result follows.

All expected values, variances and covariances below are conditional on time-0 information set. For simplicity we omit such a dependence.

To compute the leverage effect implied by the model we need to calculate

\[ \text{Cov}[v_t, X_t] = (\sigma^2 + \gamma^2 \text{Var}[s_1]) \, \text{Cov}[y_t, X_t] \]
\[ = (\beta + \gamma) \left( \sigma^2 + \gamma^2 \text{Var}[s_1] \right) \, \text{Cov}[y_t, Y_t] \]
where the second equality follows from the fact that
\[
\mathbb{E}[y_t X_t] = \mathbb{E}[\beta y_t Y_t + \gamma y_t s Y_t + \sigma y_t W(s Y_t)] \\
= \mathbb{E}[\beta y_t Y_t + \gamma y_t Y_t] = (\beta + \gamma)\mathbb{E}[y_t Y_t]
\]

\[
\mathbb{E}[y_t]\mathbb{E}[X_t] = \mathbb{E}[y_t]\mathbb{E}[\beta Y_t + \gamma s Y_t + \sigma W(s Y_t)] = \mathbb{E}[y_t]\mathbb{E}[\beta Y_t + \gamma Y_t] = (\beta + \gamma)\mathbb{E}[y_t]\mathbb{E}[Y_t].
\]

Since
\[
\text{Cov}[y_t, Y_t] = \mathbb{E}\left[y_t \int_0^t u du\right] - \mathbb{E}[y_t] \mathbb{E}\left[\int_0^t u du\right]
\]
we have that
\[
\text{Cov}[v_t, X_t] = (\beta + \gamma) \left(\sigma^2 + \gamma^2 \mathbb{V}[s_t]\right) \int_0^t \text{Cov}[y_t, y_t] du
\]
and it follows that, irrespective of the model that is used for \(y_t\), the leverage effect is determined by the sign of
\[
(\beta + \gamma)\text{Cov}[y_t, y_t].
\]

In the SV1 model we have that
\[
(\beta + \gamma)\text{Cov}[y_t, y_t] = (\beta + \gamma)e^{-\kappa_y(t-u)}\mathbb{V}[y_u]
\]
so that the leverage effect is entirely determined by the sign of \(\beta + \gamma\).

In the SV2 model the situation is slightly more complex. Using the fact that
\[
y_t = e^{-\kappa_y(t-u)} y_u + \kappa_y \int_u^t e^{-\kappa_y(t-s)} m_s ds + \int_u^t e^{-\kappa_y(t-s)} \sigma y \sqrt{s} dW_s
\]
it can be shown that
\[
\text{Cov}[y_u, y_t] = e^{-\kappa_y(t-u)} \mathbb{V}[y_u] + \kappa_y \int_u^t e^{-\kappa_y(t-s)} \text{Cov}[y_u, m_s] ds
\]
and a further calculation gives
\[
\text{Cov}[y_u, m_s] = \text{Cov}[y_u, \mathbb{E}[m_s | F_u]]
\]
\[
= \text{Cov}\left[y_u, e^{-\kappa_m(s-u)} m_u + (1 - e^{-\kappa_m(s-u)})\right]
\]
\[
= e^{-\kappa_m(s-u)} \text{Cov}[y_u, m_u]
\]
for all \(u \leq s\) so that the same result as in SV1 will hold provided that \(\text{Cov}[y_u, m_u] \geq 0\). The specification of the model implies that
\[
y_u - \mathbb{E}[y_u] = e^{-\kappa_y u} (M^y_u + A_u - \mathbb{E}[A_u])
\]
\[
m_u - \mathbb{E}[m_u] = e^{-\kappa_m u} M^m_u = m_u - e^{-\kappa_m u} m_0 - \kappa_m \int_0^u e^{-\kappa_m(u-s)} ds
\]
where
\[
A_u = \kappa_y \int_0^u e^{\kappa_x} m_x dx
\]
and the processes \((M^y_u, M^m_u)\) are orthogonal martingales with initial value zero. Using these
expressions to compute the covariance we obtain that

\[
\text{Cov}[y_u, m_u] = \mathbb{E} \left[ (y_u - \mathbb{E}[y_u]) (m_u - \mathbb{E}[m_u]) \right] \\
= \mathbb{E} \left[ e^{-(\kappa_y + \kappa_m)u} M^m_u \left( M^y_u + A_u - \mathbb{E}[A_u] \right) \right] \\
= e^{-\kappa_y u} \mathbb{E} \left[ e^{-\kappa_m u} M^m_u A_u \right] = e^{-\kappa_y u} \mathbb{E} [A_u (m_u - \mathbb{E}[m_u])] \\
= \kappa_y \mathbb{E} \left[ \int_0^u e^{-\kappa_y (u-x)} m_x (m_u - \mathbb{E}[m_u]) \, dx \right] \\
= \kappa_y \mathbb{E} \left[ \int_0^u e^{-\kappa_y (u-x)} (m_x - \mathbb{E}[m_x]) (m_u - \mathbb{E}[m_u]) \, dx \right] \\
= \kappa_y \int_0^u e^{-\kappa_y (u-x)} \text{Cov}[m_x, m_u] \, dx = \kappa_y \int_0^u e^{-(\kappa_y + \kappa_m)(u-x)} \mathbb{V}[m_x] \, dx
\]

where the last equality follows from the fact that

\[
\text{Cov}[m_x, m_u] = \text{Cov}[m_x, \mathbb{E}[m_u | \mathcal{F}_x]] \\
= \text{Cov} \left[ m_x, e^{-\kappa_m (u-x)} m_x + \left( 1 - e^{-\kappa_m (u-x)} \right) \right] \\
= e^{-\kappa_m (u-x)} \text{Cov} \left[ m_x, m_x \right] = e^{-\kappa_m (u-x)} \mathbb{V}[m_x]
\]

for all \( x \leq u \).

**B Series Solution for the Characteristic Function**

We use the light notation in Appendix A, and write the SV2 model as

\[
\begin{align*}
\frac{dy_t}{\sqrt{y_t}} &= \kappa_y \left( \theta_{y,c} + \theta_{y,d} m_t - y_t \right) \, dt + \sigma_y \sqrt{y_t} \, dW^y_t \\
\frac{dm_t}{m_t} &= \kappa_m \left( \theta_m - m_t \right) \, dt + \sigma_m \sqrt{m_t} \, dW^m_t.
\end{align*}
\]

The specification above encompasses both \( \mathbb{P} \)- and \( \mathbb{Q} \)-dynamics. For example \( \theta_{y,c} = 0 \) and \( \theta_{y,d} = \theta_m = 1 \) under \( \mathbb{P} \). The parameter specification under \( \mathbb{Q} \) is in (26).

The characteristic function of the time change \((T - t) \mapsto \int_t^T y_s \, ds\) is defined by

\[
f(t, y_t, m_t; u) = \mathbb{E} \left[ e^{u \int_t^T y_s \, ds} \right] = e^{A(T-t;u) + B(T-t;u) y_t + C(T-t;u) m_t}
\]

for a given \( u \in \mathbb{C} \) and some unknown functions \( A, B \) and \( C \) such that

\[
A(0;u) = B(0;u) = C(0;u) = 0.
\] (27)

Combining the fact that the process

\[
e^u \int_0^t y_s \, ds f(t, y_t, m_t; u)
\]

is a martingale with a standard separation of variables argument shows that these functions must solve the system of ordinary differential equations given by

\[
\begin{align*}
A'(\tau; u) &= \kappa_y \theta_{y,c} B(\tau; u) + \kappa_m \theta_m C(\tau; u) \\
B'(\tau; u) &= u - \kappa_y B(\tau; u) + \frac{1}{2} \sigma_y^2 B(\tau; u)^2 \\
C'(\tau; u) &= \kappa_y \theta_{y,d} B(\tau; u) - \kappa_m C(\tau; u) + \frac{1}{2} \sigma_m^2 C(\tau; u)^2
\end{align*}
\]

subject to (27). With the exception of the function \( B(\tau; u) \), which actually solves an autonomous equation, it is not possible to derive an explicit solution to this system. Instead
we construct a power series solution by postulating that
\[ H(\tau; u) = \sum_{k=0}^{\infty} h_k(u) \tau^k, \quad (h, H) \in \{(a, A), (b, B), (c, C)\}. \]

To determine the sequence \((a_k(u), b_k(u), c_k(u))_{k=0}^{\infty}\) of unknown coefficients we start by observing that the boundary condition (27) implies
\[ a_0(u) = b_0(u) = c_0(u) = 0, \quad u \in \mathbb{C}. \]

Substituting the conjectured series solution into the system of differential equations, using Cauchy’s product formula
\[ \left( \sum_{k=0}^{\infty} h_k(u) \tau^k \right)^2 = \sum_{k=0}^{\infty} \left( \sum_{\ell=0}^{k} h_\ell(u) h_{k-\ell}(u) \right) \tau^k \]
to compute the squares, and matching terms shows that the unknown coefficients can be computed recursively as follows:
\[ a_1(u) = u - b_1(u) = c_1(u) = 0 \]
\[ b_{k+1}(u) = \frac{1}{1 + k} \left[ -\kappa_y b_k(u) + \frac{1}{2} \sigma_y^2 \sum_{\ell=0}^{k} b_\ell(u) b_{k-\ell}(u) \right] \]
\[ c_{k+1}(u) = \frac{1}{1 + k} \left[ \kappa_y \theta_y d b_k(u) - \kappa_m c_k(u) + \frac{1}{2} \sigma_m^2 \sum_{\ell=0}^{k} c_\ell(u) c_{k-\ell}(u) \right] \]
and
\[ a_{k+1}(u) = \frac{1}{1 + k} \left[ \kappa_y \theta_{y,c} b_k(u) + \kappa_m \theta_m c_k(u) \right]. \]

We confirm the accuracy of our series solution in two ways. First, the function \(B(\tau; u)\) solves an autonomous differential equation, which has an analytic solution, and the equation is the same for both SV1 and SV2 models. Second, the characteristic function of SV1 models has an analytic solution. For the range of parameter values that we estimate or calibrate, the analytic solution and the series solution are virtually the same when the number of terms in the series solution is at least three and the time horizon \(\tau\) is less than one year. To be on the safe side, we implemented the series solution using five terms. We also experimented with 10 and 15 terms, and results were unchanged.

C Fourier Inversion
This section presents the method we use to recover the probability densities of time-changed Lévy models from their characteristic functions. To achieve high accuracy and overcome the so-called Gibbs phenomenon, we enrich the COS Method (Fang and Oosterlee, 2008) with an exponential damping.

The COS method is...
References


