Abstract

Time-changed Lévy processes, which consist of a Lévy process that runs on a stochastic time, can effectively capture the main empirical regularities of asset returns such as jumps, stochastic volatility, and leverage. However, neither their transitional density nor characteristic function is available in closed-form when the Lévy process and time-change are not independent. For a given solution to a linear parabolic partial differential equation, we derive a closed-form expression for the transitional density, moment generating function, and characteristic function of the time-changed Lévy processes, when the Lévy and time-change process are correlated. Our approach can be applied to virtually all models in the option pricing literature.
Correlated time-changed Lévy processes*

Hasan Fallahgoul†  Kihun Nam‡

August 28, 2018

Abstract

Time-changed Lévy processes, which consist of a Lévy process that runs on a stochastic time, can effectively capture the main empirical regularities of asset returns such as jumps, stochastic volatility, and leverage. However, neither their transitional density nor characteristic function is available in closed-form when the Lévy process and time-change are not independent. For a given solution to a linear parabolic partial differential equation, we derive a closed-form expression for the transitional density, moment generating function, and characteristic function of the time-changed Lévy processes, when the Lévy and time-change process are correlated. Our approach can be applied to virtually all models in the option pricing literature.

Keywords: Lévy jumps, time changes, business time.

JEL classification: G10, G12, G13

---

*We thank Julien Hugonnier for motivating this paper and for fruitful discussions. We also thank Yan Dolinsky, Loriano Mancini, and Stoyan Stoyanov for their comments. The Centre for Quantitative Finance and Investment Strategies has been supported by BNP Paribas.

†Hasan A. Fallahgoul, Monash University, School of Mathematics and Centre for Quantitative Finance and Investment Strategies, 9 Rainforest Walk, 3800 Victoria, Australia. E-mail: hasan.fallahgoul@monash.edu.

‡Kihun Nam, Monash University, School of Mathematics and Centre for Quantitative Finance and Investment Strategies, 9 Rainforest Walk, 3800 Victoria, Australia. E-mail: kihun.nam@monash.edu.
1 Introduction

Time-changed Lévy (TCL) processes consist of a Lévy process that runs on a stochastic time. TCL processes have been used as a flexible class of models for modeling tail risk of asset returns and option pricing. They can effectively capture main empirical regularities of asset returns such as jumps, stochastic volatility, and leverage. The leverage effect refers to the correlation between an asset return and its changes of volatility. Carr and Wu (2004) have developed a novel methodology for capturing the leverage effect via a TCL process. The leverage effect in their methodology is estimated by the correlation between the Lévy and activity rate processes.

A Lévy process $X = (X_t)_{t \geq 0}$ is characterized by its characteristic function
\[
\mathbb{E}[\exp(iuX_t)] = \exp(t \Psi_{X_t}(u))
\] (1)
where $i = \sqrt{-1}$, $u$ is real, and $\Psi_{X_t}(u)$ is the cumulant exponent of $X_t$. In order to capture mean-reversion and stochastic volatility regularities of asset returns, one might change the calendar time, i.e., $t$, in (1) with a nondecreasing right-continuous process $T = (T_t)_{t \geq 0}$ with left-limits. The $T$ and $X_T$ are called an activity rate process (or a business time) and a TCL process, respectively. The transitional density for a TCL process is not available in closed-form in general. Instead, the characteristic functions of TCL processes have been used for empirical investigations. If the Lévy process $X$ and the activity rate process $T$ are independent, the characteristic function of the TCL process can be expressed by the Laplace transform of $T_t$ by conditioning since the cumulant exponent of $X$ is linear respect to the calendar time. However, when $X$ and $T$ are dependent, it is not clear how one can compute the characteristic function of the TCL process. Carr and Wu (2004) compute the characteristic function of the TCL process when there is a correlation between the Lévy and activity rate process, by using the Optional Stopping Theorem and a complex-valued measure change.

In their calculation, they assume $T_t$ is a stopping time for each $t$. They use an affine process, i.e., Cox-Ingersoll-Ross, for the instantaneous activity rate (the time-derivative of the activity rate process $T$). However, $T_t$ is not a stopping time under its natural filtration.

---


2 Intuitively, a natural estimate for the leverage effect is in using empirical correlations between the daily returns and the changes of daily volatility. However, Ait-Sahalia et al. (2013) show that such natural estimate yields nearly zero correlation, which contradicts many economic reasons for expecting the estimated correlation to be negative.

The condition $T_t$ being a stopping time for every $t$ is extremely restrictive in applications. For example, if the instantaneous activity rate is given by a stochastic differential equation (SDE) as in Section 4 of Carr and Wu (2004) or in Section 2 of Huang and Wu (2005), there is no guarantee for $T_t$ to be a stopping time under a given filtration. One possibility to overcome this problem is to enlarge the filtration so that $T_t$ becomes a stopping time for each $t$. However, the increment of $X$ may no longer be independent under this enlarged filtration since $X$ and $T$ are not independent. Hence, $X$ cannot be a Lévy process anymore. On the other hand, if one can extend their result to the case where $T$ is just an adapted process, the model becomes flexible enough to incorporate virtually all proposed models in option pricing literature. In this case, all specified option pricing models in Huang and Wu (2005) are nested into this setting.

In this article, for an adapted process $T$, we provide rigorous mathematical results to Carr and Wu (2004) and Huang and Wu (2005). By conditioning and applying tower property iteratively, we obtain the probability density function, the moment generating function (Laplace transform), and the characteristic function (Fourier transform) of a TCL process. Then, the results are applied to stochastic volatility models in Huang and Wu (2005). For the sake of mathematical simplicity, we examine the case where the time-derivative of $T$ is continuous. Our method works equally well for the case where $T$ has jumps as shown in Remark 3.4.

Since we give up the stopping time property of the time-change, we cannot expect a simple formula as in Carr and Wu (2004). On the contrary, we provide a formula involving a solution of a PDE. In particular, if the instantaneous activity rate follows a linear SDE, then it is known that a closed-form solution of the corresponding PDE exists. Even though the PDE may not have a closed-form solution, it can be easily solved numerically since it is one-dimensional nondegenerate linear parabolic PDE. In addition, since the solution of the PDE represents for the joint probability density of $(T_t, B_t)_{t \geq 0}$, where the instantaneous activity rate is a solution of SDE driven by the Brownian motion $B$, we can additionally improve the efficiency of numerical calculation using the information of marginal distribution of $T_t$ and $B_t$.

We also obtain an explicit measure change via Girsanov transform which enables us to calculate characteristic function of TCL processes as $T$ and the Lévy process are independent under this new measure (see Theorem 5.1 and remarks therein). In principle, one can transform the measure so that two stochastic processes become independent, but finding the explicit measure change is not trivial in general. Our result is in parallel to Theorem 1 of Carr and Wu (2004): when $T_t$’s are stopping times, under complex measure obtained by complex Girsanov transform, one can regard $T$ and $X$ independent in the calculation of the characteristic function for TCL process. Note that, in addition to their requirement for $T_t$
being stopping time for each $t$, the extension of Girsanov transform to the complex plane is not so trivial.\(^4\) In this article, we achieve this result with a real measure change, without assuming the stopping time property for $T_t$.

The structure of the article is as follows. In Section 2, we examine the result in Carr and Wu (2004) and Huang and Wu (2005). Then, the basic settings of our model are established in Section 3. In Section 4, we provide formulas for the probability density function of a TCL process. In Section 5, the Laplace transform (and the characteristic function) of a TCL process is given. In Section 6, we generalize the framework of Huang and Wu (2005) under our setting. Finally, we conclude the article with a summary and future research questions in Section 7. All proofs and calculations are given in Appendices.

2 Motivation

In Carr and Wu (2004), they assumed the followings in Section 2:

- The probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a standard complete filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.
- $X$ is a Lévy process which is adapted to $\mathbb{F}$ with $\mathbb{E}e^{i\theta X_t} = e^{-t\Psi_X(\theta)}$.
- $T$ is a nondecreasing right-continuous and left-limit process such that, for any given $t$, $T_t$ is a stopping time with respect to $\mathbb{F}$, $T_t$ is finite almost surely, $\mathbb{E}[T_t] = t$. In addition, we assume $T_t \xrightarrow{t \to \infty} \infty$ almost surely.

Under these assumptions, Carr and Wu (2004) define $Z_t(\theta) := \exp \left( i\theta^T X_t + t\Psi_X(\theta) \right)$, $Y_t := X_{T_t}$ and they proposed the following lemma and theorem.

**Lemma 2.1** (Lemma 1 of Carr and Wu (2004)). For every $\theta \in \mathcal{D}$, $M_t(\theta) := Z_{T_t}(\theta)$ is a complex-valued $\mathbb{P}$-martingale with respect to $(\mathcal{F}_t^{Y,T})_{t \geq 0}$, the filtration generated by the process $\{(Y_t, T_t) : t \geq 0\}$.

**Theorem 2.2** (Theorem 1 of Carr and Wu (2004)). Let $\mathbb{E}$ and $\mathbb{E}^\theta$ be expectations under measure $\mathbb{P}$ and $\mathbb{Q}(\theta)$, respectively, where $\frac{d\mathbb{Q}(\theta)}{d\mathbb{P}} _{\mid t} := M_t(\theta)$. Then we have the following formula:

$$\mathbb{E}[e^{i\theta^T Y_t}] = \mathbb{E}^\theta [e^{-T_t\Psi_X(\theta)}].$$

Above theorem implies that one may calculate characteristic function of $Y$, regarding $T$ and $X$ are independent under the new complex measure. These results are the foundation of their paper and it has been widely used in vast literature such as Huang and Wu (2005).

\(^4\)For example, Jeanblanc et al. (2009) point it out in Warning 1.7.3.2.
Their proof is based on three steps: (i) prove that $Z(\theta)$ is a martingale under $\mathbb{F}$; (ii) apply Optional Stopping Theorem for stopping time $T$ to $Z(\theta)$ to conclude $M$ is a martingale with respect to $(\mathcal{F}^Y_T, s \geq 0)$, and; (iii) prove Theorem 1 using the fact that $M(\theta)$ is a density process.

They assume that $T_t$ is a stopping time for each $t$. Unfortunately, such assumption imposes very restrictive conditions on the stochastic model for $T$. Assume that $\{T_u\}_{u \geq 0}$ are stopping times. By definition, $\{\omega \in \Omega : T_u(\omega) \leq t\} \in \mathcal{F}_t$ for any $t$ and $u$. Let us fix $t$ and $u$ so that $t \leq u$. Then, for any $0 < \varepsilon \leq s < t$, we have

$$P(s - \varepsilon < T_u \leq s | \mathcal{F}_t) = E[I_{T_u \leq s} - I_{T_u \leq s - \varepsilon} | \mathcal{F}_t] = I_{s - \varepsilon < T_u \leq s}$$

Let $E := \{\omega \in \Omega : T_t(\omega) > t\}$. Since $T_u \geq T_t > t$ on $E$,

$$P(\{s - \varepsilon < T_u \leq s\} \cap E | \mathcal{F}_t) = 0$$

and therefore,

$$P(\{s - \varepsilon < T_u \leq s\} \cap E^c | \mathcal{F}_t) = I_{s - \varepsilon < T_u \leq s}$$

for all $s < t$. In other words, if the event $\{\omega \in \Omega : T_t(\omega) \leq t\} \in \mathcal{F}_t$ happens, then using the information at time $t$, we should be able to find the path of $T$ until it crosses level $t$. This implies our filtration contains the information about the future path of $T$ from time $t$ to the first $t$-level crossing time of $T$. If one removes the stopping time property of $T_t$, then Lemma 1 and Theorem 1 of their paper becomes invalid because the Optional Stopping Theorem breaks down. On the other hand, if one enlarges the underlying filtration $\mathbb{F}$ to make $\{T_t\}_{t \geq 0}$ stopping times, then, since $X$ and $T$ are not independent, the increment of $X$ may be no longer independent under this enlargement. This breaks the Lévy property of $X$. It is not easy, if not impossible, to find appropriate $X$ and the underlying filtration $\mathbb{F}$ for $T$ in Table 2 of Section 4 of the Carr and Wu (2004) such that $X$ is a Lévy process and $T_t$'s are stopping time simultaneously.

Another limitation of their article is that, even though they got a formula which is $E[e^{i\theta Y_t}] = E[e^{-T_t \Psi X(\theta)}]$, the actual calculation of the left-hand side has not been improved since one needs to find $M_t(\theta)$ which consists of $Y_t$ and $T_t$. Note that the calculation of the original expression only requires the information about the distribution of $Y_t$.

Carr and Wu (2004) do not discuss the empirical performance of the proposed generalized Fourier transform and they leave it for future research, e.g., Huang and Wu (2005). Huang and Wu (2005) investigate the specifications of different option pricing models under TCL processes. In their specifications, the jump components, the source of stochastic volatility, and the dynamics of the volatility processes are varied. Although, they derive the characteristic function for all models via generalized Fourier transform, however, none of the stochastic models they consider for stochastic volatility is a stopping time. Even if we ignore this fact that those stochastic volatility models are not a stopping time process, still one
cannot derive a closed-form formula for some specifications.  

3 Time-changed Lévy processes

We use the TCL process (2) to model the uncertainty of the economy. Denote the price of an asset such as stock or a currency at time $t$ by $S_t$. Then, the price process is given by an exponential affine function of uncertainty $Y_t$,

$$S_t = S_0 e^{\theta Y_t}$$

where $S_0$ denotes the price at time 0, which is known and fixed. We wish to write a Lévy process in the form of

$$X_t = \alpha J_t + \beta Z_{J_t}$$

(3)

for a nondecreasing right-continuous process with left-limits given by $J$ and a Brownian motion $Z$ which is independent of $J$. The process $J$ is called a subordinator. Many well-known Lévy processes can be obtained from equation (3). For example, we can recover the variance-gamma (Madan and Seneta (1990)), normal-inverse Gaussian (Barndorff-Nielsen (1997)), and CGMY (Carr et al. (2003)) process by assuming $J_t$ is a Gamma, inverse Gaussian, and tempered stable process, respectively.

Remark 3.1. The Lévy process $X$ is not the TCL process defined in Carr and Wu (2004), i.e., $Y$ in (2), although it is obtained by a time-change. The time-change in (3) is for capturing jump regularity of the log returns, while the time-change in Carr and Wu (2004) is for capturing mean-reversion and stochastic volatility of the log return.

Definition 3.2. A time-changed Lévy (TCL) process for a Lévy process $X$ in (3) is given by (2), where $T$ is a nondecreasing right-continuous process with left-limits.

In general, the random time-change (or stochastic time) $T$ can be modeled as a nondecreasing semimartingale. For simplicity, let us suppress jumps and assume absolute continuity in time. In other words,

$$T_t = \alpha_t + \int_0^t v_s \, ds$$

where $v$, a right-continuous process with left-limits, is the instantaneous activity rate. The random variables $T_t$ and $t$ are called the business and calendar time, respectively.

The transitional density of process $Y$ in (2) is not available in closed-form. However, if the business time $T_t$ is independent of $X_t$, the characteristic function of (2) is just the

---

5Detailed information can be found in Section E of Huang and Wu (2005).
6Detailed information about a subordinator process can be found in Cont and Peter (2004).
7The related subordinator for other processes of Table 1 of Carr and Wu (2004) can be obtained by using the same technique as Madan and Yor (2006).
Laplace transform of $T_t$ evaluated at the cumulant exponent $\Psi_X(u)$ of $X$. Therefore, it has a closed-form expression if the cumulant exponent of $X$ and the Laplace transform for $T_t$ are both available in closed-form.

When the Lévy process $X$ and stochastic time process $T$ are not independent, it is not obvious how one can compute the characteristic function of $Y_t$ unless $T_t$’s are stopping time for all $t$. However, as we pointed out, the condition for $T_t$ being a stopping time for every $t$ is too restrictive in applications. Our primary objective is to find formulas, as explicitly as possible, for the probability density function, the characteristic function, and the moment generating function of $Y_t$, when the Lévy process $X$ and business time $T_t$ are not independent and $T_t$ be just a random time adapted to the filtration.

To be more specific, we assume the following conditions in this article. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a standard filtration $\mathcal{F}$, two independent Brownian motions $B$ and $W$, and an adapted nondecreasing Lévy process $J$ that is independent to these Brownian motions. For $\rho \in [-1, 1]$, let $Z_t := \rho B_t + \sqrt{1 - \rho^2} W_t$. For $\mu, \sigma : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$, consider the processes $v_t$ and $T_t$ given by the following SDE:

$$
\begin{align*}
geq d v_t &= \mu(t, v_t) dt + \sigma(t, v_t) dB_t, & v_0 = 1, \\
geq d T_t &= v_t dt, & T_0 = 0.
\end{align*}
$$

Here, we assume appropriate conditions for $\mu$ and $\sigma$ to guarantee the existence of a unique strong solution such that $v$ is nonnegative, $\mathbb{E} T_t = t$, and $T_t \xrightarrow{t \to \infty} \infty$ a.s. Since $(v, T)$ is Markov, let us define $\Phi : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{C}$ to be the function satisfying

$$
\mathbb{E} \left[ e^{i \xi T_t} | \mathcal{F}_s \right] = \exp(\Phi(t, s, \xi, v_s, T_s)), \quad s \leq t.
$$

Remark 3.3. Under our assumptions, the $X$ defined in (3) is not adapted to the filtration $\mathcal{F}$ in general. The adaptedness of $X$ to $\mathcal{F}$ is not necessary: even if $X$ is adapted to $\mathcal{F}$, its time-changed process $Y_t = X_{T_t}$ will not be adapted to $\mathcal{F}$ in general. If adaptedness of $Y$ is needed, one needs to consider the filtration generated by $Y$ after its construction.

Remark 3.4. The dynamics of the instantaneous activity rate process $v$ can be generalized to incorporate jumps. For example, we can let $(v, T)$ be the unique solution of

$$
\begin{align*}
geq d v_t &= \mu(t, v_t) dt + \sigma(t, v_t) dB_t + \int_{\mathbb{R}} \zeta(t, x, v_t) N(dt, dx), & v_0 = v_{0-} > 0 \\
geq d T_t &= v_t dt, & T_0 = 0.
\end{align*}
$$

where $N$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with mean $\text{Leb} \otimes \text{Leb}$ that is independent of $J$, and $\zeta : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a jointly measurable function. In this case, we only need to generalize the Fokker-Plank PDE (Chapman-Kolmogorov forward equation), appears in Section 4, 5, and 6, to incorporate jumps.\(^8\) The rest of argument remains the same. For

\(^8\)For example, see Chapter 7 of Hanson (2007)
mathematical simplicity, we will focus on the case where there is no jump in $v$ since we can still capture the major properties of option prices with this simple model.

**Remark 3.5.** The case where $J$ is a pure jump process is of special interest since it is the simplest model that captures extreme events (producing heavy tail property). It is known that if $X$ and $\tilde{X}$ are adapted Lévy processes with $[X, \tilde{X}] = 0$, then $X$ and $\tilde{X}$ are independent.\(^9\) This result implies that if $J$ is a pure jump Lévy process that is adapted to the filtration, then the $J$ is automatically independent with respect to $(B, W, Z)$. Note that, if $\rho \neq 0$, the TCL process $Z_J$ is not independent to $B$ even though $Z_J$ is a pure jump Lévy process while $B$ is a continuous Lévy process. Therefore, when $\rho \neq 0$, there is no filtration that makes $Z_J$ and $B$ adapted simultaneously. Note that, as shown in Remark 3.3, we focus on the filtration such that $B$ is adapted while our $X$ is non-adapted Lévy process.

We end this section with our notation. For any random variable $\xi$, we will use notation $f_\xi$ as a probability density distribution or probability density "function" (PDF), that is,

$$\mathbb{P}(\xi \in A) := \int_A f_\xi(x)dx.$$ 

We will also use $f_{X|Y}$ as the conditional probability density function of $X$ with respect to $Y$. We will also use notation $\int$ for integral on the whole domain unless otherwise stated.

## 4 The distribution of the TCL process

Let us first assume $\alpha = 0$ and $\beta = 1$ in (3) so that $Y_t = Z_{t \tau_t}$. In order to find the distribution of $Z_{t \tau_t}$, one can either try to calculate PDF or an integral transform of it. In this section, we discuss the calculation of the probability density distribution directly.

The PDF of a TCL process can be expressed by the joint distribution of the time-change and Brownian motion that drives the activity rate.

**Proposition 4.1.** Under the assumptions in the previous section, we have the following formula

$$f_{Z_{t \tau_t}}(z) = \iint \frac{1}{\sqrt{2\pi j(1-\rho^2)}} e^{-\frac{(z-b)^2}{2(1-\rho^2)}} f_{B_j, T_t}(b, y) f_{J_y}(j) dy db dj.$$

Therefore, in order to calculate the probability density distribution of $Z_{t \tau_t}$, one needs to calculate the joint distribution function $B_j$ and $T_t$. When $j \leq t$, then $f_{B_j, T_t}(b, y) = \int f_{T_t|B_j, T_j}(y|b, \tilde{y}) f_{B_j, T_j}(b, \tilde{y}) d\tilde{y}$. In this case, using the Markovian property of $(v, T, B)$ we can calculate $f_{T_t|B_j, T_j}$. On the other hand, if $j \geq t$,

$$f_{B_j, T_t}(b, y) = \int f_{B_j|B_t, T_t}(b|z, y) f_{B_t, T_t}(z, y) dz$$

\(^9\)For example, see Theorem 11.43 of He et al. (1992).
Therefore, one only needs to calculate the joint PDF of $B_t$ and $T_t$. Unfortunately, even when $(v, T)$ is an affine process, $(v, T, B)$ is not affine and therefore, a closed-form representation of the characteristic function is not available. Alternatively, we can use the Fokker-Plank PDE. For $t \leq j$, let $q : (t, x, y, z) \rightarrow q(t, x, y, z)$ be the unique solution of the following Fokker-Plank PDE for $(v, T, B)^{10}$:

$$
\frac{\partial}{\partial t} q = \frac{1}{2} \frac{\partial^2}{\partial x^2}[\sigma^2 q] + \frac{1}{2} \frac{\partial^2}{\partial z^2} q - \frac{1}{2} \frac{\partial^2}{\partial x}[\mu q] - x \frac{\partial}{\partial y} q \tag{5}
$$

$q(0, x, y, z) = \delta(y) \delta(z)$. 

**Remark 4.2.** Assume that $\mu(t, x) = \kappa(1 - x)$ and $\sigma(t, x) = \sigma_v \sqrt{x}$. In this case, $v$ follows CIR process and $(v, T)$ forms a two dimensional affine process. Even though no closed-form solution of PDE (5) is known, we know the PDF of $(v_1, T_1 - t, B_t / \sqrt{t})$ converges asymptotically as $t \rightarrow \infty$. Such limit is given by the following elliptic PDE:

$$
\frac{1}{2} \frac{\partial^2}{\partial x^2} \left[ \sigma^2 \tilde{q}(t, x, y, z) \right] + \frac{1}{2} \frac{\partial^2}{\partial z^2} \tilde{q}(t, x, y, z) = 0.
$$

For numerical computation of (5), one may approximate $q(t, x, y, z)$ with $\tilde{q}(t, x, y + t, \sqrt{t}z)$ when $t$ is large enough.

Given the Fokker-Planck PDE (5), one can calculate the joint distribution of the $T_t$ and $B_j$, which is given by the following proposition.

**Proposition 4.3.** Let $q$ be the unique solution of (5). Then,

$$
f_{T_t, B_j}(y, z) = \begin{cases} 
\frac{1}{\sqrt{2\pi(j-t)^2}} \iint e^{-\frac{(s-\bar{x})^2}{2(j-t)}} q(t, x, y, z) dx dz, & t \leq j \\
\frac{1}{2\pi} \iint \int \exp(\Phi(t, j, \xi, \bar{y}) - i\xi \eta) q(j, x, \bar{y}, z) dx dy dz, & t \geq j.
\end{cases}
$$

The Fokker-Planck PDE (5) is three dimensional, but it can be reduced to one-dimensional PDE if we use Fourier transform for variables $y$ and $z$. This dimensional reduction is useful in numerical computation. In other words, the Fourier transform of (5) is given by

$$
\hat{\partial} \hat{q} = \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma^2 \hat{q}] - \frac{1}{2} \frac{\partial^2}{\partial x} [\mu - i\eta \sigma \hat{q}] - \left( \frac{\eta^2}{2} + ix \xi \right) \hat{q}, \tag{6}
$$

$$
\hat{q}(0, x, \xi, \eta) = \delta_{v_0}(x).
$$

Using $\hat{q}$, the PDF of $Z_{J_{t_1}}$ is given by the following theorem.

**Theorem 4.4.** Let $\hat{q}$ be the solution of (6). Then,

$$
f_{Z_{J_{t_1}}}(z) = \frac{1}{(2\pi)^2} \iint \hat{f}_{T_t, B_j}(\xi, \eta) e^{rac{\eta x - (1-\rho^2)\eta^2}{2\sigma^2}} \left( \int e^{i\xi y} f_{J_{t_1}}(j) dy \right) d\xi d\eta dj.
$$

---

$^{10}$The existence of a unique solution is guaranteed by the existence of unique solution for (4).
where
\[
\hat{f}_{T_i,B_j}(\xi, \eta) = \begin{cases} 
  e^{-\frac{(\xi - \eta)^2}{2}} \int \hat{q}(t, x, \xi, \eta) dx, & t \leq j \\
  \frac{1}{2\pi} \iint (\int \exp (\Phi(t, j, -\xi, x, y) + i\bar{\xi}y) dy) \hat{q}(j, x, \xi, \eta) d\bar{\xi}dx, & t \geq j.
\end{cases}
\]

**Remark 4.5.** When \((v, T)\) has a closed-form solution,\(^{11}\) then one can express \(q(t, x, y, z)\) using Brownian bridge from 0 to \(z\). We can define path functional \(\mathcal{V}, \mathcal{W} : C[0, t] \rightarrow \mathbb{R}\) such that \(v_t = \mathcal{V}(B_{[0,t]})\) and \(T_t = \mathcal{W}(B_{[0,t]})\). Then, \(f_{v_t,T_t|B_t}(x, y, z) = f_{\tilde{v}_t,\tilde{T}_t}(x, y)\), where \(\tilde{v}_t = \mathcal{V}(\tilde{B}_{[0,t]}), \tilde{T}_t = \mathcal{W}(\tilde{B}_{[0,t]}), \) and \(\tilde{B}\) is a Brownian bridge from 0 to \(z\) on time \([0, t]\), that is,
\[
\tilde{B}_s = \tilde{B}_s - \frac{s(\tilde{B}_t - z)}{t}
\]
where \(\tilde{B}\) is a Brownian motion. Then,
\[
q(t, x, y, z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} f_{\tilde{v}_t,\tilde{T}_t}(x, y).
\]

For general \(Y\) of the form (3), we can easily extend the previous theorem.

**Theorem 4.6.**
\[
f_{Y_t}(y) = \frac{\sqrt{1 - \rho^2}}{(2\pi)^2\rho} \iint \hat{f}_{T_i,B_j}(\xi, \eta) e^{-\frac{(\xi - \eta)^2}{2\sigma^2}} \left( \int e^{i\xi y f_{J_{y}}(j)} dy \right) d\xi d\eta dj.
\]

where
\[
\hat{f}_{T_i,B_j}(\xi, \eta) = \begin{cases} 
  e^{-\frac{(\xi - \eta)^2}{2}} \int \hat{q}(t, x, \xi, \eta) dx, & t \leq j \\
  \frac{1}{2\pi} \iint (\int \exp (\Phi(t, j, -\xi, x, y) + i\bar{\xi}y) dy) \hat{q}(j, x, \xi, \eta) d\bar{\xi}dx, & t \geq j.
\end{cases}
\]

5 The moment generating function of the TCL process

It is noteworthy that the Laplace transform of \(Z_{J_{T_{t}}}\) can be factored to the characteristic functions of \(W\), the inverse of \(J\) and measure-changed \(T\). Let \(v_0^{j,r} = 1, T_0^{j,r} = 0\) and
\[
dt u^{(j,r)}(u, v^{(j,r)}) = (\mu(u, v^{(j,r)}) - r\rho \sigma(u, v^{(j,r)}) I_{u \leq j}) du + \sigma(u, v^{(j,r)}) dB_u \\
dT u^{(j,r)} = v_u^{(j,r)} du.
\]

Note that \((u^{(j,r)}, T^{(j,r)})\) is the weak solution of (4) under a measure \(Q\) that makes \(B_t - r\rho I_{[0,j]}(t)\) a Brownian motion. We have the following theorem.

**Theorem 5.1.** For \(T^{(j,r)}\) given by SDE (7), we have
\[
\mathbb{E} e^{-r Z_{J_{T_{t}}}} = \frac{1}{2\pi} \iint e^{-x^2/2} \left( \int e^{i\xi y f_{J_{y}}(j)} dy \right) \mathbb{E} \left[ \exp \left( -i\xi T_{T_{t}}^{(j,r)} \right) \right] d\xi dj.
\]

\(^{11}\)For example, assume \(\mu(t, x) = \kappa(1 - x)\) and \(\sigma(t, x) = \sigma_x\). Then, we have a closed-form solution \((v_t, T_t)\), where
\[
v_t = e^{-\kappa t - \sigma_x^2 t/2 + \sigma_x B_t} \left( 1 + \kappa \int_0^t e^{\kappa s + \sigma_x^2 s/2 - \sigma_x B_s} ds \right), \quad T_t = \int_0^t v_u du.
\]
Remark 5.2. Note that $e^{r^2 j/2}$ is the Laplace transform of Brownian motion at time $j$. 

Remark 5.3. For the characteristic function of $Z_{T^t}$, we have the following formula:

$$\mathbb{E}e^{i\theta Z_{T^t}} = \frac{1}{2\pi} \int \int e^{-\theta^2 j/2} \left( \int f_{T^t}(j)e^{i\xi y} dy \right) \mathbb{E} \left[ e^{-i\xi T} \mathcal{E}_t^{\theta,j} \right] d\xi dj$$

where

$$\mathcal{E}_t^{\theta,j} := \exp \left( i\theta \rho B_{j\wedge t} + \frac{\theta^2 \rho^2 (j \wedge t)}{2} \right).$$

Here, note that $e^{-\theta^2 j/2}$ is the characteristic function of Brownian motion at time $j$. Heuristically, $\mathbb{E} \left[ e^{-i\xi T} \mathcal{E}_t^{\theta,j} \right]$ is the expectation of $e^{-i\xi T}$ under the complex measure given by $\mathcal{E}_t^{\theta,j} d\mathbb{P}$.

However, rigorous validation of such statement is tricky (see Warning 1.7.3.2 of Jeanblanc et al. (2009)). The actual derivation is provided in Appendix.

Theorem 5.1 implies that one can transform the dependency of $Z$ and $T$ in terms of measure change in $T$: note that the distribution of $T$ under $\mathbb{P}$ is the same as the distribution of $T_{(j,r)}$ under $\mathbb{Q}$ and $T_{(j,r)} = T$ if $\rho = 0$. In this sense, Theorem 5.1 is analogous to Theorem 1 of Carr and Wu (2004) which finds a (complex) measure so that one can calculate the characteristic function of TCL as the time-change and Lévy process are independent.

Assuming that $J$ is strictly increasing, there exists the inverse $F(\omega)$ of $J(\omega)$ for each $\omega \in \Omega$ and then the Laplace transform can be expressed in terms of characteristic functions of $W, F$, and $T_{(j,r)}$.

Corollary 5.4. In addition to the conditions in Theorem 5.1, assume that $J$ is strictly increasing process and define $F_j$ so that $F_j(\omega)$ is the inverse of $J_y(\omega)$ for each $\omega \in \Omega$. Let us define $\chi(a, X) := \mathbb{E}e^{iaX}$ be the characteristic function for any random variable $X$. Then,

$$\mathbb{E}e^{-rZ_{T^t}} = \frac{1}{2\pi} \int \int \chi(ir, W_j) \chi(\xi, F_j) \chi(-\xi, T_{(j,r)}^t) d\xi dj.$$ 

Note that, in Theorem 5.1, the only ambiguous term is $\mathbb{E} \left[ \exp \left( -i\xi T_{(j,r)}^t \right) \right]$. It can be calculated in a closed-form for only highly restricted cases. In general, we need to solve a Fokker-Plank PDE to calculate it. Note that the PDF of $(v_{(j,r)}^t, T_{(j,r)}^t)$ under $\mathbb{P}$ does not depend on $j$ if $t \leq j$. Let the PDF of $(v_{(j,r)}^t, T_{(j,r)}^t)$ be $G(t, x, y)$ when $t \leq j$. Then, $G$ is a (weak) solution of the following Fokker-Plank PDE:

$$(\partial u G)(u, x, y) = \frac{1}{2} \partial_{xx} \left[ |\sigma(u, x)|^2 G(u, x, y) \right] - \partial_x \left[ (\mu(u, x) - r\rho \sigma(u, x)) G(u, x, y) \right]$$

(8) 

$$- x \partial_y G(u, x, y)$$

$$G(0, x, y) = \delta_{v_0}(x) \delta(y).$$

Hence, the Fourier transform of $T_{(j,r)}^t$ can be expressed using $G$. 

10
Proposition 5.5. If \((8)\) has a unique solution \(G\),

\[
\mathbb{E} \left[ \exp \left( -i \xi T_t^{(j,r)} \right) \right] = \begin{cases} 
\iint \exp(\Phi(t, j, -\xi, x, y)) G(j, x, y) dxdy & \text{if } t > j \\
\iint \exp(-i\xi y) G(t, x, y) dxdy & \text{if } t \leq j.
\end{cases}
\]

It is noteworthy that if \(\Phi(t, s, \xi, x, y)\) is affine with respect to \(y\), then the Fokker-Plank PDE can be reduced to one-dimension using Fourier transform.

Proposition 5.6. Assume that \(\Phi(t, s, \xi, x, y) = \phi(t, s, \xi, x) + i\psi(t, s, \xi, x) y\) and there is a unique solution to

\[
(\partial_u \hat{G})(u, x, \eta) = \frac{1}{2} \partial_{xx} \left[ |\sigma(u, x)|^2 \hat{G}(u, x, \eta) \right] - \partial_x \left[ (\mu(u, x) - r \rho \sigma(u, x)) \hat{G}(u, x, \eta) \right] - i\eta \hat{G}(u, x, \eta).
\]

with \(\hat{G}(0, x, \eta) = \delta_v(x)\). In this case,

\[
\mathbb{E} \left[ \exp \left( -i \xi T_t^{(j,r)} \right) \right] = \begin{cases} 
\int \exp(\Phi(t, j, -\xi, x, y)) \hat{G}(j, x, -\psi(t, j, -\xi, x)) dx & \text{for } t > j \\
\int \hat{G}(t, x, \xi) dx & \text{for } t \leq j.
\end{cases}
\]

Remark 5.7. For characteristic function \(\mathbb{E}[e^{i \theta Z_{J_t}}]\), we have a formula that is corresponding to the case where \(r = -i\theta\), that is,

\[
\mathbb{E}e^{i \theta Z_{J_t}} = \frac{1}{2\pi} \iint e^{-\theta^2 j/2} \left( \int f_{J_b}(j)e^{i\xi y} dy \right) \mathbb{E} \left[ e^{-i \xi T_t^{(j,r)}} \xi_t^{(j,r)} \right] d\xi dj.
\]

\[
\mathbb{E} \left[ e^{-i \xi T_t^{(j,r)}} \right] = \begin{cases} 
\iint \exp(\Phi(t, j, -\xi, x, y)) G^\theta(j, x, y) dxdy & \text{if } t > j \\
\iint \exp(-i\xi y) G^\theta(t, x, y) dxdy & \text{if } t \leq j.
\end{cases}
\]

where \(G^\theta\) is the solution of

\[
(\partial_u G^\theta)(u, x, y) = \frac{1}{2} \partial_{xx} \left[ |\sigma(u, x)|^2 G^\theta(u, x, y) \right] - \partial_x \left[ (\mu(u, x) + i\theta \rho \sigma(u, x)) G^\theta(u, x, y) \right] - x \partial_y G^\theta(u, x, y)
\]

\(G^\theta(0, x, y) = \delta_v(x)\delta_{T_0}(y)\).

As we noted earlier, one cannot simply change \(r = -i\theta\) to extend the Girsanov transform. See appendix for derivation and details of the above statement.

Let \(Y_t = \alpha J_t + \beta Z_{J_t}\). The following theorem gives the moment generating function for the TCL process (2).

Theorem 5.8. Let \(Y_t := \alpha J_t + \beta Z_{J_t}\). Then, the Laplace transform of \(Y\) is given by

\[
\mathbb{E}e^{-rY_t} = \frac{1}{2\pi} \iint e^{r^2 \beta^2 j/2 - r\alpha j} \left( \int e^{i\xi y} f_{J_b}(j) dy \right) \mathbb{E} \left[ \exp \left( -i \xi T_t^{(j,r)} \right) \right] d\xi dj.
\]

Remark 5.9. Note that, for a constant \(j\), the Laplace transform of \(\alpha j + \beta W_j\) is given by \(e^{r^2 \beta^2 j/2 - r\alpha j}\).
6 Relation to the jump and stochastic volatility specifications of Huang and Wu (2005)

Huang and Wu (2005) provide a framework that encompasses almost all of the models proposed in the option pricing literature. Their framework is based on two ingredients: jump structure and stochastic volatility component. In this section, we discuss the relationship between our framework and Huang and Wu (2005), regarding jump structure and stochastic volatility specification. We show that all specified option pricing models in Huang and Wu (2005) are nested into our framework.

6.1 Jump structure
Any Lévy jump process (or any Lévy process) is uniquely defined by its Lévy triplet: location, scale, and Lévy measure. The structure of the jump is captured by its Lévy measure, i.e., \( \nu(dx) \), which controls the arrival rate of jumps of size \( x \in \mathbb{R}^0 \) (the real line excluding zero). Lévy jump processes can be categorized into three categories: (i) finite activity; (ii) infinite activity with finite variation, and; (iii) infinite activity with infinite variation.

In this section, we define a Lévy process by changing the time of a Brownian motion with a subordinator, which is a pure jump process, i.e. equation (3). The advantage of this definition is that one can construct any jump process from (3) only by specifying the subordinator \( J_t \). For example, if \( J_t \) is the gamma process, then \( X_t \) is the variance gamma process. Therefore, equation (3) encompasses all specifications for the jump structure in Huang and Wu (2005).

6.2 Stochastic volatility
Huang and Wu (2005) consider the return process before time-changes as the sum of a Brownian motion with constant drift and a pure jump process. They specify four different stochastic volatility models, by changing the time of the Brownian motion and/or jump component of the return process. Four stochastic volatility models of theirs are: (SV1) stochastic volatility from diffusion component; (SV2) stochastic volatility from jump component; (SV3) joint contribution from jump and diffusion, where the stochastic time-change is the same for both diffusion and jump component, and; (SV4) joint contribution from jump and diffusion processes with different stochastic time-changes.

This section will discuss how to employ our result to four stochastic volatility specifications in Huang and Wu (2005). In these models, for a given filtration \( \mathbb{F} \), the process \( X_t \) is an adapted Lévy process. Then, by Lévy -Itô decomposition theorem, there are continuous Lévy process \( X^c \) and pure-jump Lévy process \( X^j \) which are independent and \( X = X^c + X^j \).

\[\text{Any Lévy process can be expressed as this. See equation (3) of Huang and Wu (2005).}\]
Huang and Wu (2005) studied the effect of time-changes $T^c$ and $T^j$ which are applied to $X^c$ and $X^j$, respectively. Throughout the section, since Huang and Wu (2005) assumed that $T^c, T^j$, and $X$ are adapted to the same filtration, we can assume the following without losing generality:

$$X^c_i = a^c_1 t + a^c_2 B^c_i + a^c_3 B^j_i + a^c_4 W^c_i$$
$$X^j_i = a^j_1 J_i + a^j_2 W^j_i$$
$$dv^c_i = \mu^c(1 - v^c_i) dt + \sigma^c \sqrt{v^c_i(1 - \rho^2)} dB^c_i + \rho \sigma^c \sqrt{v^c_i} dB^j_i$$
$$dv^j_i = \mu^j(1 - v^j_i) dt + \sigma^j \sqrt{v^j_i} dB^j_i$$

$$T^c_i = \int_0^t v^c_s ds$$
$$T^j_i = \int_0^t v^j_s ds.$$  

Here, $(a^c_i, \mu^c, \sigma^c)_{i \in \{c,j\}, i=1,2,...,}$ and $\rho$ are given constants, $B^c, B^j, W^c, W^j$ are independent Brownian motions, and $J$ is a pure jump process. This is possible because, under the adaptedness condition in Huang and Wu (2005), $X^j$ is always independent to any other Brownian motion by Remark 3.5.

Under these assumptions, one can calculate the Laplace transform of $Y_i := X^c_{T^c_i} + X^j_{T^j_i}$, for specific choices of constants that correspond to the stochastic volatility models of Huang and Wu (2005). For notational convenience, let us denote $L_J(r, t) := \mathbb{E}e^{-rJ_i}$. Note that $J$ is given by our model and

$$\mathbb{E}\left[\exp\left(-rX^j_{T^j_i}\right) \mid T^j_i\right] = \mathbb{E}\left[\exp\left(-ra^j_1 J_{T^j_i} - ra^j_2 W^j_{T^j_i}\right) \mid T^j_i\right]$$
$$= \mathbb{E}\left[\exp\left(-ra^j_1 J_{T^j_i}\right) \mathbb{E}\left[\exp\left(-ra^j_2 W^j_{T^j_i}\right) \mid J_{T^j_i}, T^j_i \right] \mid T^j_i\right]$$
$$= \mathbb{E}\left[\exp\left(-ra^j_1 J_{T^j_i} + \frac{r^2[a^j_2]^2}{2} J_{T^j_i} \mid T^j_i\right)\right]$$
$$= L_J\left(ra^j_1 - \frac{r^2[a^j_2]^2}{2}, T^j_i\right).$$

In addition,

$$\mathbb{E}\exp\left(-rX^c_{T^c_i}\right) = \mathbb{E}\exp\left(-ra^c_1 T^c_i + a^c_2 B^c_{T^c_i} + a^c_3 B^j_{T^c_i} + a^c_4 W^c_{T^c_i}\right)$$
$$= \mathbb{E}\left[\exp\left(-r\left(a^c_1 T^c_i + a^c_2 B^c_{T^c_i} + a^c_3 B^j_{T^c_i}\right)\right) \mathbb{E}\left[\exp\left(-ra^c_4 W^c_{T^c_i}\right) \mid B^c, B^j \right]\right]$$
$$= \mathbb{E}\left[\exp\left(-r\left((a^c_1 - r[a^c_2]^2/2) T^c_i + a^c_2 B^c_{T^c_i} + a^c_3 B^j_{T^c_i}\right))\right].$$

Note that, for $N := |(\rho a^c_2 - \sqrt{1 - \rho^2 a^c_2} \sqrt{1 - \rho^2 a^c_2})|, B := \sqrt{1 - \rho^2 B^c + \rho B^j},$  

$$\tilde{B} := \frac{1}{N}\left((\rho a^c_2 - \rho \sqrt{1 - \rho^2 a^c_2}) B^c + ((1 - \rho^2) a^c_2 - \rho \sqrt{1 - \rho^2 a^c_2}) B^j\right),$$
we know B and \( \tilde{B} \) are independent Brownian motion with

\[
\tilde{B}_t := (a_1^c - r|a_4^c|^2/2)t + a_2^c B_t^c + a_3^c B_t^j
\]

\[
= (a_1^c - r|a_4^c|^2/2)t + (a_2^c \sqrt{1 - \rho^2 + \rho \alpha_5^c}) B_t + N \tilde{B}_t.
\]

The Laplace transform of \( X_t^c \) is given by

\[
\mathbb{E} \exp(-r \tilde{B}_t) = \mathbb{E} \left[ \exp \left( -r(\alpha_5^c - \rho \tilde{\alpha}_5^c) B_t - r|a_4^c|^2/2 \right) \mathbb{E} \left[ e^{-rN \tilde{B}_t} | B \right] \right]
\]

\[
= \mathbb{E} \left[ \exp \left( -r(\alpha_5^c - \rho \tilde{\alpha}_5^c) B_t - r(\alpha_5^c - r(|a_4^c|^2 + \beta^c)/2) \right) \right]
\]

If we let \( J_t = t, \alpha = a_4^c - r(|a_4^c|^2 + \beta^c)/2, \beta = a_5^c \sqrt{1 - \rho^2 + \rho \alpha_5^c} \) and \( Z = B \) in Theorem 5.8, we get

\[
\mathbb{E} e^{-rX_t^c} = \frac{1}{2\pi} \int e^{-\beta^2 j/2 - \rho \alpha j + i\xi j} \mathbb{E} \left[ \exp \left( -i \tilde{J}_t \right) \right] d\xi dj.
\]

where

\[
d\tilde{v}_t = \left[ \mu^c (1 - \tilde{v}_u) - r/\sqrt{\tilde{v}_u} u \right] du + \sigma^c \sqrt{\tilde{v}_u} dB_u, \quad \text{Eq} \tag{10}
\]

\[
\tilde{J}_t := \int_0^t \tilde{v}_u du.
\]

### 6.3 When \( T^j \) and \( (X^c, T^c) \) are independent

The case where \( T^j \) is independent from \( X^c \) and \( T^c \) includes SV1 (\( v_0^j = 1, \mu^j = \sigma^j = \rho = 0 \)) and SV4 (\( \rho = 0 \)) models in the Huang and Wu (2005). In this case, we have \( \rho = 0 \) in the previous subsection and

\[
\mathbb{E} e^{-rY_t} = \mathbb{E} \left[ \exp \left( -rX_t^c \right) \mathbb{E} \left[ \exp \left( -rX_t^j \right) \right] \right]
\]

\[
= \frac{1}{2\pi} \mathbb{E} \left[ L_J \left( r a_1^c - \frac{r^2 |a_2^c|^2}{2}, T_t^j \right) \right] \int e^{-\beta^2 j/2 - \rho \alpha j + i\xi j} \mathbb{E} \left[ \exp \left( -i \tilde{J}_t \right) \right] d\xi dj
\]

where \( \tilde{J} \) is given by (10).

### 6.4 When \( T^c \) and \( (X^c, T^j) \) are independent

This case becomes the SV2 model of Huang and Wu (2005) when \( v_0^c = 1 \) and \( \rho = a_2^c = \mu^c = \sigma^c = 0 \). Let \( \rho = 0 \) and \( a_2^c = 0 \). Then,

\[
\mathbb{E} e^{-rY_t} = \mathbb{E} \left[ \exp \left( -r \left( X_t^j + a_3^c T_t^c + a_5^c B_t^j \right) \right) \right] \mathbb{E} \left[ \exp \left( -r a_2^c W_t^c \right) | X^j, B^c, B^j \right]
\]

\[
= \mathbb{E} \left[ \exp \left( -r \left( (a_1^c - r|a_4^c|^2/2) T_t^c + X_t^j + a_5^c B_t^j \right) \right) \right]
\]

\[
= \mathbb{E} \left[ \exp \left( -r \left( (a_1^c - r|a_4^c|^2/2) T_t^c + a_5^c B_t^j \right) \right) L_J \left( r a_1^c - \frac{r^2 |a_2^c|^2}{2}, T_t^j \right) \right].
\]

Note that

\[
F(t, s) := \mathbb{E} \left[ a_3^c B_t^j L_J \left( r a_1^c - \frac{r^2 |a_2^c|^2}{2}, T_t^j \right) \right]
\]
\[
\int \int a_3^c z L J \left( r a_1^j - \frac{r^2|a_2^j|^2}{2}, y \right) f_{T_i^j, B_i^j}(y, z) dy dz
\]

where \( f_{T_i^j, B_i^j} \) can be calculated by Proposition 4.3. Then,

\[
E e^{-\gamma t_i} = E \left[ \exp \left( -r \left( (a_1^e - r|a_4^e|^2/2)T_i^e \right) \right) \mathcal{F}(t, T_i^e) \right].
\]

### 6.5 When \( T^c = T^j \)

This is the case for SV3 of Huang and Wu (2005) if \( \rho = 1, a_2^e = 0, \mu^e = \mu^j, \) and \( \sigma^e = \sigma^j \). Then,

\[
E e^{-\gamma t_i} = E \left[ \exp \left( -r X_i^c \right) \mathcal{E} \left[ \exp \left( -r X_j^c \right) | T_j^c, X_i^c \right] \right]
\]

\[
= E \left[ \exp \left( -r \left( a_1^c T_i^j + a_3^c B_j^j + a_4^c W_i^c \right) \right) L J \left( r a_1^j - \frac{r^2|a_2^j|^2}{2}, T_i^j \right) \right]
\]

\[
= E \left[ \exp \left( -r \left( (a_1^c - r|a_4^c|^2/2)T_i^j + a_3^c B_i^j \right) \right) L J \left( r a_1^j - \frac{r^2|a_2^j|^2}{2}, T_i^j \right) \right].
\]

The last expression can be calculated using the following formula and our result on calculating \( f_{T_i, B_j} \) using Proposition 4.3.

\[
E g(T_i^j) \exp \left( -r B_i^j \right) = \int \int g(y) e^{-rz} f_{T_i^j, B_i^j}(y, z) dy dz
\]

\[
= \int \int g(y) e^{-rz} f_{T_i^j, B_i^j}(y, z) dy dz.
\]

### 7 Summary and future research

We derived an analytic formula based on a solution of a PDE for the transitional density, moment generating function, and characteristic function of a time-changed Lévy process, where there is a correlation between the Lévy and activity rate process. Given the generality of the specifications in Huang and Wu (2005), our approach is applicable to virtually all of the models proposed in the option pricing literature.

In order to find the transitional density of a TCL process, our method consists of the following step: (i) solve (5) or (6), and; (ii) apply Theorem 4.4. On the other hand, if one wants to find the Laplace transform of the TCL process, our method consists of the following step: (i') solve (8) or (9) based on the model; (ii') find \( E \left[ \exp \left( -i\xi T_t^{(j,r)} \right) \right] \) using Proposition 5.5 or 5.6, and; (iii') apply Theorem 5.1. Each method has their own strength and weakness. The former approach require more computation power for calculating PDE but the latter approach needs inverse transform.

There are two directions for future investigation. First of all, one can conduct an empirical implementation of the theoretical results and investigate the empirical performance the correlated time-changed Lévy processes in time series analysis and option pricing. In
this case, one also needs to analyze the numerical aspects of PDEs appeared in this article. In other words, one needs to find an efficient and accurate numerical method to find PDEs appeared in this article and analyzes the convergence speed and estimate the error. Another direction of the future research is to explore the theoretical properties and empirical advantages of a TCL process when the activity rate process is a stopping time. In this case, Carr and Wu (2004) provide an analytic formula for the characteristic function of a TCL process when the Lévy and activity rate process are not independent, and also, the activity rate process is a stopping time.

**A Proofs**

*Proof of Proposition 4.1* Note that

\[ f_{W J T_t, B J T_t, J T_t}(w, b, j) = f_{W J_t}(w) f_{B J_t}(b, j) \]

since \( W_j \) and \( (B_j, J T_t) \) are independent. Therefore,

\[
\begin{align*}
    f_{Z J T_t}(z) &= \frac{1}{\rho \sqrt{1 - \rho^2}} \int \int f_{W J_t, B J_t, J T_t} \left( \frac{z - b}{\sqrt{1 - \rho^2}}, \frac{b}{\rho}, j \right) \, dj \, db \\
    &= \frac{1}{\rho \sqrt{1 - \rho^2}} \int \int f_{W_j} \left( \frac{z - b}{\sqrt{1 - \rho^2}} \right) f_{B_j, J T_t} \left( \frac{b}{\rho}, j \right) \, db \, dj \\
    &= \frac{1}{\sqrt{1 - \rho^2}} \int \int f_{W_j} \left( \frac{z - \rho \tilde{b}}{\sqrt{1 - \rho^2}} \right) f_{B_j, J T_t} \left( \tilde{b}, j \right) \, d\tilde{b} \, dj
\end{align*}
\]

where \( \tilde{b} = b/\rho \). On the other hand, since \( J \) is independent to \( T \), we have

\[
f_{B_j, J T_t}(b, j) = \int f_{B_j, J_{y, T_t}}(b, j, y) \, dy
\]

\[
= \int f_{B_j|J_{y, T_t}}(b|j, y) f_{J_{y|T_t}}(j|y) f_{T_t}(y) \, dy
\]

\[
= \int f_{B_j|J_{y, T_t}}(b|j, y) f_{J_y}(j) f_{T_t}(y) \, dy
\]

Since \( \sigma(B_j, T_t) \) and \( J_y \) are independent, we have \( f_{B_j|J_{y, T_t}} = f_{B_j|T_t} \) and therefore,

\[
f_{B J_t, J T_t}(b, j) = \int f_{J_y}(j) f_{B_j, J T_t}(b, y) \, dy.
\]

In sum,

\[
f_{Z J T_t}(z) = \int \int \frac{1}{\sqrt{2\pi j(1 - \rho^2)}} e^{-\frac{(z - \rho \tilde{b})^2}{2(1 - \rho^2)}} f_{B_j, J T_t}(b, y) f_{J_y}(j) \, dy \, db \, dj.
\]
Proof of Proposition 4.3 Note that
\[ q(t, x, y, z) := f_{v_t, T_t, B_t}(x, y, z); \quad t \leq j \]
is a solution to (5). For \( t \leq j \),
\[
\begin{align*}
    f_{T_t, B_j}(y, z) &= \int \int f_{B_j | B_t, T_t}(z | z, x, y) f_{B_t, v_t, T_t}(z, x, y) dx \, dz \\
    &= \int \int f_{B_j - B_t}(z - z) f_{B_t, v_t, T_t}(z, x, y) dx \, dz \\
    &= \frac{1}{\sqrt{2\pi(j-t)}} \int \int e^{-\frac{(z-y)^2}{2(j-t)}} q(t, x, y, z) dx \, dz
\end{align*}
\]
since \( T_t, B_t \in \mathcal{F}_t^B \) are independent to \( B_j - B_t \). On the other hand, for \( t \geq j \), we have
\[
\begin{align*}
    \mathbb{E} \left[ e^{i\xi T_t} | B_j \right] &= \int e^{i\xi y} f_{T_t | B_j}(y | z) dy \bigg|_{z = B_j} \\
    &= \int \sqrt{2\pi j} \exp \left( \Phi(t, j, \xi, x, \tilde{y}) \right) q(j, x, y, z) dx \, dy \bigg|_{z = B_j}
\end{align*}
\]
Therefore,
\[
\begin{align*}
    f_{Z_{JT_t}}(z) &= f_{T_t | B_j}(y | z) f_{B_j}(z) \\
    &= \frac{1}{2\pi} \int e^{-i\xi y} \int \exp \left( \Phi(t, j, \xi, x, \tilde{y}) \right) q(j, x, y, z) dx \, dy \, d\tilde{y} \, d\xi \\
    &= \frac{1}{2\pi} \int \int \exp \left( \Phi(t, j, \xi, x, \tilde{y}) - i\xi y \right) q(j, x, y, z) dx \, dy \, d\tilde{y} \, d\xi.
\end{align*}
\]
\[
\square
\]

Proof of Theorem 4.4 Note that, for \( \hat{f}_{T_t, B_j}(\xi, \eta) := \int \int e^{-i\xi y - i\eta z} f_{T_t, B_j}(y, z) dy \, dz \),
\[
\begin{align*}
    f_{B_j, T_i}(b, y) &= \frac{1}{(2\pi)^2} \int e^{i\xi y + i\eta b} \hat{f}_{T_t, B_j}(\xi, \eta) d\xi \, d\eta.
\end{align*}
\]
Therefore,
\[
\begin{align*}
    f_{Z_{JT_t}}(z) &= \int \int \int \frac{1}{\sqrt{2\pi j(1-\rho^2)}} e^{-\frac{(z-y)^2}{2(1-\rho^2)}} f_{B_j, T_i}(b, y) f_{J_y}(j) dy \, db \, dj \\
    &= \frac{1}{(2\pi)^2} \int \int \int \int \frac{1}{\sqrt{2\pi j(1-\rho^2)}} e^{-\frac{(z-y)^2}{2(1-\rho^2)}} e^{i\xi y + i\eta b} \hat{f}_{T_t, B_j}(\xi, \eta) f_{J_y}(j) \, d\xi \, d\eta \, dy \, db \, dj.
\end{align*}
\]
Then our claim follows by the same argument in the proof of Theorem 4.4.

\[ \square \]

Proof of Theorem 5.1

On the other hand, if \( t \leq j \),

\[
\hat{f}_{T_i, B_j}(\xi, \eta) = \frac{1}{\sqrt{2\pi(j - t)}} \int \int \int \int e^{-i\xi y - i\eta z} e^{-\frac{(y - \rho) \cdot b}{2\rho^2}} q(t, x, y, z) dx dy dz
\]

\[ = e^{-\frac{(j-t)^2}{2\rho^2}} \int \int \int e^{-i\xi y - i\eta z} q(t, x, y, z) dx dy dz
\]

\[ = e^{-\frac{(j-t)^2}{2\rho^2}} \int \hat{q}(t, x, \xi, \eta) dx. \]

In the case where \( t > j \), by noting the calculation of \( \mathbb{E}[e^{-i\xi T_t}|B_j] \) in the proof of Proposition 4.3,

\[
\hat{f}_{T_i, B_j}(\xi, \eta) = \mathbb{E}e^{-i\xi T_t - i\eta B_j} = \mathbb{E} \left[ e^{-i\eta B_j} \mathbb{E} \left[ e^{-i\xi T_t}|B_j\right] \right]
\]

\[ = \int e^{-i\eta z} \left( \int \sqrt{2\pi} \exp \left( \Phi(t, j, -\xi, x, y) + \frac{x^2}{2j} \right) q(j, x, y, z) dx dy \right) f_{B_j}(z) dz
\]

\[ = \int \int \exp \left( \Phi(t, j, -\xi, x, y) \right) \left( \int e^{-i\eta z} q(j, x, y, z) dz \right) dx dy
\]

\[ = \frac{1}{2\pi} \int \int \exp \left( \Phi(t, j, -\xi, x, y) \right) \left( \int e^{i\xi y} \hat{q}(j, x, \xi, \eta) d\xi dz \right) \hat{q}(j, x, \xi, \eta) d\xi dx.
\]

\[ \square \]

Proof of Theorem 4.6 Note that

\[ f_{Y_t}(\bar{y}) = \frac{1}{\beta} \int \int f_{W_i, B_j, T, J_t} \left( \frac{1}{\beta} \sqrt{1 - \rho^2} (\bar{y} - \rho \beta b - \alpha j), b, j \right) dbdj
\]

\[ = \frac{1}{\beta} \int \int f_{W_i} \left( \frac{1}{\beta} \sqrt{1 - \rho^2} (\bar{y} - \rho \beta b - \alpha j) \right) f_{B_j, T, J_t} (b, j) dbdj
\]

\[ = \frac{1}{\beta} \int \int \frac{1}{\sqrt{2\pi j}} \exp \left( -\frac{(\bar{y} - \rho \beta b - \alpha j)^2}{2\beta^2 j (1 - \rho^2)} \right) f_{B_j, T, J_t}(b, y) f_{J_t}(j) dy db dj
\]

Then our claim follows by the same argument in the proof of Theorem 4.4.

\[ \square \]

Proof of Theorem 5.1 Let

\[ q(t, x, y, z) := f_{v_t, T_i, B_t}(x, y, z); \quad j \geq t \]

\[ p(j, t, x, y, b) := f_{v_t, T_i | B_t}(x, y | b); \quad j \leq t \]

18
\[
\mathbb{E}e^{-rZ_{j,t}} = \mathbb{E} \left[ e^{-r\rho B_{j,t}} \mathbb{E} \left[ e^{-r\sqrt{1-\rho^2} W_{j,t}} \middle| J, B \right] \right]
\]

\[
= \int_0^t \int \int \int \exp \left( -r\rho b + \frac{r^2(1-\rho^2)j}{2} \right) f_{B_i,T_i}(b,y)f_{J_y}(j)dydbdj \]

\[
= \int_0^t \frac{1}{\sqrt{2\pi}j} \int \int \int \exp \left( -r\rho b + \frac{r^2(1-\rho^2)j}{2} - \frac{b^2}{2j} \right) p(j,t,x,y,b)f_{J_y}(j)dxdydbdj \]

\[
+ \int_t^\infty \frac{1}{\sqrt{2\pi(j-t)}} \int \int \int \int \exp \left( -r\rho b + \frac{r^2(1-\rho^2)j}{2} - \frac{(b-z)^2}{2(j-t)} \right) q(t,x,y,z)f_{J_y}(j)dxdydzdbdj
\]

Note that

\[
\frac{1}{\sqrt{2\pi(j-t)}} \int e^{-r\rho b - \frac{(b-z)^2}{2(j-t)}} db = \exp \left( -r\rho z + \frac{r^2\rho^2(j-t)}{2} \right)
\]

and therefore, when \( j > t \), by integrating with respect to \( b \),

\[
\frac{1}{\sqrt{2\pi(j-t)}} \int \int \int \int \int \exp \left( \frac{r^2(1-\rho^2)j}{2} - r\rho z + \frac{r^2\rho^2(j-t)}{2} \right) q(t,x,y,z)f_{J_y}(j)dxdydz
\]

\[
= \int \int \int \int \exp \left( -r\rho z + \frac{r^2(j-\rho^2t)}{2} \right) q(t,x,y,z)f_{J_y}(j)dxdydz
\]

By changing the integration parameter \( b \) to \( z \) for the first term, we have,

\[
\mathbb{E}e^{-rZ_{j,t}} = \int_0^t \frac{1}{\sqrt{2\pi}j} \int \int \int \exp \left( -r\rho z + \frac{r^2(1-\rho^2)j}{2} - \frac{z^2}{2j} \right) p(j,t,x,y,z)f_{J_y}(j)dxdydzdj
\]

\[
+ \int_t^\infty \int \int \int \int \int \exp \left( -r\rho z + \frac{r^2(j-\rho^2t)}{2} \right) q(t,x,y,z)f_{J_y}(j)dxdydzdj
\]

\[
= \int \int \int \int \exp \left( -r\rho z + \frac{r^2(j-(j\wedge t)\rho^2)}{2} \right) Q(j,t,x,y,z)f_{J_y}(j)dxdydzdj
\]

where

\[
Q(j,t,x,y,z) := \frac{1}{\sqrt{2\pi j}}e^{-\frac{z^2}{2j}}p(j,t,x,y,z)1_{j\leq t} + q(t,x,y,z)1_{j > t}
\]

\[
= f_{t,B_{j\wedge t}}(x,y,z).
\]

If we simplify above equation by considering \( g(y,j) := f_{J_y}(j) \) as a distribution with variables \((y,j)\),

\[
\mathbb{E}e^{-rZ_{N_{j,t}}} = \int e^{r^2j/2}e^{-r^2\rho^2(j\wedge t)/2} \mathbb{E} \left[ g(T_t,j) e^{-r\rho B_{j\wedge t}} \right] dj.
\]

Let us denote \( \mathbb{E}^{(j,r)} \) be the expectation under changed measure

\[
dP^{(j,r)} = \exp \left( -r\rho B_{j\wedge t} - \frac{1}{2} r^2 \rho^2 (j \wedge t) \right) dP =: \mathcal{F}_t^{(j,r)} dP
\]

and \( f^{(j,r)}_\xi \) to be the probability density of a random variable \( \xi \) under \( P^{(j,r)} \). Under \( P^{(j,r)} \), since \( \mathcal{F}_t^{(j,r)} \) is uniform integrable martingale by Novikov condition, \( B^{(j,r)} := B + r\rho 1_{[0,j]}(\cdot) \) is
a Brownian motion on $[0, t]$ such that
\[ \sigma(B_s^{(j,r)} : s \leq u) = \sigma(B_s : s \leq u) \]
for all $u \leq t$. Note that, for any $\tilde{j}$,
\[ \mathbb{E} \left[ g(T_t, \tilde{j}) \xi_t^{(j,r)} \right] = \mathbb{E}^{(j,r)} \left[ g(T_t, \tilde{j}) \right] = \int g(y, \tilde{j}) \tilde{f}^{(j,r)}_{T_t}(y) dy = \int f_{j, \tilde{j}}(\tilde{j}) \tilde{f}^{(j,r)}_{T_t}(y) dy. \]
Therefore,
\[ \mathbb{E} e^{-rZ_{j,T_t}} = \int e^{\frac{r^2}{2} \int g(T_t, j) \, dj} = \int e^{\frac{r^2}{2} \int f_{j, \tilde{j}}(\tilde{j}) \tilde{f}^{(j,r)}_{T_t}(y) dy dy \, dj} \]
Note that, for $\mathbb{P}^{(j,r)}$-Brownian motion $B^{(j,r)} := B + \int_0^t r \rho 1_{[0, j]}(s) ds$,
\[
dv_u = (\mu(u, v_u) - \rho \sigma(u, v_u) 1_{[0, j]}(u)) du + \sigma(u, v_u) dB^{(j,r)}_u \]
\[dT_u = v_u du \]
Then, the distribution of $(v_t^{(j,r)} , T_t^{(j,r)})$ under $\mathbb{P}$ and the distribution of $(v_t, T_t)$ under $\mathbb{P}^{(j,r)}$ are identical. Therefore,
\[ \mathbb{E}^{(j,r)} [\exp(-i\xi T_t)] = \mathbb{E} \left[ \exp \left( -i\xi T_t^{(j,r)} \right) \right]. \]
In sum, we have
\[
\mathbb{E} e^{-rZ_{j,T_t}} = \int e^{\frac{r^2}{2} \int f_{j, \tilde{j}}(\tilde{j}) \frac{1}{2\pi} \exp(-i\xi T_t^{(j,r)})} \, d\xi dy dy \, dj = \frac{1}{2\pi} \int e^{\frac{r^2}{2} \int f_{j, \tilde{j}}(\tilde{j}) \exp(i\xi T_t)} \mathbb{E} \left[ \exp(-i\xi T_t^{(j,r)}) \right] \, d\xi dy \, dj \]
\[ \square \]

**Proof of Proposition 5.5** The second part of the claim is obvious since $G(t, x, y, z)$ is the PDF of $(v_t^{(j,r)}, T_t^{(j,r)})$ when $t \leq j$. Consider the case when $t > j$. Since SDE (7) is identical to (4) for $u > j$, we have
\[ \mathbb{E} \left[ \exp \left( -i\xi T_t^{(j,r)} \right) \big| \mathcal{F}_j \right] = \exp(\Phi(t, j, -\xi, v_j^{(j,r)}, T_j^{(j,r)})) \]
by our definition of $\Phi$. If we take the expectation on both side, we have the first part of the claim. \[ \square \]

**Proof of Proposition 5.6** Note that
\[ \mathbb{E} \exp(\Phi(t, j, -\xi, v_j^{(j,r)}, T_j^{(j,r)})) = \mathbb{E} \exp \left( \phi(t, j, -\xi, v_j^{(j,r)}) + i\psi(t, j, -\xi, v_j^{(j,r)}) T_j^{(j,r)} \right) \]
\[ \int \int \exp (\phi(t, j, -\xi, x) + i\psi(t, j, -\xi, x)y) G(j, x, y) dy dx \]

\[ = \int e^{\phi(t, j, -\xi, x)} \int \exp (i\psi(t, j, -\xi, x)y) G(j, x, y) dy dx \]

Note that
\[ \hat{G}(u, x, \eta) := \int e^{-i\eta y} G(u, x, y) dy \]

satisfies (9) and
\[ \int \exp (i\psi(t, j, -\xi, x)y) G(j, x, y) dy = \hat{G}(j, x, -\psi(t, j, -\xi, x)) \]
\[ \int \exp (-i\xi y) G(t, x, y) dy = \hat{G}(t, x, \xi). \]

\[ \square \]

**Proof of Theorem 5.8** We can follow the steps from previous results. For \( Q \) defined as in the proof of Theorem 5.1,
\[ \mathbb{E} e^{-rY_t} = \mathbb{E} \left[ e^{-r\alpha J_t - r\beta \rho B_{J_t} T_t} | J, B \right] \]
\[ = \mathbb{E} \left[ e^{-r\alpha J_{T_t} - r\beta \rho B_{J_{T_t}} T_t} \mathbb{E} \left[ e^{-r\beta \sqrt{1 - \rho^2} W_{J_{T_t}} T_t} | J, B \right] \right] \]
\[ = \mathbb{E} \left[ e^{-r\alpha J_{T_t} - r\beta \rho B_{J_{T_t}} T_t} + \frac{r^2 \beta^2 (1 - \rho^2)}{2} J_{T_t} \right] \]
\[ = \mathbb{E} \left[ \exp \left( -r\beta \rho B_{J_{T_t}} + \left( \frac{r^2 \beta^2 (1 - \rho^2)}{2} - r\alpha \right) J_{T_t} \right) \right] \]
\[ = \int \int \int \exp \left( -r\alpha j - r\beta \rho b + \frac{r^2 \beta^2 (1 - \rho^2)}{2} j \right) f_{B_{J_t} T_t}(b, y) f_{J_t}(j) dy db dj \]
\[ = \int \int \int \exp \left( -r\alpha j - r\beta \rho z + \frac{r^2 \beta^2 (j - (j \wedge t) \rho^2)}{2} \right) Q(j, t, x, y, z) f_{J_t}(j) dx dy dz dj \]
\[ = \int e^{r^2 \beta^2 j/2 - r \alpha j} e^{-r^2 \beta^2 \rho^2 (j \wedge t)/2} \mathbb{E} \left[ g(T_t, j) e^{-r\beta \rho B_{J_t} T_t} \right] dj \]
\[ = \frac{1}{2\pi} \int \int e^{r^2 \beta^2 j/2 - r \alpha j} \left( \int e^{i\xi y} f_{J_t}(j) dy \right) \mathbb{E} \left[ \exp \left( -i\xi T_t^{(j, r\beta)} \right) \right] d\xi dj. \]

Here, \( g(y, j) := f_{J_t}(j) \). \[ \square \]
B Proof of Remark 5.3

The argument is analogous to the proof of Theorem 5.1. From our calculation in the proof of Theorem 5.1, we have

\[
E e^{i\theta Z_{j\nu t}} = \iint \int \exp \left( i\theta \rho b - \frac{\theta^2 (1 - \rho^2) j}{2} \right) f_{B_j, T_j}(b, y) f_{J_y}(j) dy db dj
\]

\[
= \int_0^t \frac{1}{\sqrt{2\pi j}} \iint \int \exp \left( i\theta \rho b - \frac{\theta^2 (1 - \rho^2) j}{2} \right) \frac{\sqrt{\pi}}{2j} p(j, t, x, y, b) f_{J_y}(j) dx dy db dj
\]

\[
+ \int_t^\infty \frac{1}{\sqrt{2\pi (j - t)}} \iint \int \exp \left( i\theta \rho b - \frac{\theta^2 (1 - \rho^2) j}{2} \right) \frac{\sqrt{\pi}}{2j} q(t, y, z, f_{J_y}(j) dx dy dz db dj
\]

Note that

\[
\frac{1}{\sqrt{2\pi (j - t)}} \int e^{i\theta \rho b - \frac{(b - z)^2}{2(\theta^2 - \rho^2)}} db = \exp \left( i\theta \rho z - \frac{\theta^2 (j - t)}{2} \right)
\]

and therefore, when \( j > t \), by integrating with respect to \( b \),

\[
\frac{1}{\sqrt{2\pi (j - t)}} \iint \int \exp \left( -\frac{\theta^2 (1 - \rho^2) j}{2} + i\theta \rho z - \frac{\theta^2 (j - t)}{2} \right) q(t, x, y, z) f_{J_y}(j) dx dy dz db
\]

\[
= \iint \int \exp \left( i\theta \rho z - \frac{\theta^2 (j - \rho^2 t)}{2} \right) q(t, x, y, z) f_{J_y}(j) dx dy dz
\]

By changing the integration parameter \( b \) to \( z \) for the first term, we have,

\[
E e^{i\theta Z_{j\nu t}} = \int_0^t \frac{1}{\sqrt{2\pi j}} \iint \int \exp \left( i\theta \rho z - \frac{\theta^2 (1 - \rho^2) j}{2} - \frac{z^2}{2j} \right) p(j, t, x, y, z) f_{J_y}(j) dx dy dz dj
\]

\[
+ \int_t^\infty \iint \int \exp \left( i\theta \rho z - \frac{\theta^2 (j - \rho^2 t)}{2} \right) q(t, x, y, z) f_{J_y}(j) dx dy dz dj
\]

\[
= \iint \int \exp \left( i\theta \rho z - \frac{\theta^2 (j - \rho^2 t)}{2} \right) Q(j, t, x, y, z) f_{J_y}(j) dx dy dz dj
\]

where

\[
Q(j, t, x, y, z) := \frac{1}{\sqrt{2\pi j}} e^{-\frac{z^2}{2j}} p(j, t, x, y, z) 1_{j \leq t} + q(t, x, y, z) 1_{j > t}
\]

\[
= f_{y, T_j, B_j jN}(x, y, z).
\]

If we simplify above equation by considering \( g(y, j) := f_{J_y}(j) \) as a distribution with variables \( (y, j) \),

\[
E e^{i\theta Z_{j\nu t}} = \int e^{-\theta^2 j/2} g^{\theta^2 (j \wedge t)/2} \mathbb{E} \left[ g(T_j, j) e^{i\theta B_j jN} \right] dj
\]

\[
= \int e^{-\theta^2 j/2} \mathbb{E} \left[ g(T_j, j) \mathcal{E}_t^\theta \right] dj.
\]

Then, we have the following proposition by the same technique in the proof of Theorem 5.1.
Proposition B.1. 
\[ E \left[ g(T_t, j) E^{\theta,j}_t \right] = \frac{1}{2\pi} \int \left( \int g(y, j) e^{i\xi y} dy \right) E \left[ e^{-i\xi T_t} E^{\theta,j}_t \right] d\xi \]

Proof. Note that 
\[ E \left[ g(T_t, j) E^{\theta,j}_t \right] = \int \int g(y, j) e^{i\varphi_0 + \frac{\varphi^2}{2} t} f_{T_t, B_{j\wedge t}}(y, b) dy db \]
\[ = \frac{1}{2\pi} \int \int \int g(y, j) e^{i\xi y} e^{i\varphi_0 + \frac{\varphi^2}{2} t} e^{-i\xi \bar{y}} f_{T_t, B_{j\wedge t}}(\bar{y}, b) d\bar{y} d\xi d\bar{y} db \]
\[ = \frac{1}{2\pi} \int \left( \int g(y, j) e^{i\xi y} dy \right) E \left[ e^{-i\xi T_t} E^{\theta,j}_t \right] d\xi \]

Therefore,
\[ E e^{i\theta Z_{N_t}} = \frac{1}{2\pi} \int \int e^{-\varphi^2/2} \left( \int f_{v_t}(j) e^{i\xi y} dy \right) E \left[ e^{-i\xi T_t} E^{\theta,j}_t \right] d\xi dj \]

C Calculating \( E \left[ e^{-i\xi T_t} E^{\theta,j}_t \right] \) using Fokker Planck equation

Lemma C.1. For \( t > j \),
\[ E \left[ e^{-i\xi T_t} E^{\theta,j}_t \right] = E \left[ \exp(\Phi(t, j, \xi, v_j, T_j)) E^{\theta,j}_j \right] \]

Proof. For \( t > j \), we have \( E^{\theta,j}_t = E^{\theta,j}_j \in F_j \) and therefore,
\[ E \left[ \exp (-i\xi T_t) E^{\theta,j}_t \bigg| F_j \right] = E^{\theta,j}_j E \left[ \exp (-i\xi T_t) \big| F_j \right] \]
\[ = E^{\theta,j}_j \exp(\Phi(t, j, \xi, v_j, T_j)) \]
by our definition of \( \Phi \). We prove the lemma by taking the expectation on both side. \( \square \)

Above lemma tell us that we only need to know distribution of \((v_t, T_t, B_t)\) for \( t \leq j \) to calculate \( E \left[ e^{-i\xi T_t} E^{\theta,j}_t \right] \).

Proposition C.2. Let
\[ G^\theta(t, x, y) := \int \exp \left( i\theta \rho z + \frac{\theta^2 \rho^2 t}{2} \right) f_{v_t, T_t, B_t}(x, y, z) dz. \]
Then, \( G^\theta \) satisfies
\[ (\partial_u G^\theta)(u, x, y) = \frac{1}{2} \partial_{xx} \left[ |\sigma(u, x)|^2 G^\theta(u, x, y) \right] - \partial_x \left[ (\mu(u, x) + i\theta \rho \sigma(u, x)) G^\theta(u, x, y) \right] \]
\[ - x \partial_y G^\theta(u, x, y) \]
\[ G^\theta(0, x, y) = \delta_{v_0}(x) \delta_{T_0}(y) \]
Proof. Note that $H(t, x, y, z) := f_{v_t, T_t, B_t}(x, y, z)$ satisfies the Fokker Planck PDE:
\[
\partial_u H = \frac{1}{2} \partial_{xx} [\sigma^2 H] + \frac{1}{2} \partial_{zz} [\sigma H] - \partial_x [\mu H] - \partial_y [x H].
\]
If we multiply both sides with $\exp \left( i \theta \rho z + \frac{\theta^2 \rho^2 u}{2} \right)$ and integrate with respect to $z$, then the left hand side becomes
\[
\int \exp \left( i \theta \rho z + \frac{\theta^2 \rho^2 u}{2} \right) (\partial_u H)(u, x, y, z)dz = \partial_u G^\theta (u, x, y) - \frac{\theta^2 \rho^2}{2} G^\theta (u, x, y).
\]
On the other hand, we have
\[
\int \exp \left( i \theta \rho z + \frac{\theta^2 \rho^2 u}{2} \right) (\partial_z H)(u, x, y, z)dz = -\frac{i \theta \rho}{2} G^\theta (u, x, y)
\]
by integration by parts. Therefore, the right-hand side becomes
\[
\frac{1}{2} \partial_{xx} [\sigma^2 G^\theta] - \frac{\theta^2 \rho^2}{2} G^\theta - i \theta \rho \partial_x [\sigma G^\theta] - \partial_x [\mu G^\theta] - x \partial_y [G^\theta].
\]
We proved the claim. \hfill \qed

Proposition C.3. For solution $G^\theta$ of (11),
\[
E \left[ e^{-i \xi T_t} \xi_t^{\theta,j} \right] = \begin{cases} \int \int \exp \left( \Phi(t, j, \xi, x, y) \right) G^\theta (j, x, y) dxdy & \text{if } t > j \\ \int \int \exp \left( -i \xi y \right) G^\theta (t, x, y) dxdy & \text{if } t \leq j \end{cases}
\]
If $\Phi(t, s, \xi, x, y)$ is affine with respect to $y$, then the Fokker-Planck PDE can be simplified.

Proposition C.4. Assume that
\[
\Phi(t, s, \xi, x, y) = \phi(t, s, \xi, x) + i \psi(t, s, \xi, x) y
\]
In this case,
\[
E \left[ \exp \left( -i \xi T_t \right) \xi_t^{\theta,j} \right] = \begin{cases} \int e^{\phi(t, j, \xi, x)} \hat{G}^\theta (j, x, y) dx & \text{for } t > j \\ \int \hat{G}^\theta (t, x, \xi) dx & \text{for } t \leq j \end{cases}
\]
where $\hat{G}^\theta$ satisfies $\hat{G}^\theta (0, x, \eta) = \delta_{00} (x)$ and
\[
(\partial_u \hat{G}^\theta)(u, x, \eta) = \frac{1}{2} \partial_{xx} \left[ |\sigma(u, x)|^2 \hat{G}^\theta (u, x, \eta) \right] - \partial_x \left[ (\mu(u, x) + i \theta \rho \sigma(u, x)) \hat{G}^\theta (u, x, \eta) \right] - i \eta \hat{G}^\theta (u, x, \eta).
\]
Proof. Let us define
\[
\hat{G}^\theta (t, x, \eta) := \int e^{-i \eta y} G(t, x, y, z) dy.
\]
Then, it satisfies the Fourier transform of (11), that is,
\[
(\partial_u \hat{G}^\theta)(u, x, \eta) = \frac{1}{2} \partial_{xx} \left[ |\sigma(u, x)|^2 \hat{G}^\theta (u, x, \eta) \right] - \partial_x \left[ (\mu(u, x) + i \theta \rho \sigma(u, x)) \hat{G}^\theta (u, x, \eta) \right]
\]
\[-i\eta \hat{G}^\theta(u, x, \eta)\].

Note that
\[
\int\int \exp \left( \Phi(t, j, \xi, x, y) \right) G^\theta(j, x, y) dx dy
= \int e^{i\phi(t, s, \xi, x)} \int e^{i\psi(t, s, \xi, x)} y G^\theta(j, x, y) dy dx
\]
By the definition of \(\hat{G}^\theta\),
\[
\int e^{i\psi(t, j, \xi, x)} y G^\theta(j, x, y) dy = \hat{G}^\theta(j, x, -\psi(t, j, \xi, x))
\]
and the claim is proved. \(\square\)
References


Madan, D., and M. Yor. 2006. CGMY and Meixner subordinators are absolutely continuous with respect to one sided stable subordinators. *Prepublication du Laboratoire de Probabilites et Modeles Aléatoires* .

