Presenter: **Associate Professor Burkard Polster, Monash University - *2015***

Title: ***Using maths to solve the Rubik's cube™ (20:57)***

Time	***Dialogue***
00:15 00:57	So in 1980 the world changed. Rubik came out with his cubes and from one day to the next basically the world changed into one big huge Rubik's cube. … (Laughter). I was there so I saw it happen. I was actually going to high school and I got hold of one of those Rubik's cubes and the next 3 weeks were a blur. Basically I was there just trying to solve this thing. And after 3 weeks I could do the Rubik's cube, I could do anything. Give me a Rubik's cube anything scrambled and I could do it. That's pretty much the end of it for most people. A lot of people can do the Rubik's cube these days so is it now solved once and for all? Well definitely not for real mathematicians like myself and actually most people who do the Rubik's cube, what they do is they kind of go out, Google it, solve the Rubik's cube and take someone else's recipe, memorise it and then that's what they unleash on a Rubik's cube but that's not what I call solving and actually that's not what I did at the time. So when I first got my Rubik's cube, I didn't look up anything. I just tried for three weeks to come up with a method myself to actually solve Rubik's cubes.
01:36 02:07	Just in general, that was pretty much my first really, really, really big maths problem that I solved. Rubik's cube is inherently very mathematical. It's got permutations in the background, algebra structures, and lots of complexity of course. So it's very mathematical and a three week problem, I solved it and that very much got me going as a mathematician. I know a lot of people and my colleagues who had similar experiences, not only that but a Rubik's cube in itself, everybody was playing with it so from one day to the next there was this big Mount Everest in front of us, in front of anybody who wanted to engage with this thing in a serious way. We had to conquer it. We had to know everything about it. … and, well you would think that by now we would know everything about the Rubik's cube and that's not it at all. In fact one of the main questions has only been solved recently and I'm going to tell you a little bit about this.
02:38 03:04	Everybody knows that (solving) the Rubik's cube is hard OK? You know that the Rubik's cube is hard? You've tried? Just how hard is it? OK... Just how hard is it? Well, I have a whole collection of these things here so let me just show you quickly how hard something like this is. So there is a Rubik's cube here and I'm just going to take it apart for you. At the time what people actually recommended if you can't do it, is you take the stickers off and then put them on again, but it's actually not a good idea at all, (… laughter) because those stickers will never really stay where they are supposed to be. So what you can do is kind of twist this thing like 45 degrees and you dig in with your finger like that and then you can actually take it apart like that, and those things actually come together quite easily and I'm going to explode the thing , so it going. It's almost gone. So there …there …right. (… laughter) and the only thing left over is this middle part and I will just show you all the pieces now here.
03:40 04:04	So we have got a cube, so we've got eight corner pieces and we've got twelve edges so we've got edge pieces and then we also got this 3D cross in the middle and that thing is completely welded together and that doesn't move at all. When you operate the cube this will kind of stay fixed in the middle. OK just imagine all those pieces lying around, put on a blindfold and lets just figure out how we can put this thing together while I grope around the table grab the first piece and putting into the cross and grab the second piece and putting into the cross and lets just keep on going until I have just got something cubical in my hand. OK?

Time	*Dialogue*
04:18 04:52	That won't look like the cube on the right, guaranteed. (… laughter) There is a very, very small chance of that happening. So how many different ways can you put this thing together? It's actually quite easy to calculate. You take a corner piece, there are 8 corner positions and you put them somewhere and there are seven left and the second corner can go to seven different places, OK and then next one six and so on, 8 X 7 X 6 X 5 and so on. You can calculate that and you can orient them in a couple of different ways. Anyway pretty straight forward calculation and you get … well a few combinations (… laughter) so you can put the cube into that many different configurations. OK, that's quintillions. So let's give you an idea. If you measure the surface of the earth in millimetres squared that's about the number you get. Right? So if you shrink a cube to 1mm by 1mm by 1mm and take that many of them you can cover the surface of the earth with those mini cubes. So that's a lot of cubes and a big huge number.
05:24 05:58	So this number there is really easy to come up with but only some of these configurations can actually be solved with legal moves and so the real number of configurations is a bit smaller and there is a nice relationship between the two numbers. You just divide the first number by twelve and you get the number of configurations. So if you do this blindfolded experiment and you ask what is the probability that you actually get something that can be solved? It's one in twelve. So you get a one in twelve chance. So to figure out what the twelve does, that's some really, really nice mathematics. So the first number is really easy to get but to get the twelve to justify that takes some smart thinking. All right, now. So we've got all these things to deal with like anybody who designs an algorithm, like me all those years ago when I designed my algorithm to take care of any Rubik's cube my algorithm, my recipe has to be able to take care of all those possibilities. Right so that sounds like a really, really formable task.
06:29 07:05	Now when the Rubik's cube came up, mathematicians were pretty much the first ones who really got hold of this thing. I still have some colleagues, some older colleagues who tell me that the thing came out, I just left my work, I went to Hungary and I got this thing (… laughter) and I got a couple of other ones and brought them back to the UK actually and it was a big, big hit with mathematicians and they were the first ones to come up with some really good recipes for solving these things. So … what else did they do well? There is the Mount Everest. What is a really really good maths question to ask? Well the really, really good maths questions beyond finding a method that works for everything is to find the best method to solve the Rubik's cube. So what do I mean by this? Well, just imagine you have all those different configurations. You put the first configuration into one Rubik's cube and the second configuration into another Rubik's cube and just line them all up. Yeah, line them all up.
07:30 08:01	And now we look at the first guy and say, "what is the minimum number of moves it takes to do this one, to solve this one, OK"? And then you do the same for the second one, the third one and so you basically have these different configurations that you have got numbers hovering above them, right and then you look at all those numbers and you see what is the worst case scenario right? So that one might take 15, that one might take 17, what's the worst? What is the worst it can take to solve this? If you have got some really super human being, how many moves will it take? Hmmm, There are some super human beings in Melbourne. The champion speed 'cuber', he does it in about, a Rubik's cube of 3 by 3 by 3 in an average of 7 second or something like this. How many moves do you think they take on average? Well about 50 to 60 in less than ten seconds. You don't even see the moves. It's spectacular but it gets a lot better if you go for optimal. Optimal is really asking for God's algorithm. What would God do if he had to do (solve) the Rubik's cube?

Time	*Dialogue*
08:41 09:13	Ok. So how do you do something like this right? You've got this huge, huge, huge number of possibilities so how do you conquer something like this? Well just recently we were facing a similar sort of problem and at that point it was really a matter of life and death to solve this. I'm assuming a lot of people will now guess what it was? It was kind of figuring out how to crack the Enigma cipher machine. OK. So the Enigma cipher machine was used by the Germans. That's my guys. You can probably tell by the accent? So they were enciphering all their messages with this machine. Now on the slide you can see three arrows. The three arrows point at three rotors. The rotors have the 26 letters of the alphabet on it and they can be put into the machine in all possible ways. Basically, you know it is sometimes three rotors or sometimes it is five rotors and with the really big machines you come up to about a million possibilities of just putting the rotors in the machine.
09:49 10:26	So that is a lot of possibilities but it actually gets a lot worse because you've also got the green arrow. The Green arrow points at a plug board and you see the letters of the alphabet again there and you can actually connect up pairs of letters of the alphabet with cables, right, and that switches around things worse … right. So if you take all those possibilities together, like how you put the rotors in and how you put the plugs in you get about the same number of positions again as a Rubik's cube. So it is bad, right? And what makes it worse is that the Germans actually set up this machine every day in a different way. So it works like in the morning everybody who has one of those machines sets it up in exactly the same way. Then throughout the day messages get transmitted using that setting, they kind of float around, they are publically available and can be intercepted but they are of no use if you don't know how the Enigma machine is set up in the first place. So you intercept all these messages and you look at them and you somehow have to infer from these messages how the machine is set up and then you are OK but you have to this once a day otherwise it is pretty useless.
11:00 11:21	OK, so how do you do this, how do you do this? Well, it's a combination of kind of brute force computing and being really, really, really smart. OK, (… laughter) and actually the people who did the work and you have probably seen the movie recently, right? (*The Imitation Game*, 2014 is an American historical drama thriller *film* directed by Morten Tyldum based on the real life story of legendary cryptanalyst *Alan* Turing), were actually mathematicians and most of them were pure mathematicians. So people think they do stuff that will never be applied, no, no, no, in this case and I will come to this is a second it was really pure mathematics that really held the key to this problem. So what they did was, they didn't have computers at the time, so they had to invent them basically. So they actually invented the first computers and what did these computers do? Well they basically went through all of the possible rotor positions.
11:50 12:16	So at Betchley Park (*Bletchley Park* was the central site for British codebreakers during World War II) they had rooms full of basically replicas of these rotors and they kind of just click away there for hours going through all of the positions. OK? And then for every rotor position what they could do is kind of exclude this particular rotor position together with all the possible ways of plugging in things at the bottom in kind of one go. OK, so that kind of parcelling up of the problem, so there's these zillions and gadzillions of possibilities, so you parcel them up in a really, really smart way and you take care of these parcels in like one instant. You just look at it and the current flows, it's possible. The current doesn't flow it's not possible and the smart thinking behind it is actually something that anyone who has done a bit of pure mathematics knows, it's called a proof by contradiction. So you use that sort of technique for example to show that the square root of 2 is an irrational number.

Time	***Dialogue***
12:48 13:12	That's pretty far away from any real world application but it actually held the key to kind of being able to look at this problem and in one instant go all this stuffs not possible. OK? So that's what they did and actually the same sort of approach worked recently for the Rubik's cube. So again for the Rubik's cube we have got all these possibilities, now it's a matter of just parcelling them up in the right way. Having a really ingenious way of dealing with these individual parcels and then unleashing some computers on them. Actually *Google* donated like heaps of hours of computing time and they crunched through all the things and now we know … that … the worst case scenario is 20 (moves) and that actually gives you (refer to image) the complete list of possibilities. So you can see that he numbers kind of ramp up and they kind of peak there somewhere and there is kind of a sudden drop down to 20.
13:50 14:19	So these really worst case positions there is actually quite a few of them still, yeah, but if someone actually scrambles up the cube for you and you ask, well, how many moves can I expect here, if a super human being is unleashed on this scrambled up cube what can I expect here in terms of moves? It's about 18. About 18. That's where things bunch up here in the diagram. OK. So that is kind of a lot of hand waving, right and some kind of smart thinking in the background that takes care of this huge problem. I actually want to leave you with something that you can actually use yourself. Like one of those really ingenious mathematical insights that you can bring to bear on the original problem. OK? So, which will basically enable anybody here to design their own recipe. Ok. So they design their own recipe. Now most people when they first touch a Rubik's cube and let me just get another one here as I have destroyed one already, so there. That's an intact one.
14:56 15:34	So the first time you scramble it up and then you try really, really hard for a couple of hours and then what most people who are really serious about this manage to do is, they can fix the first layer. OK? They can fix the first layer. These nine pieces that you see here … those everybody can do. Alright, now my claim is if you get to this stage, if you are persistent enough to get to this stage where you can solve the first layer, then you can solve the whole cube. Now most people actually give up at this stage. Why do you give up? (… laughter). Because anything you do now, pretty much anything, well you can do this. Right, and that won't destroy the first layer, but anything else will cut through the first layer and actually destroy what have just, you know put together like in an hour. So basically what you need in this point in time is like sequences of moves that only touch very small parts of the cube, but don't touch anything else. OK? So things that manipulate only very small parts of the cube. So how do we do this? How do we do this? OK. So here's the idea, the basic idea.
16:05 16:34	It's called a '*Commutator*' in Mathematics. It comes up quite a bit in abstract algebra. Here's what we do. So if you are a master of the first layer and you are all masters of the first layer and I can see you are masters of the first layer. What you can do is, you can do anything you want with the first layer. So if you don't worry about the bottom being kind of scrambled up, you can take these two guys and kind of swap them around. You can take this one out and flip it and don't have anything else here in the top layer change, right? For example let's just focus on this one bit here. This guy here. This guy is an edge piece and what we want to do is switch it over. Now you may not realise this but I assure you if you are a master of the first layer, you can do this. I'm just going to switch this over now. So what I'm going to do is I'm just going to go slowly here … and … now in the top layer, you can see the top layer is perfectly alright. The bottom is kind of messed up. That's bad to start with, but you know that's fine. Now how do we fix up, fix up the cube? Well we'll run this whole thing backwards. OK?

Time	***Dialogue***
17:13 17:39	So if I run what I've just done backwards it's going to restore the cube. OK. Let me show you. I'll run it backwards, so we go like that … and it's back to normal. Right? So forwards, backwards, forwards, backwards. So you just practice this for a while. OK? So you design your move of doing this will be different to my move but it doesn't matter. You run it forward, you run it backwards, you run it forwards, you run it backwards until you are really, really OK with that. So let's run it forward now. You've practiced this for an hour right? Now we run it forward so OK, you are in, you are in. Now we take notice of what happens when we run this backwards. When we run this backwards this guy is flipped and the bottom is restored. Right? That is what it does. OK. Now the trick is before we run it backwards we give it a bit of a twist. OK. We give it a bit of a twist. Alright, now if I run this backwards now what is going to happen to the bottom? It's going to get restored. (… laughter)
18:14 18:55	Sounds like a bit of magic but I didn't change anything essential here. So whatever happens when I run things backwards really just refers to the positions here. This guy gets flipped and nothing else on top happens and the bottom gets restored. So what I will do is make a twist like this and now I run it backwards. So let me run this backwards now (… pause) and there and I just have to untwist this thing there and you can see now the magic's happened, just based on my move or your move. Right? What's happened then is that everything here has stayed unchanged the only thing that has changed is these two guys have been flipped over, OK? And with a move like that you can actually fix the orientation of all the edge pieces of the cube. You can do something similar for twisting corners for example. So as a master of the first layer you can design a move that kind of twists one of these guys and then you kind of do this, untwist, and do that and you've got a move that twists this one and that one and you can use that to fix up the orientation of all the corners. OK?
19:31 20:06 20:40	And then the same as the other thing for moving positions so all you have to do is design a total of 4 different magic moves like this and you can put them together and make your own recipe for the Rubik's cube. Not only for this one. I've got about 400 of these and this is the main insight that gets you into all of them. I've got things like this. So a few layers here (… laughter) you know and once you scramble that one up its going to take a while to fix that but basically it's the same idea, right? So we've got our top layer here and once we are masters of the top layer we can do certain things. Anyway what that shows you is, or I wanted to show you is that although a problem can be incredibly complex it is sometimes these tiny ingenious ideas that reduce it to something manageable and that was really the case for these recipes that we are designing here and also for a really, really big problem and OK that's the Rubik's cube for you. Thanks. (…applause)